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FIXED POINT RESULTS FOR CONTRACTIVE MAPPINGS IN COMPLEX VALUED FUZZY METRIC SPACES

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Abstract. The aim of this paper is to introduce the notion of complex valued fuzzy metric spaces. We prove some fixed point results of contractive mappings on complex valued fuzzy metric spaces. Some examples are presented to support the results proved herein. Our results extend various results in the existing literature.
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1. Introduction

Fixed point theory is one of the well known traditional theories in mathematics that has a broad set of applications. Existence of fixed point of mappings satisfying certain contractive conditions can be employed to prove the existence of solution to nonlinear equations. The Banach's contraction principle, the most widely used fixed point theorem in all of mathematical analysis, is constructive in nature and is one of the most useful tools in solving existence problems in many branches of mathematics. This theorem, which has been extended in many directions, has many applications in mathematics and other related disciplines as well. These generalizations were obtained either by improving the contractive conditions or by imposing some additional conditions on the ambient space. There have been a number of generalizations of metric spaces such as, rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, D-metric spaces and cone metric spaces. Recently, Azam et al. [1] have introduced a new approach in metric fixed point theory by replacing the set of positive real numbers with complex number system endowed with an order structure. In this way, they have introduced the concept of complex valued metric space.

On the other hand, Zadeh [19] introduced the concept of fuzzy sets. This fact initiated an intense research activity leading to the development of the interesting theory of fuzzy sets and systems. The notion of fuzzy metric space was initiated by Kramosil and Michálek [11]. George and Veeramani [7] modified the concept of fuzzy metric space given in [11] to obtain a Hausdorff topology induced by such fuzzy metric. Grabiec [8] initiated the fixed point theory in fuzzy metric spaces. For the definitions, examples and some basic properties of fuzzy metric spaces necessary to this work, we refer to [7, 8, 10, 11] and the references mentioned therein.

In this paper, we extend the concept of fuzzy metric space to complex valued fuzzy metric space. We also obtain some fixed point results in complex valued fuzzy metric spaces.

2. Complex valued fuzzy metric spaces

In this section, complex valued fuzzy metric spaces are defined and some properties of such spaces are proved.

In what follows, $\mathbb{C}$ denotes the complex number system over the field of real numbers. We set $P = \{(a,b) \colon 0 \le a < \infty, 0 \le b < \infty\} \subset \mathbb{C}$. The elements $(0,0)$, $(1,1) \in P$ are denoted by θ and ℓ, respectively.

Define a partial ordering $\preceq$ on $\mathbb{C}$ by $c_1 \preceq c_2$ (or, equivalently, $c_2 \succeq c_1$) if and only if $c_2 - c_1 \in P$. We write $c_1 \prec c_2$ (or, equivalently, $c_2 \succ c_1$) to indicate that $\mathrm{Re}(c_1) < \mathrm{Re}(c_2)$ and $\mathrm{Im}(c_1) < \mathrm{Im}(c_2)$ (see, also, [1]). A sequence $\{c_n\}$ in $\mathbb{C}$ is said to be monotonic with respect to $\preceq$ if either $c_n \preceq c_{n+1}$ for all $n \in \mathbb{N}$ or $c_{n+1} \preceq c_n$ for all $n \in \mathbb{N}$.

We define the closed unit complex interval by $I = \{(a,b) \colon 0 \le a \le 1, 0 \le b \le 1\}$, and the open unit complex interval by $I_O = \{(a,b) \colon 0 < a < 1, 0 < b < 1\}$. P_θ denotes the set $\{(a,b) \colon 0 < a < \infty, 0 < b < \infty\}$. It is obvious that, for $c_1, c_2 \in \mathbb{C}$, $c_1 \prec c_2$ if and only if $c_2 - c_1 \in P_\theta$.

For $A \subset \mathbb{C}$, if there exists an element $\inf A \in \mathbb{C}$ such that it is a lower bound of A, that is, $\inf A \preceq a$ for all $a \in A$ and $u \preceq \inf A$ for every lower bound $u \in \mathbb{C}$ of A, then $\inf A$ is called the greatest lower bound or infimum of A. Similarly, we define $\sup A$, the least upper bound or supremum of A, in usual manner.

Remark 2.1. Let $c_n \in P$ for all $n \in \mathbb{N}$ then:

(a) If the sequence $\{c_n\}$ is monotonic with respect to $\preceq$ and there exists $\alpha, \beta \in P$ such that $\alpha \preceq c_n \preceq \beta$, for all $n \in \mathbb{N}$, then there exists $c \in P$ such that $\lim_{n\to\infty} c_n = c$.

(b) Although the partial ordering $\preceq$ is not a linear (total) order on $\mathbb{C}$, the pair $(\mathbb{C}, \preceq)$ is a lattice.

(c) If $S \subset \mathbb{C}$ is such that

$$\text{there exist } \alpha, \beta \in \mathbb{C} \text{ with } \alpha \preceq s \preceq \beta \text{ for all } s \in S,$$

then $\inf S$ and $\sup S$ both exist.

Definition 2.2. Let X be a nonempty set. A complex fuzzy set M is characterized by a mapping with domain X and values in the closed unit complex interval I.

Definition 2.3. A binary operation $*\colon I \times I \to I$ is called a complex valued $t-$norm if:

(1) $c_1 * c_2 = c_2 * c_1$;
(2) $c_1 * c_2 \preceq c_3 * c_4$ whenever $c_1 \preceq c_3, c_2 \preceq c_4$;
(3) $c_1 * (c_2 * c_3) = (c_1 * c_2) * c_3$;
(4) $c * \theta = \theta, c * \ell = c$;

for all $c, c_1, c_2, c_3, c_4 \in I$.

The following are some examples of complex valued $t-$norms.

Example 2.4. Let the binary operations $*_1, *_2, *_3 \colon I \times I \to I$ be defined, respectively, by

(1) $c_1 *_1 c_2 = (a_1a_2, b_1b_2)$, for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$;
(2) $c_1 *_2 c_2 = (\min\{a_1, a_2\}, \min\{b_1, b_2\})$, for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$;
(3) $c_1 *_3 c_2 = (\max\{a_1 + a_2 - 1, 0\}, \max\{b_1 + b_2 - 1, 0\})$, for all $c_1 = (a_1, b_1)$, $c_2 = (a_2, b_2) \in I$.

Then $*_1, *_2$ and $*_3$ are complex valued $t-$norms.

Indeed, if $I_{\mathbb{R}} = [0, 1]$ is the closed unit real interval and $\star_1, \star_2 \colon I_{\mathbb{R}} \times I_{\mathbb{R}} \to I_{\mathbb{R}}$ are two $t-$norms, then $*\colon I \times I \to I$ defined by

$$c_1 * c_2 = (a_1 \star_1 a_2, b_1 \star_2 b_2), \text{ for all } c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I,$$

is a complex valued $t-$norm.

Example 2.5. Define $*\colon I \times I \to I$ as follows:

$$c_1 * c_2 = \begin{cases} (a_1, b_1), & \text{if } (a_2, b_2) = \ell; \\ (a_2, b_2), & \text{if } (a_1, b_1) = \ell; \\ \theta, & \text{otherwise,} \end{cases}$$

for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$. Then $*$ is a complex valued $t-$norm.

Note that, in the example above, $c_1 * c_2$ cannot be expressed as $(a_1 \star_1 a_2, b_1 \star_2 b_2)$, where $\star_1, \star_2 \colon I_{\mathbb{R}} \times I_{\mathbb{R}} \to I_{\mathbb{R}}$ are two $t-$norms.

Definition 2.6. Let X be a nonempty set, $*$ a continuous complex valued $t-$norm and M a complex fuzzy set on $X^2 \times P_\theta$ satisfying the following conditions:

(1) $\theta \prec M(x, y, c)$;
(2) $M(x, y, c) = \ell$ for every $c \in P_\theta$ if and only if $x = y$;
(3) $M(x, y, c) = M(y, x, c)$;
(4) $M(x, y, c) * M(y, z, c') \preceq M(x, z, c + c')$;
(5) $M(x, y, \cdot)\colon P_\theta \to I$ is continuous;

for all $x, y, z \in X$ and $c, c' \in P_\theta$.
Then the triplet $(X, M, *)$ is called a complex valued fuzzy metric space and M is called a complex valued fuzzy metric on X. A complex valued fuzzy metric can be thought of as the degree of nearness between two points of X with respect to a complex parameter $c \in P_\theta$.

Example 2.7. Let (X, d) be any metric space. Define $*$ by $c_1 * c_2 = (a_1a_2, b_1b_2)$, for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$. Let the complex fuzzy set M be given by

$$M(x, y, c) = \frac{a+b}{a+b+d(x, y)}\ell$$

for all $x, y \in X$, $c = (a, b) \in P_\theta$. Then $(X, M, *)$ is a complex valued fuzzy metric space.

Indeed, if (X, d) is a metric space and $f\colon P_\theta \to (0, \infty)$ is a continuous and nondecreasing function, that is, $c_1 \preceq c_2$ implies $f(c_1) \leq f(c_2)$, then $(X, M, *)$ is a complex valued fuzzy metric space, where $c_1 * c_2 = (a_1a_2, b_1b_2)$ for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$ and

$$M(x, y, c) = \frac{f(c)}{f(c)+d(x, y)}\ell$$

for all $x, y \in X, c \in P_\theta$.
Similarly, it is obvious that, for any metric space (X, d), the triplet $(X, M, *)$ is a complex valued fuzzy metric space, where

$$M(x, y, c) = \left[\exp\left(\frac{d(x, y)}{f(c)}\right)\right]^{-1}\ell$$

for all $x, y \in X, c \in P_\theta$, $c_1 * c_2 = (a_1a_2, b_1b_2)$ for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$ and $f\colon P_\theta \to (0, \infty)$ is a continuous and nondecreasing function.

The following example is inspired by [6] and Example 4.6 [9].

Example 2.8. Let $X = \mathbb{N}$ (the set of all natural numbers). Define $*$ by $c_1 * c_2 = (a_1a_2, b_1b_2)$, for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$. Let the complex fuzzy set M be given by

$$M(x, y, c) = \begin{cases} \dfrac{x}{y}\ell, & \text{if } x \leq y; \\ \dfrac{y}{x}\ell, & \text{if } y \leq x, \end{cases}$$

for all $x, y \in X$, $c \in P_\theta$. Then $(X, M, *)$ is a complex valued fuzzy metric space.

Lemma 2.9. *Let $(X, M, *)$ be a complex valued fuzzy metric space. If $c, c' \in P_\theta$ and $c \prec c'$, then $M(x, y, c) \preceq M(x, y, c')$ for all $x, y \in X$.*

Proof. Suppose that $c, c' \in P_\theta$ are such that $c \prec c'$. It follows that $c' - c \in P_\theta$ and

$$M(x, y, c) * M(y, y, c' - c) \preceq M(x, y, c').$$

Hence $M(x, y, c) \preceq M(x, y, c')$. □

Let $(X, M, *)$ be a complex valued fuzzy metric space. An open ball $B(x, r, c)$ with center $x \in X$ and radius r, $r \in I_O$, $c \in P_\theta$ is defined by

$$B(x, r, c) = \{y \in X : \ell - r \prec M(x, y, c)\}.$$

The collection $\{B(x, r, c) : x \in X, r \in I_O, c \in P_\theta\}$ is a neighborhood system for the topology τ on X induced by the complex valued fuzzy metric M.

Definition 2.10. Let $(X, M, *)$ be a complex valued fuzzy metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if for each $r \in I_O$ and $c \in P_\theta$ there exists $n_0 \in \mathbb{N}$ such that

$$\ell - r \prec M(x_n, x, c) \quad \text{for all} \quad n > n_0.$$

Lemma 2.11. *Let $(X, M, *)$ be a complex valued fuzzy metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $\lim_{n\to\infty} M(x_n, x, c) = \ell$ holds for all $c \in P_\theta$.*

Proof. Suppose that $\lim_{n\to\infty} M(x_n, x, c) = \ell$ for all $c \in P_\theta$. Let $c \in P_\theta$ be fixed. For $r \in I_O$, there exists a real number $\epsilon > 0$ such that $z \in \mathbb{C}$, $|z| < \epsilon$ implies that $z \prec r$. For this $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|\ell - M(x_n, x, c)| < \epsilon \quad \text{for all} \quad n > n_0.$$

Therefore $\ell - M(x_n, x, c) \prec r$, that is, $\ell - r \prec M(x_n, x, c)$ for all $n > n_0$.

Conversely, consider $c \in P_\theta$ fixed. Let $\epsilon > 0$ be given and for all $r \in I_O$ there exists $n_0 \in \mathbb{N}$ such that $\ell - r \prec M(x_n, x, c)$ for all $n > n_0$. Choose $r \in I_O$ such that $|r| < \epsilon$, then we have $|\ell - M(x_n, x, c)| < |r| < \epsilon$ for all $n > n_0$. Thus $\lim_{n\to\infty} M(x_n, x, c) = \ell$ holds for all $c \in P_\theta$. ☐

Definition 2.12. Let $(X, M, *)$ be a complex valued fuzzy metric space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, c) = \ell \quad \text{for all } c \in P_\theta. \tag{2.1}$$

The complex valued fuzzy metric space $(X, M, *)$ is called complete if every Cauchy sequence in X converges in X.

Remark 2.13. Usually, two definitions of Cauchy sequence are used in the framework of fuzzy metric spaces. One is due to Grabiec [8] and the other is due to George and Veeramani [6]. Similarly, Grabiec [8] and George and Veeramani [6] used different definitions of completeness of fuzzy metric spaces (for details, see [18] and the references mentioned therein). Our definitions of a Cauchy sequence and completeness can be viewed as extensions of the corresponding definitions due to George and Veeramani [6] to the complex valued fuzzy metric spaces.

Lemma 2.14. *Let $(X, M, *)$ be a complex valued fuzzy metric space. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for each $r \in I_O$ and $c \in P_\theta$ there exists $n_0 \in \mathbb{N}$ such that*

$$\ell - r \prec M(x_n, x_m, c) \quad \textit{for all} \quad n, m > n_0. \tag{2.2}$$

Proof. Suppose that (2.1) holds and let $c \in P_\theta$ be fixed. Similarly to the proof of Lemma 2.11, for each $r \in I_O$, there exists $n_0 \in \mathbb{N}$ such that $\ell - \inf_{m>n} M(x_n, x_m, c) \prec r$, for all $n > n_0$, that is, $\ell - r \prec \inf_{m>n} M(x_n, x_m, c)$ for all $n > n_0$. This implies that $\ell - r \prec \inf_{m>n} M(x_n, x_m, c) \preceq M(x_n, x_m, c)$ for all $n > n_0$ and $m > n$. On the other hand, if $m = n > n_0$, $\ell - r \prec \ell = M(x_n, x_m, c)$ and, if $n_0 < m < n$, $\ell - r \prec \inf_{n>m} M(x_m, x_n, c) \preceq M(x_m, x_n, c) = M(x_n, x_m, c)$. Hence, $\ell - r \prec M(x_n, x_m, c)$ for all $n, m > n_0$ and (2.2) holds.

Conversely, consider $c \in P_\theta$ fixed. Let $\epsilon > 0$ be given. We know that, for all $r \in I_O$ there exists $n_0 \in \mathbb{N}$ such that $\ell - r \prec M(x_n, x_m, c)$ for all $n, m > n_0$. Thus, $\ell - 2r \prec \ell - r \preceq \inf_{m>n} M(x_n, x_m, c)$ for all $n > n_0$, i.e., $\ell - \inf_{m>n} M(x_n, x_m, c) \prec 2r$ for all $n > n_0$. We choose $r \in I_O$ such that $|r| < \frac{\epsilon}{2}$, then we have $|\ell - \inf_{m>n} M(x_n, x_m, c)| < 2|r| < \epsilon$ for all $n > n_0$, and (2.1) holds. □

The proof of the following remark is straightforward.

Remark 2.15. Let $c_n, c'_n, z \in P$, for all $n \in \mathbb{N}$, then:

(1) If $c_n \preceq c'_n \preceq \ell$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = \ell$, then $\lim_{n\to\infty} c'_n = \ell$.

(2) If $c_n \preceq z$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = c \in P$, then $c \preceq z$.

(3) If $z \preceq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = c \in P$, then $z \preceq c$.

Let $(X, M, *)$ be a complex valued fuzzy metric space. A self mapping T on X is called a fuzzy Banach contraction if

$$\ell - M(Tx, Ty, c) \preceq k\,[\ell - M(x, y, c)] \tag{2.3}$$

for all $x, y \in X, c \in P_\theta$, where k is a real number such that $k \in [0, 1)$. Here, k is called the fuzzy contractive constant of T.

3. Fixed point theorems

In this section, we prove some fixed point results for self-mappings on complex valued fuzzy metric spaces satisfying certain contractive conditions.

Theorem 3.1. *Let $(X, M, *)$ be a complete complex valued fuzzy metric space and $T\colon X \to X$ a fuzzy Banach contraction with fuzzy contractive constant k. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. First, we show that $\{x_n\}$ is a Cauchy sequence.

For $n \in \mathbb{N}$ and fixed $c \in P_\theta$, define

$$A_n := \{M(x_n, x_m, c)\colon m > n\} \subset I.$$

Since $\theta \prec M(x_n, x_m, c) \preceq \ell$ for all $n \in \mathbb{N}$, by Remark 2.1, the infimum, $\inf A_n = \alpha_n$ (say) exists for all $n \in \mathbb{N}$. For $c \in P_\theta$ and $n, m \in \mathbb{N}$ with $m > n$, by (2.3), we have

$$\begin{aligned} \ell - M(x_{n+1}, x_{m+1}, c) &= \ell - M(Tx_n, Tx_m, c) \\ &\preceq k\left[\ell - M(x_n, x_m, c)\right] \\ &\preceq \ell - M(x_n, x_m, c), \end{aligned} \tag{3.1}$$

which implies that

$$M(x_n, x_m, c) \preceq M(x_{n+1}, x_{m+1}, c) \quad \text{for all} \quad n \in \mathbb{N},\ m > n.$$

Therefore, by definition, we have

$$\theta \preceq \alpha_n \preceq \alpha_{n+1} \preceq \ell, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.2}$$

Thus, $\{\alpha_n\}$ is a monotonic sequence in P and, using Remark 2.1 and (3.2), there exists $\ell_1 \in P$ such that

$$\lim_{n\to\infty} \alpha_n = \ell_1. \tag{3.3}$$

The inequality (3.1) implies that

$$(1-k)\ell + kM(x_n, x_m, c) \preceq M(x_{n+1}, x_{m+1}, c)$$

for all $m > n$, and so $(1-k)\ell + k\alpha_n \preceq \alpha_{n+1}$, for every $n \in \mathbb{N}$, which with (3.3) gives

$$(1-k)\ell \preceq (1-k)\ell_1.$$

Because $k \in [0, 1)$ and using Remark 2.15, we must have $\ell_1 = \ell$. Thus,

$$\lim_{n\to\infty} \alpha_n = \ell.$$

Hence,

$$\lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, c) = \ell, \quad \text{for all} \quad c \in P_\theta. \tag{3.4}$$

Thus, from (3.4), we have proved that $\{x_n\}$ is a Cauchy sequence in X. By completeness of X and Lemma 2.11, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, c) = \ell \ \text{ for all } \ c \in P_\theta. \tag{3.5}$$

For any $c \in P_\theta, n \in \mathbb{N}$, it follows from (2.3) that

$$\ell - M(Tx_n, Tu, c) \preceq k\left[\ell - M(x_n, u, c)\right],$$

that is

$$\ell(1-k) + kM(x_n, u, c) \preceq M(Tx_n, Tu, c). \tag{3.6}$$

Now, for any $c \in P_\theta$,

$$\begin{aligned} M(u, Tu, c) &\succeq M(u, x_{n+1}, c/2) * M(x_{n+1}, Tu, c/2) \\ &= M(u, x_{n+1}, c/2) * M(Tx_n, Tu, c/2). \end{aligned}$$

By taking the limit as $n \to \infty$ and using (3.5), (3.6) and Remark 2.15, we obtain that $M(u, Tu, c) = \ell$ for all $c \in P_\theta$, that is, $Tu = u$.

If $v \in X$ is another fixed point of T and there exists $c \in P_\theta$ such that $M(u, v, c) \neq \ell$, then it follows from (2.3) that

$$\ell - M(u, v, c) = \ell - M(Tu, Tv, c) \preceq k\left[\ell - M(u, v, c)\right], \tag{3.7}$$

where at least one component of $M(u, v, c)$ is less than one. For that $c \in P_\theta$, denote $M(u, v, c) = (a, b)$, then, if $a < 1$, from (3.7), we have that $1 - a \leq k(1 - a) < 1 - a$, which is a contradiction and, if $b < 1$, we have $1-b \leq k(1-b) < 1-b$, a contradiction again. Hence $M(u, v, c) = \ell$ for all $c \in P_\theta$, that is, $u = v$. □

Remark 3.2. In Theorem 3.1, the contraction condition for the mapping T, i.e., (2.3), can be replaced by the following one, with analogous proof:

$$\ell - M(Tx, Ty, c) \preceq k(c)\,[\ell - M(x, y, c)]$$

for all $x, y \in X, c \in P_\theta$, where k is a real function $k : P_\theta \to [0, 1)$.

The following example illustrates the applicability of Theorem 3.1.

Example 3.3. Let $I_{\mathbb{R}} = [0, 1]$ and $X = I_{\mathbb{R}} \times \{0\} \cup \{0\} \times I_{\mathbb{R}}$. Define $*$ by

$$c_1 * c_2 = (\max\{a_1 + a_2 - 1, 0\}, \max\{b_1 + b_2 - 1, 0\})$$

for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$. Define $d\colon X \times X \to \mathbb{C}$ as follows:

$$d((x, 0), (y, 0)) = |x - y|(\alpha, 1), \quad d((0, x), (0, y)) = |x - y|(1, \beta)$$

$$\text{and} \qquad d((x, 0), (0, y)) = d((0, y), (x, 0)) = (\alpha x + y, x + \beta y),$$

where α, β are fixed nonnegative real constants. Then (X, d) is a complex valued metric space. Furthermore, it is possible to define

$$M(p, q, c) = \ell - \frac{d(p, q)}{1 + \alpha + \beta + ab} \quad \text{for all} \quad p, q \in X, c = (a, b) \in P_\theta.$$

Then $(X, M, *)$ is a complete complex valued fuzzy metric space. Let $\gamma, \delta \in [0, 1]$ be fixed real constants and $T\colon X \to X$ be a mapping defined by

$$T(x, 0) = (0, \gamma x), \quad \text{and} \quad T(0, x) = (\delta x, 0).$$

Moreover, suppose that the constants α, β, γ, δ are chosen in such a way that it is possible to take the constant $k \geq 0$ with $k = \max\left\{\frac{\gamma}{\alpha}, \gamma\beta, \delta\alpha, \frac{\delta}{\beta}\right\} < 1$. Then T satisfies the condition (2.3). Here, we are considering α, $\beta > 0$. In the case where $\alpha = 0$ and $\beta \neq 0$, for the validity of (2.3), we have to select $\gamma = 0$ and $0 \leq k = \frac{\delta}{\beta} < 1$; in the case $\beta = 0$ and $\alpha \neq 0$, we have to select $\delta = 0$ and $0 \leq k = \frac{\gamma}{\alpha} < 1$; and, if $\alpha = \beta = 0$, then we have to select $\gamma = \delta = 0$ and any $0 \leq k < 1$.

Thus, all the conditions in Theorem 3.1 are satisfied. Hence, T has a unique fixed point, which is $u = (0, 0) \in X$.

Corollary 3.4. *Let $(X, M, *)$ be a complete complex valued fuzzy metric space. Suppose that $T\colon X \to X$ satisfies*

$$\ell - M(T^n x, T^n y, c) \preceq k\,[\ell - M(x, y, c)]$$

for all $x, y \in X, c \in P_\theta$, where $k \in [0, 1)$ is a constant and n is some positive integer. Then T has a unique fixed point in X.

Proof. As T^n satisfies all the conditions of Theorem 3.1, so T^n has a unique fixed point u in X. But $T^nTu = TT^nu = Tu$ gives that Tu is another fixed point of T^n. By uniqueness, we have $Tu = u$. Since a fixed point of T is also a fixed point of T^n, therefore the fixed point of T is unique. □

For given $r \in I_O, c \in P_\theta$ and $x_0 \in X$, we set $B[x_0, r, c] = \{x \in X : \ell - r \preceq M(x_0, x, c)\}$. In the following, we prove the existence of a fixed point for a mapping which satisfies a restricted contraction condition.

Corollary 3.5. *Let $(X, M, *)$ be a complete complex valued fuzzy metric space such that $c \preceq c * c$ for all $c \in I_O$. Suppose that $T : X \to X$ satisfies:*

i) *There exist $x_0 \in X$ and $r \in I_O$ such that $\ell - r \preceq M(x_0, Tx_0, c)$ for all $c \in P_\theta$.*

ii) *There exists $k \in [0, 1)$ such that, for all $x, y \in B[x_0, r, c], c \in P_\theta$*

$$\ell - M(Tx, Ty, c) \preceq k\left[\ell - M(x, y, c)\right]. \tag{3.8}$$

Then T has a unique fixed point in $B[x_0, r, c]$.

Proof. We only need to prove that $B[x_0, r, c]$ is complete and $Tx \in B[x_0, r, c]$ for all $x \in B[x_0, r, c], c \in P_\theta$.

Suppose that $\{x_n\}$ is a Cauchy sequence in $B[x_0, r, c]$. Then, by completeness of X and by Lemma 2.11, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, c) = \ell \text{ for all } c \in P_\theta.$$

Now, for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$M\left(x_0, u, c + \frac{c}{m}\right) \succeq M(x_0, x_n, c) * M\left(x_n, u, \frac{c}{m}\right).$$

As $x_n \in B[x_0, r, c]$, for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} M(x_n, u, c) = \ell$ for all $c \in P_\theta$, and using the properties of the complex valued $t-$norm and Remark 2.15, we have

$$M\left(x_0, u, c + \frac{c}{m}\right) \succeq (\ell - r) * \ell = \ell - r, \text{ for every } m \in \mathbb{N}.$$

Taking the limit as $m \to \infty$ and by Remark 2.15, we obtain $M(x_0, u, c) \succeq \ell - r$. Therefore, $u \in B[x_0, r, c]$.

For every $x \in B[x_0, r, c]$, it follows from (3.8) that

$$\ell - M(Tx_0, Tx, c) \preceq k[\ell - M(x_0, x, c)],$$

that is,

$$\ell(1 - k) + kM(x_0, x, c) \preceq M(Tx_0, Tx, c).$$

Therefore, for all $m \in \mathbb{N}$, we have

$$\begin{aligned} M\left(x_0, Tx, c + \frac{c}{m}\right) &\succeq M\left(x_0, Tx_0, \frac{c}{m}\right) * M(Tx_0, Tx, c) \\ &\succeq (\ell - r) * [\ell(1 - k) + kM(x_0, x, c)] \\ &\succeq (\ell - r) * [\ell(1 - k) + k(\ell - r)] \\ &= (\ell - r) * (\ell - kr) \succeq (\ell - r) * (\ell - r) \\ &\succeq \ell - r. \end{aligned}$$

Taking the limit as $m \to \infty$ and by Remark 2.15, we obtain $M(x_0, Tx, c) \succeq \ell - r$. Hence $Tx \in B[x_0, r, c]$. □

Remark 3.6. In Corollary 3.5, condition ii) can be replaced by the following weaker one:

ii*) There exists $k \in [0, 1)$ such that, for all $x, y \in B[x_0, r, c], c \in P_\theta$.

$$\ell - M(Tx_0, Tx, c) \preceq k\left[\ell - M(x_0, x, c)\right]. \tag{3.9}$$

Let $\{c_n\}$ be a sequence in P, then the sequence $\{c_n\}$ is said to diverge to ∞ as $n \to \infty$, and we write $\lim\limits_{n\to\infty} c_n = \infty$, if for all $c \in P$ there exists $n_0 \in \mathbb{N}$ such that $c \preceq c_n$ for all $n > n_0$.

Theorem 3.7. *Let $(X, M, *)$ be a complete complex valued fuzzy metric space such that, for any sequence $\{c_n\}$ in P_θ with $\lim\limits_{n\to\infty} c_n = \infty$, we have $\lim\limits_{n\to\infty} \inf\limits_{y\in X} M(x, y, c_n) = \ell$, for all $x \in X$. If $T\colon X \to X$ satisfies that*

$$M(Tx, Ty, \lambda c) \succeq M(x, y, c) \tag{3.10}$$

for all $x, y \in X$ and $c \in P_\theta$, where $\lambda \in (0, 1)$, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrarily fixed. Define a sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1} \quad \text{for all} \quad n \in \mathbb{N}.$$

If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence.

For $n \in \mathbb{N}$ and $c \in P_\theta$ fixed, we define

$$A_n := \{M(x_n, x_m, c)\colon m > n\} \subset I.$$

Since $\theta \prec M(x_n, x_m, c) \preceq \ell$ for all $m \in \mathbb{N}$ with $m > n$, by Remark 2.1, the infimum, $\inf A_n = \alpha_n$ (say) exists for all $n \in \mathbb{N}$. For $c \in P_\theta$ and $n, m \in \mathbb{N}$ with $m > n$, we have, by (3.10) and Lemma 2.9,

$$M(x_{n+1}, x_{m+1}, c) = M(Tx_n, Tx_m, c) \succeq M(x_n, x_m, c/\lambda) \succeq M(x_n, x_m, c), \tag{3.11}$$

which implies that

$$M(x_n, x_m, c) \preceq M(x_{n+1}, x_{m+1}, c) \quad \text{for all} \quad n, m \in \mathbb{N} \ \text{ with } \ m > n.$$

Therefore, by definition, we have

$$\theta \preceq \alpha_n \preceq \alpha_{n+1} \preceq \ell, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.12}$$

Thus, $\{\alpha_n\}$ is a monotonic sequence in P and, using Remark 2.1 and (3.12), there exists $\ell_1 \in P$ such that

$$\lim_{n\to\infty} \alpha_n = \ell_1. \tag{3.13}$$

Again, by (3.11), we have, for $c \in P_\theta$ and $n \in \mathbb{N}$,

$$\alpha_{n+1} = \inf_{m>n} M(x_{n+1}, x_{m+1}, c) \succeq \inf_{m>n} M(x_n, x_m, c/\lambda).$$

Similarly, we get, for $c \in P_\theta$ and $n, m \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} M(x_{n+1}, x_{m+1}, c) &\succeq M(x_n, x_m, c/\lambda) = M(Tx_{n-1}, Tx_{m-1}, c/\lambda) \\ &\succeq M(x_{n-1}, x_{m-1}, c/\lambda^2) \\ &= M(Tx_{n-2}, Tx_{m-2}, c/\lambda^2) \succeq M(x_{n-2}, x_{m-2}, c/\lambda^3) \\ &\succeq \cdots \succeq M(x_0, x_{m-n}, c/\lambda^{n+1}), \end{aligned}$$

hence, for all $c \in P_\theta$ and $n \in \mathbb{N}$,

$$\alpha_{n+1} = \inf_{m>n} M(x_{n+1}, x_{m+1}, c) \succeq \inf_{m>n} M(x_0, x_{m-n}, c/\lambda^{n+1}) \succeq \inf_{y \in X} M(x_0, y, c/\lambda^{n+1}).$$

Since $\lim_{n\to\infty} c/\lambda^{n+1} = \infty$, by (3.13) and the hypothesis, we have

$$\ell_1 \succeq \lim_{n\to\infty} \inf_{y \in X} M(x_0, y, c/\lambda^{n+1}) = \ell. \tag{3.14}$$

By (3.13) and (3.14), we have that

$$\lim_{n\to\infty} \alpha_n = \ell.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. By completeness of X and Lemma 2.11, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, c) = \ell \quad \text{for all} \quad c \in P_\theta. \tag{3.15}$$

For any $c \in P_\theta$, it follows, from (3.10), that

$$\begin{aligned} M(u, Tu, c) &\succeq M(u, x_{n+1}, c/2) * M(x_{n+1}, Tu, c/2) \\ &= M(u, x_{n+1}, c/2) * M(Tx_n, Tu, c/2) \\ &\succeq M(u, x_{n+1}, c/2) * M(x_n, u, c/2\lambda). \end{aligned}$$

Taking the limit as $n \to \infty$ and using (3.15) and Remark 2.15, we obtain that $M(u, Tu, c) = \ell$ for all $c \in P_\theta$, that is, $Tu = u$.

If $v \in X$ is another fixed point of T and there exists $c \in P_\theta$ such that $M(u, v, c) \neq \ell$, then it follows from (3.10) that

$$\begin{aligned} M(u, v, c) &= M(Tu, Tv, c) \succeq M(u, v, c/\lambda) = M(Tu, Tv, c/\lambda) \\ &\succeq M(u, v, c/\lambda^2) \succeq \cdots \succeq M(u, v, c/\lambda^n), \end{aligned}$$

for all $n \in \mathbb{N}$. Using that $\lim_{n\to\infty} c/\lambda^n = \infty$ and $M(u, v, c/\lambda^n) \succeq \inf_{y \in X} M(u, y, c/\lambda^n)$, it follows from the above inequality that $M(u, v, c) \succeq \ell$, which is a contradiction. Hence, $M(u, v, c) = \ell$ for all $c \in P_\theta$, that is, $u = v$, and the uniqueness follows. □

Note that, in the proof of Theorem 3.7, for the uniqueness of fixed point it is enough that, for any sequence $\{c_n\}$ in P_θ with $\lim_{n\to\infty} c_n = \infty$, we have $\lim_{n\to\infty} M(x, y, c_n) = \ell$ for all $x, y \in X$. This condition is trivially derived from the assumptions of Theorem 3.7 since, for any sequence $\{c_n\}$ in P_θ with $\lim_{n\to\infty} c_n = \infty$ and for all $x, y \in X$, $\lim_{n\to\infty} M(x, y, c_n) \geq \lim_{n\to\infty} \inf_{z \in X} M(x, z, c_n) = \ell$. Notice that the converse of this fact is not true in general. This remark and the rest of the proof of Theorem 3.7 suggest that we can give a more general statement for this fixed point result, as follows.

Theorem 3.8. *Let $(X, M, *)$ be a complete complex valued fuzzy metric space such that, for any sequence $\{c_n\}$ in P_θ with $\lim_{n\to\infty} c_n = \infty$, we have $\lim_{n\to\infty} M(x, y, c_n) = \ell$ for all $x, y \in X$. Moreover, suppose that there exists $x_0 \in X$ such that, for any sequence $\{c_n\}$ in P_θ with $\lim_{n\to\infty} c_n = \infty$, we have*

$$\lim_{n\to\infty} \inf_{y\in C_{x_0}} M(x_0, y, c_n) = \ell, \tag{3.16}$$

where C_{x_0} represents the set of T-iterates of x_0, that is, $C_{x_0} = \{T^k(x_0) \,:\, k \in \mathbb{N}\}$.

If $T\colon X \to X$ satisfies that $M(Tx, Ty, \lambda c) \succeq M(x, y, c)$ for all $x, y \in X$ and $c \in P_\theta$, where $\lambda \in (0, 1)$, then T has a unique fixed point in X.

Proof. We define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, where $x_0 \in X$ is the element given by the statement. The condition (3.16) helps to guarantee that $\{x_n\}$ is a Cauchy sequence, since, for $c \in P_\theta$ and $n \in \mathbb{N}$,

$$\begin{aligned}\alpha_{n+1} &= \inf_{m>n} M(x_{n+1}, x_{m+1}, c) \succeq \inf_{m>n} M(x_0, x_{m-n}, c/\lambda^{n+1})\\ &= \inf_{m>n} M(x_0, T^{m-n}(x_0), c/\lambda^{n+1}) = \inf_{y\in C_{x_0}} M(x_0, y, c/\lambda^{n+1})\end{aligned}$$

and, then,

$$\lim_{n\to\infty} \alpha_{n+1} \succeq \lim_{n\to\infty} \inf_{y\in C_{x_0}} M(x_0, y, c/\lambda^{n+1}) = \ell.$$

The proof is finished similarly to that of Theorem 3.7. □

Proceeding similarly to the proofs of Theorems 3.7 and 3.8, we deduce the following result.

Theorem 3.9. *Let $(X, M, *)$ be a complete complex valued fuzzy metric space and $T\colon X \to X$ be such that:*

- *$M(Tx, Ty, \lambda(c)\cdot c) \succeq M(x, y, c)$ for all $x, y \in X$ and $c \in P_\theta$, where $\lambda : P_\theta \to (0, 1)$.*
- *$\lim_{n\to\infty} M\left(x, y, \dfrac{c}{(\lambda(c))^n}\right) = \ell$ for all $x, y \in X$ and $c \in P_\theta$.*
- *There exists $x_0 \in X$ such that*

$$\lim_{n\to\infty} \inf_{y\in C_{x_0}} M\left(x_0, y, \frac{c}{(\lambda(c))^n}\right) = \ell \quad \forall c \in P_\theta,$$

where C_{x_0} represents the set of T-iterates of x_0, that is, $C_{x_0} = \{T^k(x_0) \,:\, k \in \mathbb{N}\}$.

Then T has a unique fixed point in X.

Example 3.10. Let $X = [0, 1]$. Define $*$ by

$$c_1 * c_2 = (a_1a_2, b_1b_2)$$

for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$. Let the complex fuzzy set M be given by

$$M(x, y, c) = \frac{ab}{ab + |x - y|}\ell$$

for all $x, y \in X$, $c = (a, b) \in P_\theta$. Then $(X, M, *)$ is a complete complex valued fuzzy metric space. Besides, for any sequence $\{c_n\}$ in P_θ, $c_n = (a_n, b_n)$, such that

$\lim_{n\to\infty} c_n = \infty$ and for each $x \in X$ fixed, we have, using that $|x-y| \leq 1$, for all $y \in [0,1]$, that

$$\ell \succeq \inf_{y\in X} M(x,y,c_n) = \inf_{y\in X} \frac{a_n b_n}{a_n b_n + |x-y|}\ell = \inf_{y\in X} \frac{1}{1+\frac{|x-y|}{a_n b_n}}\ell$$
$$= \frac{1}{1+\frac{\sup_{y\in X}|x-y|}{a_n b_n}}\ell \succeq \frac{1}{1+\frac{1}{a_n b_n}}\ell,$$

so that

$$\ell \succeq \lim_{n\to\infty} \inf_{y\in X} M(x,y,c_n) \succeq \lim_{n\to\infty} \frac{1}{1+\frac{1}{a_n b_n}}\ell = \ell.$$

Define a mapping $T\colon X \to X$ by $Tx = a_0x^2 + b_0$ for all $x \in X$, where $a_0, b_0 \in \left(0, \frac{1}{2}\right)$. Note that T satisfies (3.10) with $\lambda = \sqrt{2a_0} \in (0,1)$, and all the conditions of Theorem 3.7 hold. Moreover, $u = \dfrac{1-\sqrt{1-4a_0b_0}}{2a_0}$ is the unique fixed point of T in X.

The following example shows that the assumption "for any sequence $\{c_n\}$ in P_θ with $\lim_{n\to\infty} c_n = \infty$, we have $\lim_{n\to\infty} \inf_{y\in X} M(x,y,c_n) = \ell$, for all $x \in X$" of Theorem 3.7 is not superfluous.

Example 3.11. Let $X = \{n\ell\colon n \in \mathbb{N}\}$. Define $*$ by $c_1 * c_2 = (a_1a_2, b_1b_2)$ for all $c_1 = (a_1,b_1), c_2 = (a_2,b_2) \in I$ and the complex valued fuzzy set M by

$$M(n\ell, m\ell, c) = \begin{cases} \dfrac{n}{m}\ell, & \text{if } n \leq m; \\ \dfrac{m}{n}\ell, & \text{if } m \leq n, \end{cases}$$

for all $n, m \in \mathbb{N}$, $c \in P_\theta$. Then $(X, M, *)$ is a complete complex valued fuzzy metric space. Consider the mapping $T\colon X \to X$ defined by $T(n\ell) = (n+1)\ell$ for all $n \in \mathbb{N}$. Consider the sequence $\{c_n\}$ in P_θ such that $c_n = (n,n)$ for all $n \in \mathbb{N}$. It is obvious that $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} M(x,y,c_n) \neq \ell$ for all $x, y \in X$ with $x \neq y$. Moreover, for each $x = k\ell$, $y = m\ell \in X$ fixed and $n \in \mathbb{N}$, we get

$$M(x,y,c_n) = M(k\ell, m\ell, c_n) = \begin{cases} \frac{k}{m}\ell & \text{if } k \leq m, \\ \frac{m}{k}\ell & \text{if } m \leq k, \end{cases},$$

then, for each $x = k\ell \in X$ and $n \in \mathbb{N}$,

$$\inf_{y\in X} M(x,y,c_n) = \inf_{m\in\mathbb{N}} M(k\ell, m\ell, c_n) = \theta,$$

so that

$$\lim_{n\to\infty} \inf_{y\in X} M(x,y,c_n) = \theta \neq \ell.$$

Note that (3.10) is satisfied for arbitrary $\lambda \in (0,1)$, but T has no fixed point in X.

For this example, conditions in Theorem 3.8 are not satisfied either since, for the same choice of $\{c_n\}$ and for every $x_0 = k_0\ell \in X$,

$$\lim_{n\to\infty} \inf_{y\in C_{x_0}} M(x_0, y, c_n) = \inf_{m>k_0} M(k_0\ell, m\ell, c_n) = \theta \neq \ell,$$

due to $C_{x_0} = \{m\ell : m > k_0\}$.

If (X, ρ) is a bounded metric space, that is, there exists $K > 0$ such that $d(x, y) < K$ for all $x, y \in X$, and $\star$ is the Łukasiewicz $t-$norm, then $(X, M_\rho, \star)$ is a fuzzy metric space (see [10]), where $M_\rho(p, q, t) = 1 - \dfrac{\rho(p, q)}{g(t)}$ for all $p, q \in X, t > 0$ and $g\colon \mathbb{R}^+ \to (K, \infty)$ is a nondecreasing continuous function.

Similarly, suppose that (X, d_c) is a complex valued metric space such that there exists $C = (k_1, k_2) \in P_\theta$ with $d_c(p, q) \prec C$ for all $p, q \in X$. If $G\colon P_\theta \to (L, \infty)$ is a nondecreasing continuous function, where $L = \max\{k_1, k_2\}$, and $*$ is defined by $c_1 * c_2 = (a_1 \star a_2, b_1 \star b_2)$ for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$, then $(X, M_{d_c}, *)$ is a complex valued fuzzy metric space, where $M_{d_c}(p, q, c) = \ell - \dfrac{d_c(p, q)}{G(c)}$ for all $p, q \in X$ and $c \in P_\theta$.

The complex valued fuzzy metric M_{d_c} is an extension of the fuzzy metric M_ρ to complex values. Besides, if the complex valued metric space (X, d_c) is complete, then the complex valued fuzzy metric space $(X, M_{d_c}, *)$ is complete. Indeed, suppose that (X, d_c) is complete and consider $\{x_n\}$ a Cauchy sequence in $(X, M_{d_c}, *)$, then

$$\lim_{n\to\infty} \inf_{m>n} M_{d_c}(x_n, x_m, c_1) = \lim_{n\to\infty} \inf_{m>n} \left(\ell - \frac{d_c(x_n, x_m)}{G(c_1)}\right) = \ell, \text{ for all } c_1 \in P_\theta.$$

This implies that

$$\lim_{n\to\infty} \left(\ell - \frac{\sup_{m>n} d_c(x_n, x_m)}{G(c_1)}\right) = \ell, \text{ for all } c_1 \in P_\theta,$$

that is,

$$\lim_{n\to\infty} \sup_{m>n} d_c(x_n, x_m) = \theta.$$

For a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for every $n > n_0$,

$$\left|\sup_{m>n} d_c(x_n, x_m) - \theta\right| < \varepsilon.$$

Since $\theta \preceq d_c(x_n, x_m) \preceq \sup_{m>n} d_c(x_n, x_m)$, for $n, m \in \mathbb{N}$ with $m > n$, we have

$$|d_c(x_n, x_m) - \theta| \leq \left|\sup_{m>n} d_c(x_n, x_m) - \theta\right| < \varepsilon, \text{ for every } n, m \in \mathbb{N},\ m > n > n_0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d_c), so that there exists $x \in X$ the limit of $\{x_n\}$ in (X, d_c). The convergence is also in $(X, M_{d_c}, *)$, since

$$\lim_{n\to\infty} M(x_n, x, c_1) = \lim_{n\to\infty} \left(\ell - \frac{d_c(x_n, x)}{G(c_1)}\right) = \ell, \text{ for all } c_1 \in P_\theta.$$

In the following example, we compare our results with the corresponding fixed point results in usual fuzzy metric spaces. It shows the effect of the complex valued fuzzy metric on the contractive condition of the mapping, that is, a contractive mapping on $(X, M_{d_c}, *)$ may not be contractive in the setup of $(X, M_\rho, \star)$.

Example 3.12. Let $I_{\mathbb{R}} = [0,1]$ and $X = I_{\mathbb{R}} \times \{0\} \cup \{0\} \times I_{\mathbb{R}}$. Let $*$ be defined by

$$c_1 * c_2 = (\max\{a_1 + a_2 - 1, 0\}, \max\{b_1 + b_2 - 1, 0\})$$

for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$. Define $d_c\colon X \times X \to \mathbb{C}$ by

$$d_c((x,0),(y,0)) = |x-y|(2,1), \quad d_c((0,x),(0,y)) = |x-y|\left(1, \frac{3}{5}\right)$$

and

$$d_c((x,0),(0,y)) = d_c((0,y),(x,0)) = \left(2x + y, x + \frac{3}{5}y\right).$$

Obviously, (X, d_c) is a complex valued metric space. Let M_{d_c} be given by

$$M_{d_c}(p,q,c_1) = \ell - \frac{5d_c(p,q)}{18 + 5ab} \ \text{ for all } \ p,q \in X, c_1 = (a,b) \in P_\theta.$$

Then $(X, M_{d_c}, *)$ is a complete complex valued fuzzy metric space. Let $T\colon X \to X$ be a mapping defined by

$$T(x,0) = (0,x), \qquad \text{and} \qquad T(0,x) = \left(\frac{2}{5}x, 0\right).$$

Note that T satisfies the condition (3.10) with any $1 > \lambda \geq \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$. Hence, all the conditions of Theorem 3.7 are satisfied, since, for each $x \in X$ fixed and for any sequence $\{c_n\}$ in P_θ, $c_n = (a_n, b_n)$, with $\lim_{n\to\infty} c_n = \infty$, we have

$$\inf_{y\in X} M(x,y,c_n) = \inf_{y \in X}\left(\ell - \frac{5d_c(x,y)}{18 + 5a_nb_n}\right) = \left(\ell - \frac{5\sup_{y\in X} d_c(x,y)}{18 + 5a_nb_n}\right) \overset{n\to\infty}{\to} \ell,$$

due to $d_c(x,y) \prec (\frac{16}{5}, \frac{9}{5})$, for every x, $y \in X$. Besides, conditions in Theorem 3.1 hold as well since (2.3) is satisfied for $1 > k \geq \frac{4}{5}$. Indeed, (2.3) is reduced to

$$\frac{5d_c(Tx,Ty)}{18+5ab} \preceq k\left[\frac{5d_c(x,y)}{18+5ab}\right],$$

for x, $y \in X$ and $c_1 = (a,b) \in P_\theta$, or, equivalenly, $d_c(Tx,Ty) \preceq kd_c(x,y)$, for x, $y \in X$. By distinguishing the different cases, this condition is satisfied for $1 > k \geq \frac{4}{5}$.

Moreover, T has a unique fixed point $u = (0,0) \in X$.

On the other hand, if ρ is the usual metric on X, then $(X, M_\rho, \star)$ is a fuzzy metric space, where $M_\rho(p,q,t) = 1 - \dfrac{\rho(p,q)}{g(t)}$ for all $p,q \in X$, $t > 0$, and $g\colon \mathbb{R}^+ \to (K,\infty)$, with $K > \sqrt{2}$, is an increasing continuous function. Then T is not a fuzzy contractive mapping in the fuzzy metric space $(X, M_\rho, \star)$, neither in the sense of Gregori and

Sapena [9] nor in the sense of Grabiec [8]. Indeed, the contractivity condition in Definition 3.5 [9] is written as

$$\frac{1}{1-\frac{\rho(Tp,Tq)}{g(t)}}-1\le k\left(\frac{1}{1-\frac{\rho(p,q)}{g(t)}}-1\right),\quad \forall p,\, q\in X \text{ and } t>0,$$

or also

$$\frac{\rho(Tp,Tq)}{g(t)}\le 1-\frac{1}{k\left(\frac{1}{1-\frac{\rho(p,q)}{g(t)}}-1\right)+1},\quad \forall p,\, q\in X \text{ and } t>0.$$

Just taking the elements $p=(x,0)$, $q=(y,0)\in X$, the above inequality is reduced to

$$\frac{|x-y|}{g(t)}\le 1-\frac{1}{k\left(\frac{1}{1-\frac{|x-y|}{g(t)}}-1\right)+1}=\frac{k|x-y|}{(k-1)|x-y|+g(t)},\quad \forall x,\, y\in[0,1] \text{ and } t>0,$$

which is trivially true for $x=y\in[0,1]$. However, if $x\neq y$ in $[0,1]$, this inequality is equivalent to $(1-k)|x-y|\ge(1-k)g(t)$ and, if $k\in(0,1)$, it leads to the contradiction $1\ge|x-y|\ge g(t)>\sqrt{2}$.

Finally, the contractivity condition in the sense of Grabiec is written as

$$1-\frac{\rho(Tp,Tq)}{g(kt)}\ge 1-\frac{\rho(p,q)}{g(t)},\quad \forall p,\, q\in X \text{ and } t>0,$$

that is,

$$\frac{\rho(Tp,Tq)}{g(kt)}\le\frac{\rho(p,q)}{g(t)},\quad \forall p,\, q\in X \text{ and } t>0.$$

Taking $p=(x,0)$, $q=(y,0)\in X$, with $x\neq y$, we get $g(kt)\ge g(t)$, for $t>0$, which is absurd, taking into account that $k\in(0,1)$ and the increasing character of g.

3.1. **Some fixed point results for nondecreasing self-mappings in partially ordered spaces.** In this section, in the context of partially ordered spaces, we show how the monotone character of the self-mapping T allows to relax the contractivity condition to its validity on comparable elements. To this purpose, we follow some of the ideas in [14] by considering some additional properties on the space X. In the following, consider $(X,\sqsubseteq)$ a partially ordered set and the following hypotheses:

(HM1) For every sequence $\{x_n\}$ in X that is monotonic increasing with respect to the partial ordering $\sqsubseteq$ and such that there exists $x\in X$ with $\lim_{n\to\infty}x_n=x$, we have $x_n\sqsubseteq x$, for every $n\in\mathbb{N}$.

(HM2) For every sequence $\{x_n\}$ in X that is monotonic decreasing with respect to the partial ordering $\sqsubseteq$ and such that there exists $x\in X$ with $\lim_{n\to\infty}x_n=x$, we have $x\sqsubseteq x_n$, for every $n\in\mathbb{N}$.

(HU) For every x, $y\in X$, there exists $z\in X$ such that z is $\sqsubseteq$-comparable both with x and y.

Theorem 3.13. *Let $(X, M, *)$ be a complete complex valued fuzzy metric space such that $\sqsubseteq$ is a partial ordering in X and let $T\colon X \to X$ be a monotone nondecreasing mapping (i.e., $x \sqsubseteq y \implies Tx \sqsubseteq Ty$) satisfying that*

$$\ell - M(Tx, Ty, c) \preceq k(c)\left[\ell - M(x, y, c)\right], \text{ for all } x, y \in X \text{ with } x \sqsubseteq y \text{ and } c \in P_\theta, \tag{3.17}$$

where k is a real function $k : P_\theta \to [0, 1)$. Suppose that one of the following conditions holds:

- *There exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$ and (HM1) is satisfied.*
- *There exists $x_0 \in X$ such that $Tx_0 \sqsubseteq x_0$ (or, equivalently $x_0 \sqsupseteq Tx_0$) and (HM2) is satisfied.*

Then T has a fixed point in X (and different fixed points are not $\sqsubseteq$-comparable). Moreover, if (HU) is satisfied, then T has a unique fixed point in X.

Proof. The proof is similar to that of Theorem 3.1. We start with the element $x_0 \in X$ given by the statement and define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$, for $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then the existence of fixed point is derived, so that we consider that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. The sequence defined $\{x_n\}$ is $\sqsubseteq$-monotone. Indeed, it follows by induction. It is easy to check that, in the case $x_0 \sqsubseteq Tx_0 = x_1$, we get $x_1 = Tx_0 \sqsubseteq Tx_1 = x_2$ and, if $x_{n-1} \sqsubseteq x_n$, then $x_n = Tx_{n-1} \sqsubseteq Tx_n = x_{n+1}$, so that $\{x_n\}$ is nondecreasing. In the case $x_0 \sqsupseteq Tx_0 = x_1$, we get $x_1 = Tx_0 \sqsupseteq Tx_1 = x_2$ and, if $x_{n-1} \sqsupseteq x_n$, then $x_n = Tx_{n-1} \sqsupseteq Tx_n = x_{n+1}$, so that $\{x_n\}$ is nonincreasing.

We prove that $\{x_n\}$ is a Cauchy sequence by defining, for $n \in \mathbb{N}$ and fixed $c \in P_\theta$, the same set A_n as in the proof of Theorem 3.1, whose infimum α_n exists.

Note that, by the symmetry of M with respect the first two variables, condition (3.17) is satisfied for every pair of $\sqsubseteq$-comparable elements. Hence, both if $x_0 \sqsubseteq Tx_0$ or $x_0 \sqsupseteq Tx_0$, the terms in the sequence $\{x_n\}$ are comparable between them and, therefore, for $c \in P_\theta$ and $n, m \in \mathbb{N}$ with $m > n$, we get, by (3.17), that

$$\begin{aligned} \ell - M(x_{n+1}, x_{m+1}, c) &= \ell - M(Tx_n, Tx_m, c) \\ &\preceq k(c)\left[\ell - M(x_n, x_m, c)\right] \\ &\preceq \ell - M(x_n, x_m, c), \end{aligned} \tag{3.18}$$

so that, similarly to the proof of Theorem 3.1, we have that $\{\alpha_n\}$ is a monotonic sequence in P and $\lim_{n\to\infty} \alpha_n = \ell_1 \in P$. Then (3.18) implies that $(1 - k(c))\ell + k(c)\alpha_n \preceq \alpha_{n+1}$, for every $n \in \mathbb{N}$, so that $(1 - k(c))\ell \preceq (1 - k(c))\ell_1$, getting that $\lim_{n\to\infty} \alpha_n = \ell$ and $\{x_n\}$ is a Cauchy sequence. The complete character of X provides, in consequence, the existence of $u \in X$ with $\lim_{n\to\infty} M(x_n, u, c) = \ell$, for all $c \in P_\theta$. By hypothesis (HM1) (resp., (HM2)), every term of the sequence x_n is $\sqsubseteq$-comparable to the limit u, therefore, by (3.17), we have for any $c \in P_\theta$ and $n \in \mathbb{N}$,

$$\ell - M(Tx_n, Tu, c) \preceq k(c)\left[\ell - M(x_n, u, c)\right],$$

i.e.,

$$\ell(1 - k(c)) + k(c)M(x_n, u, c) \preceq M(Tx_n, Tu, c). \tag{3.19}$$

For every $c \in P_\theta$ fixed, we have

$$\begin{aligned} M(u,Tu,c) &\succeq M(u,x_{n+1},c/2) * M(Tx_n,Tu,c/2) \\ &\succeq M(u,x_{n+1},c/2) * \left(\ell\left(1-k\left(\frac{c}{2}\right)\right) + k\left(\frac{c}{2}\right) M\left(x_n,u,\frac{c}{2}\right)\right), \end{aligned}$$

thus, taking the limit as $n \to \infty$, we deduce that $M(u,Tu,c) = \ell$ for each $c \in P_\theta$, so that $Tu = u$.

To prove the uniqueness of the fixed point, consider $v \in X$ another fixed point of T. If u and v are $\sqsubseteq$-comparable, then, for every $c \in P_\theta$,

$$\ell - M(u,v,c) = \ell - M(Tu,Tv,c) \preceq k(c)\left[\ell - M(u,v,c)\right],$$

so that $(1-k(c))\ell \preceq (1-k(c))M(u,v,c)$. By the properties of k, we have $M(u,v,c) = \ell$ for all $c \in P_\theta$, that is, $u = v$.

On the other hand, if u, $v \in X$ are not $\sqsubseteq$-comparable, then, by (HU), there exists $z \in X$ such that z is $\sqsubseteq$-comparable both to u and v. This implies that $T^n z$ is $\sqsubseteq$-comparable both to $u = T^n u$ and $v = T^n v$. Hence, for every $c \in P_\theta$ and every $n \in \mathbb{N}$,

$$M(u,v,c) \succeq M\left(u,T^n z,\frac{c}{2}\right) * M\left(T^n z,v,\frac{c}{2}\right) = M\left(T^n u,T^n z,\frac{c}{2}\right) * M\left(T^n z,T^n v,\frac{c}{2}\right).$$

Besides, by (3.17), since u and z are $\sqsubseteq$-comparable,

$$M\left(Tu,Tz,\tilde{c}\right) \succeq (1-k(\tilde{c}))\ell + k(\tilde{c})M\left(u,z,\tilde{c}\right), \forall \tilde{c} \in P_\theta,$$

hence, for $n \in \mathbb{N}$,

$$\begin{aligned} M\left(T^n u,T^n z,\tilde{c}\right) &\succeq (1-k(\tilde{c}))\ell + k(\tilde{c})M\left(T^{n-1}u,T^{n-1}z,\tilde{c}\right) \\ &\succeq (1-k(\tilde{c}))\ell + k(\tilde{c})\left((1-k(\tilde{c}))\ell + k(\tilde{c})M\left(T^{n-2}u,T^{n-2}z,\tilde{c}\right)\right) \\ &= (1-k(\tilde{c}))\ell + k(\tilde{c})(1-k(\tilde{c}))\ell + (k(\tilde{c}))^2 M\left(T^{n-2}u,T^{n-2}z,\tilde{c}\right) \\ &\succeq (1-k(\tilde{c}))\ell + k(\tilde{c})(1-k(\tilde{c}))\ell + (k(\tilde{c}))^2(1-k(\tilde{c}))\ell \\ &\quad + (k(\tilde{c}))^3 M\left(T^{n-3}u,T^{n-3}z,\tilde{c}\right) \\ &\succeq \cdots \succeq (1-k(\tilde{c}))\sum_{j=0}^{n-1}(k(\tilde{c}))^j\,\ell + (k(\tilde{c}))^n M\left(u,z,\tilde{c}\right),\ \forall \tilde{c} \in P_\theta \end{aligned}$$

and, similarly, for $M(T^n z,T^n v,\tilde{c})$. In consequence, for every $c \in P_\theta$ and every $n \in \mathbb{N}$,

$$\begin{aligned} M(u,v,c) &\succeq M\left(T^n u,T^n z,\frac{c}{2}\right) * M\left(T^n z,T^n v,\frac{c}{2}\right) \\ &\succeq \left[\left(1-k\left(\frac{c}{2}\right)\right)\sum_{j=0}^{n-1}\left(k\left(\frac{c}{2}\right)\right)^j\,\ell + \left(k\left(\frac{c}{2}\right)\right)^n M\left(u,z,\frac{c}{2}\right)\right] \\ &\quad * \left[\left(1-k\left(\frac{c}{2}\right)\right)\sum_{j=0}^{n-1}\left(k\left(\frac{c}{2}\right)\right)^j\,\ell + \left(k\left(\frac{c}{2}\right)\right)^n M\left(z,v,\frac{c}{2}\right)\right]. \end{aligned}$$

Using that, for each $c \in P_\theta$, $k(\frac{c}{2}) \in [0,1)$, we have that $\left(k\left(\frac{c}{2}\right)\right)^n \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(k\left(\frac{c}{2}\right)\right)^j = \sum_{j=0}^{\infty} \left(k\left(\frac{c}{2}\right)\right)^j = \frac{1}{1-k\left(\frac{c}{2}\right)}$$

for each $c \in P_\theta$ and we conclude that

$$\lim_{n\to\infty} \left[\left(1-k\left(\frac{c}{2}\right)\right) \sum_{j=0}^{n-1} \left(k\left(\frac{c}{2}\right)\right)^j \ell + \left(k\left(\frac{c}{2}\right)\right)^n M\left(u,z,\frac{c}{2}\right)\right] = \ell, \text{ for each } c \in P_\theta$$

and similarly for the other term, so that $M(u,v,c) = \ell$ for all $c \in P_\theta$, that is, $u = v$. The uniqueness of fixed point is, hence, proved under condition (HU). ☐

Theorem 3.14. *Let $(X, M, *)$ be a complete complex valued fuzzy metric space such that $\sqsubseteq$ is a partial ordering in X and let $T\colon X \to X$ be a monotone nondecreasing mapping (i.e., $x \sqsubseteq y \implies Tx \sqsubseteq Ty$) satisfying that:*

- *$M(Tx,Ty,\lambda(c)\cdot c) \succeq M(x,y,c)$ for all $x, y \in X$ with $x \sqsubseteq y$ and $c \in P_\theta$, where $\lambda : P_\theta \to (0,1)$.*
- *$\lim_{n\to\infty} M\left(x,y,\frac{c}{(\lambda(c))^n}\right) = \ell$ for all $x, y \in X$ with $x \sqsubseteq y$ and $c \in P_\theta$.*

Suppose also that one of the following conditions holds:

(i) *There exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$ and*

$$\lim_{n\to\infty} \inf_{y\in C_{x_0}} M\left(x_0,y,\frac{c}{(\lambda(c))^n}\right) = \ell \quad \forall c \in P_\theta,$$

where C_{x_0} represents the set of T-iterates of x_0, that is,

$$C_{x_0} = \{T^k(x_0) \,:\, k \in \mathbb{N}\},$$

and (HM1) holds.

(ii) *There exists $x_0 \in X$ such that $x_0 \sqsupseteq Tx_0$ and*

$$\lim_{n\to\infty} \inf_{y\in C_{x_0}} M\left(x_0,y,\frac{c}{(\lambda(c))^n}\right) = \ell \quad \forall c \in P_\theta,$$

and (HM2) holds.

Then T has a fixed point in X (and different fixed points are not $\sqsubseteq$-comparable). Moreover, if (HU) is satisfied, then T has a unique fixed point in X.

Proof. We start the sequence at x_0 given by hypotheses (i) or (ii) and define $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. We consider the general case $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. It is clear that $\{x_n\}$ is nondecreasing in case (i) and nonincreasing in case (ii) (that is, any two terms in the sequence are $\sqsubseteq$-comparable). Similarly to the proof of Theorem 3.7, we consider, for $n \in \mathbb{N}$ and $c \in P_\theta$ fixed, the set A_n with infimum α_n. By the contractivity condition over comparable elements, we have, for $c \in P_\theta$ and $n, m \in \mathbb{N}$ with $m > n$,

$$M(x_{n+1},x_{m+1},c) = M(Tx_n,Tx_m,c) \succeq M(x_n,x_m,c/\lambda(c)) \succeq M(x_n,x_m,c), \quad (3.20)$$

so that $\{\alpha_n\}$ is nondecreasing and convergent to $\ell_1 \in P$ (with $\ell_1 \preceq \ell$).

Besides, for $c \in P_\theta$ and $n, m \in \mathbb{N}$ with $m > n$, since the terms in the sequence are comparable, we get

$$M(x_{n+1}, x_{m+1}, c) \succeq M(x_0, x_{m-n}, c/(\lambda(c))^{n+1}),$$

so that, for all $c \in P_\theta$ and $n \in \mathbb{N}$,

$$\alpha_{n+1} \succeq \inf_{m>n} M(x_0, x_{m-n}, c/(\lambda(c))^{n+1}) = \inf_{y \in C_{x_0}} M(x_0, y, c/(\lambda(c))^{n+1}).$$

Therefore, $\lim_{n\to\infty} \alpha_n = \ell$ and $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists $u \in X$ such that $\lim_{n\to\infty} M(x_n, u, c) = \ell$ for all $c \in P_\theta$.

In case that (i) holds, due to (HM1), we have that $x_n \sqsubseteq u$, for every $n \in \mathbb{N}$ and, if (ii) holds, due to (HM2), we have that $x_n \sqsupseteq u$, for every $n \in \mathbb{N}$. Since x_n is comparable to u for every $n \in \mathbb{N}$, we have, for any $c \in P_\theta$ and $n \in \mathbb{N}$, that

$$\begin{aligned} M(u, Tu, c) &\succeq M(u, x_{n+1}, c/2) * M(Tx_n, Tu, c/2) \\ &\succeq M(u, x_{n+1}, c/2) * M(x_n, u, c/(2\lambda(c))). \end{aligned}$$

From this inequality, we deduce that $Tu = u$.

Finally, if $v \in X$ is another fixed point of T, we distinguish two cases. If u and v are $\sqsubseteq$-comparable, then

$$M(u, v, c) = M(Tu, Tv, c) \succeq M(u, v, c/\lambda(c)) \succeq \cdots \succeq M(u, v, c/(\lambda(c))^n), \text{ for all } n \in \mathbb{N}.$$

By hypothesis, $M(u, v, c) = \ell$ for all $c \in P_\theta$, that is, $u = v$.

On the other hand, if u and v are not $\sqsubseteq$-comparable, by (HU), there exists $z \in X$ that is $\sqsubseteq$-comparable to u and v and, hence, $T^n z$ is also $\sqsubseteq$-comparable to u and v. Therefore,

$$\begin{aligned} M(u, v, c) &= M(T^n u, T^n v, c) \succeq M(T^n u, T^n z, c/2) * M(T^n z, T^n v, c/2) \\ &\succeq M(T^{n-1} u, T^{n-1} z, c/(2\lambda(c))) * M(T^{n-1} z, T^{n-1} v, c/(2\lambda(c))) \\ &\succeq \cdots \succeq M(u, z, c/(2(\lambda(c))^n)) * M(z, v, c/(2(\lambda(c))^n)), \end{aligned}$$

for all $c \in P_\theta$, $n \in \mathbb{N}$.

Passing to the limit as $n \to \infty$, we have $M(u, v, c) = \ell$ for all $c \in P_\theta$, thus, $u = v$. □

4. Some final considerations

As final considerations, we give some hints about the relevance of complex valued fuzzy metric spaces and fixed point results on spaces with such an structure.

From the point of view of applications, many processes find a suitable formulation in mathematical terms through the establishment of a base space that might have particularities which do not fit with the restrictions of classical Banach spaces. In particular, fuzzy metric spaces are an appropriate basis for dealing with the uncertainty present in many real phenomena, being very useful to establish the distance between two elements in terms of their degree of proximity. Their applicability is widely extended to approximate reasoning, decision-making problems, or tasks of classification and data analysis [2].

Some relevant applications of fuzzy metric spaces are shown in [2] in relation to the complexity of algorithms and information systems based on access locality (see also [3]). The use of fuzzy metrics allows the authors to give the classification tasks a dynamic behavior, evolving with time.

The applications of fuzzy metrics to image filtering are also well-known. In [13], it is highlighted the relevance of having at disposal different examples of fuzzy metrics to use them in engineering applications. Besides, the authors of [13] illustrate the use of fuzzy metrics to analyze the proximity of two pixels in a color image, which is highly useful in image processing. In this sense, fuzzy metrics give additional capabilities in constrast with the classical approach, such as the possibility to quantify the distance between two elements of arbitrary type in terms of a number between 0 and 1 in a uniform scale basis, as well as allowing to deal with the uncertainty present in many practical applications, giving the possibility to work with different degrees of certainty in concordance with what happens in many fuzzy processes. In [13], the authors propose an image filter based on fuzzy metrics and compare the results obtained with other methods.

Moreover, the author of [2] presents an interesting application of fixed point results for a general type of fuzzy metrics on uncertain domains, and thus proves the existence and uniqueness of solution for the recurrence equations associated to recursive algorithms. See also [4, 16] to find some particular examples of contexts where these algorithms arise, such as object-oriented design (objects relying one on the other) or language theory (in relation with dependent rules of grammar).

On the other hand, complex spaces are proved to be of great relevance, for instance, in physics, where the question of complexification of space and time is an important issue for the unification theory. Among the different contributions to the topic, we mention the pioneering work by El Naschie [5] with the introduction of the idea of complex time and its formal definition. Based on this idea, the authors of [12] show that complex structure of time is in concordance with the consideration of the additional coordinate of the time employed by a light signal travelling between two inertial observers whose aim is to compare their own measurements of time. They also show that El Naschie's complex time can be understood as a limit when the speed of the observers attains the speed of light and they derive the inverse Lorentz transformations of special relativity directly from this structure. In [17], it is showed that the idea of time complexification gives an appropriate context to interpret the connections between quantum and classical mechanics. Another recent contribution on the topic is [15].

The use of complex fuzzy metrics can have applications to problems under different interpretations. On one hand, it may permit the consideration of a complex t variable, in relation with some physical features or just attending to two different magnitudes considered as differenced elements. This would allow computing the proximity between elements of the space as two separated values in terms of two criteria, and proceed, for instance, with some filtering process which takes into account that the corresponding pair of values remains on a certain region (instead of being close to a certain value). This approach might also be helpful in the situation where the process of study is evolving in relation with two different variables. On the other hand, this structure might allow dealing with information depending on two different types of data or registers, for instance, the criteria to compare pixels and units of sound could be different, so that the comparison can be made simultaneously at both levels through a metric that considers the two types of elements separately. Of course, the possibility of combining both criteria in only one remains, so that filters based on this type of fuzzy metrics can be an alternative in filtering processes with higher dimension entities of information.

In this context, the usefulness of fixed point results in complex fuzzy metric spaces is clear due to the relevance of this type of results to the solvability of functional problems, or even differential equations, in many fields of applications, as long as the structure of complex fuzzy metric is suitable for the particularities of the space considered.

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