Fixed Point Theory, 19(2018), No. 2, 733-750
DOI: 10.24193/fpt-ro.2018.2.55
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINT THEOREMS IN QUASI-METRIC SPACES AND THE SPECIALIZATION PARTIAL ORDER

NASEER SHAHZAD* AND OSCAR VALERO**

*Department of Mathematics, Faculty of Science
King Abdulaziz University, P.O. Box 80203, Jeddah 21859, Saudi Arabia
E-mail: nshahzad@kau.edu.sa

**Departamento de Ciencias Matemáticas e Informática
Universidad de las Islas Baleares
Ctra. de Valldemossa km. 7.5, 07122 Palma de Mallorca, Spain
E-mail: o.valero@uib.es

Abstract. In this paper we present a new fixed point theorem in quasi-metric spaces which captures the spirit of Kleene's fixed point theorem. To this end, we explore the fundamental assumptions in the aforesaid result when we consider quasi-metric spaces endowed with the specialization partial order. Thus, we introduce an appropriate notion of order-completeness and order-continuity that ensure the existence of fixed point with distinguished properties. Moreover, some fixed point theorems are derived as a particular case of our main result when the self-mappings under consideration satisfy, in addition, any type of Banach contractive condition under different quasi-metric notions of completeness.
Key Words and Phrases: quasi-metric, specialization partial order, order-completeness, fixed point, monotonicity, order-continuity, contraction.
2010 Mathematics Subject Classification: 06A06, 47H09, 47H10, 54E50, 54F05, 68N30.

1. The statement of the problem

The mathematical modelling for the development of formal methods useful in program verification is based mainly on the principle of fixed point induction in partially ordered sets (see [6], [16], [17] and [24]). Concretely, the aforementioned principle is supported by the so-called Kleene's fixed point theorem. Let us recall a few pertinent notions about partially ordered sets with the aim of introducing the famous result. To this end, we will denote by $\mathbb{N}$ the set of positive integer numbers.

Following [6], a partially ordered set is pair $(X, \leq)$, where X is a nonempty set X and $\leq$ is a partial order on X; i.e., a binary relation over X fulfilling (for all $x, y, z \in X$)

(i) $x \leq x$ (reflexivity);
(ii) $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry);
(iii) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity).

If $(X, \leq)$ is a partially ordered set and $Y \subseteq X$, then an upper bound for Y in $(X, \leq)$ is an element $x \in X$ such that $y \leq x$ for all $y \in Y$. The least upper bound for Y in $(X, \leq)$, if exists, is an element $z \in X$ which is an upper bound for Y and, in addition, satisfies that $z \leq x$ provided that $x \in X$ is an upper bound for Y. We will denote by $\uparrow_{\leq} x$, with $x \in X$, the set $\{y \in X : x \leq y\}$ and by $\downarrow_{\leq} x$ the set $\{y \in X : y \leq x\}$.

According to [4], a partially ordered set $(X, \leq)$ is said to be chain-complete provided that every increasing sequence has a least upper bound, where a sequence $(x_n)_{n\in\mathbb{N}}$ is said to be increasing whenever $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. In addition, a mapping from a partially ordered set $(X, \leq)$ into itself is said to be $\leq$-continuous if the least upper bound of the sequence $(f(x_n))_{n\in\mathbb{N}}$ is $f(x)$ for every increasing sequence $(x_n)_{n\in\mathbb{N}}$ whose least upper bound exists and is x.

Taking into account the above notions, the Kleene's fixed point theorem can be stated as follows:

Theorem 1.1. *Let $(X, \leq)$ be a chain-complete partially ordered set and let f be a $\leq$-continuous mapping from $(X, \leq)$ into itself. If there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$, then there exists $x^* \in Fix(f) = \{x \in X : f(x) = x\}$ which satisfies that $x^* \in \uparrow_{\leq} x_0$.*

Parallel to the order-theoretic foundation of the fixed point induction technique, another mathematical foundation has been developed for program verification which is, now, based in metric spaces and Banach's fixed point theorem (for a fuller treatment of the topic we refer the reader to [3]). Let us recall the aforementioned celebrated result because it will be useful for our subsequent discussion.

Theorem 1.2. *Let (X, d) be a complete metric space and let f be a mapping from X into itself such that there exists $k \in [0, 1[$ with*

$$d(f(x), f(y)) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$. Then there exists a unique $x^* \in X$ such that $Fix(f) = x^*$.

Of course we assume that the reader is familiar with the basic notions of mathematical analysis in fixed point theory. Otherwise, we refer the reader to [7].

In [11] and [20], it was argued that the fact that the topology induced by a metric is T_2 implies, in general, that metric spaces are not a suitable tool in order to support a metric foundation to the fixed point induction principle in terms of Theorem 1.1. Motivated by this fact, M.B. Smyth and S.G. Matthews worked in the development of mathematical tools for program verification that allow to present both aforesaid mathematical approaches, the order-theoretic and the metric one, under the same framework. In the cited papers, the conclusion achieved is that of introducing generalized metric concepts that are able to model at the same time the metric (topological) and order properties that a mathematical model for program verification must satisfy.

Next we recall the basic ideas about the required generalized metrics. To this end, denote by $\mathbb{R}^+$ the set of nonnegative real numbers. According to [20] (see also [9] and [11]), a quasi-metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$;
(ii) $d(x,z) \leq d(x,y) + d(y,z)$.

Clearly, a metric d on X is a quasi-metric with the additional property
(iii) $d(x,y) = d(y,x)$, for all $x, y \in X$.

Each quasi-metric d on X induces a topology $\tau(d)$ on X, which has as a base the family of open d-balls $\{B_d(x,r) : x \in X,\ r > 0\}$, where

$$B_d(x,r) = \{y \in X : d(x,y) < r\}$$

for all $x \in X$ and $r > 0$. Notice that unlike the metric case, the topology induced by a quasi-metric is only T_0 but not T_2 in general.

A quasi-metric space is a pair (X,d) such that X is a nonempty set and d is a quasi-metric on X.

The advantage of quasi-metric spaces with respect to metric spaces as tools in program verification is given mainly by two facts.

On the one hand, the mathematical models for program verification based on metric spaces, i.e. based on Theorem 1.2, can be retrieved from quasi-metric spaces. Indeed, every quasi-metric d on a nonempty set X induces in a natural way a metic d^s given by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ for all $x, y \in X$, where d^{-1} is a quasi-metric, called the conjugate quasi-metric of d, defined by $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$. Moreover, a quasi-metic space (X,d) is said to be bicomplete provided that the metric space (X,d^s) is complete. Thanks to the above relationship between metrics and quasi-metrics, the Banach fixed point theorem can be easily adapted to the quasi-metric approach in the following way.

Theorem 1.3. *Let (X,d) be a bicomplete quasi-metric space and let f be a mapping from X into itself such that there exists $k \in [0,1[$ with*

$$d(f(x), f(y)) \leq kd(x,y) \tag{1.2}$$

for all $x, y \in X$. Then there exists a unique $x^* \in X$ such that $Fix(f) = x^*$.

Obviously Theorem 1.3 allows to retrieve as a particular case the mathematical models for program verification based on Theorem 1.2. However, in [11], Matthews pointed out that Theorem 1.3 is not able to model certain situations that arise in program verification and, for this reason, another notion of generalized metric space was introduced and exploited by means of a Banach type fixed point theorem. We will not recall Matthews' theorem here because we will focus on quasi-metrics in our discussion later.

On the other hand, every quasi-metric d induces on a nonempty set X the so-called specialization partial order $\leq_d$ by $x \leq_d y \Leftrightarrow d(x,y) = 0$. Observe that this presents an advantage with respect to metric spaces because in the latter the specialization partial order is reduced to the flat order, i.e., every element of the space is order related only with itself. The specialization partial order makes that this kind of generalized metrics can establish suitable connections between partial orders and topologies (which are not T_2 in general) for modeling processes in program verification (for a fuller treatment of that topic we refer the reader to [11]). Concretely one of that connections is a fixed point theorem appropriate to permit the unfolding of mathematical models for program verification in the spirit of Kleene's fixed point theorem (Theorem 1.1). The aforesaid result was stated by J.J.M.M. Rutten in [15] (see also [8]) and says the following:

Theorem 1.4. *Let (X,d) be a complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *For all $x,y \in X$,*
$$d(f(x),f(y)) \leq d(x,y). \tag{1.3}$$
(3) *f is continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ with $x^* \in \uparrow_{\leq_d} x_0$ and x^* is the least upper bound of $Fix(f) \cap \uparrow_{\leq_d} x_0$ in $(X,\leq)$. Moreover, $Fix(f) = x^*$ whenever there exits $k \in [0,1[$ such that
$$d(f(x),f(y)) \leq kd(x,y) \tag{1.4}$$
for all $x,y \in X$.

In the preceding result the required completeness of the quasi-metric space and the continuity of the mapping are in the sense of Smyth ([21]) and Rutten ([15]), respectively. We will avoid to recall them, since they are not useful for our purpose. Nevertheless, we want to observe that this notion of continuity is not in general topological.

It is worthy to mention that J.J. Nieto and R. Rodríguez-López have made progress in reconciling order-theoretic and metric fixed point techniques in the classical case with the aim of discussing the existence of solutions to differential equations in [12] (see also [13]). Since we will refer to them later in Section 2.3 we will remember these results:

Theorem 1.5. *Let $(X,\leq)$ be a partially ordered set and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq f(x_0)$.*
(2) *There exist a metric d on X and $k \in [0,1[$ such that (X,d) is a complete metric space and*
$$d(f(x),f(y)) \leq kd(x,y) \tag{1.5}$$
for all $x,y \in X$ with $y \leq x$.
(3) *f is monotone and continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$.

The result below replaces the continuity of the mapping assumed in Theorem 1.5 by an order-theoretic property of the space.

Theorem 1.6. *Let $(X,\leq)$ be a partially ordered set and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq f(x_0)$.*
(2) *There exist a metric d on X and $k \in [0,1[$ such that (X,d) is a complete metric space and*
$$d(f(x),f(y)) \leq kd(x,y) \tag{1.6}$$
for all $x,y \in X$ with $y \leq x$.

(3) *f is monotone.*
(4) *If $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $(X, \leq)$ which converges to $x \in X$ with respect to $\tau(d)$, then x is an upper bound of $(x_n)_{n\in\mathbb{N}}$.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$.

It is worth noting that results somewhat similar to Theorems 5 and 6 were essentially obtained two decades ago in [22, 23].

Various applications of fixed point theory in generalized metric spaces have been given in [1, 2, 18, 19]. Although in [8] it is pointed out that Theorem 1.4 reconciles the order-theoretic and the metric approach for program verification, the aforementioned result uses an special and unusual notion of continuity, i.e., continuity in the sense of [15]. Since the aforementioned notion of continuity is not in general topological the main objective of this paper is to obtain a fixed point theorem in the context of quasi-metric spaces, using more standard notions of continuity, in such a way that, on the one hand, the essence of Kleene's fixed point theorem is captured and goes in the same direction as Theorem 1.4 and that, on the other hand, quasi-metric versions of Theorems 1.5 and 1.6 can be retrieved as a particular case. In order to achieve the target we focus our study on quasi-metric spaces ordered by the specialization partial order and, in addition, we introduce a notion of order-completeness for such spaces and orbitally-order-continuity for self mappings defined on such spaces. Thus we show that the assumptions in Theorems 1.4, 1.5 and 1.6 about completeness and continuity can be reduced to order-completeness and orbitally-order-continuity in the quasi-metric context in order to guarantee the existence of fixed point. Of course, the new notions are illustrated through examples. Moreover, we explore the "monotonic-convergence" property assumed in statement of Theorem 1.6, assumption (4) in the above-mentioned theorem, in the case of quasi-metric spaces. Concretely, we show that the latter spaces enjoy intrinsically a type of monotonic-convergence property when the partial order is exactly the specialization one. Furthermore, we provide examples which illustrate that the assumptions in our new results cannot be weakened. Finally, a few fixed point results in the spirit of Theorems 1.5 and 1.6 are derived from our main fixed point theorem when several quasi-metric completeness are considered.

2. The new fixed point result

In this section we provide the promised fixed point theorem in quasi-metric spaces ordered by the specialization partial order.

2.1. **The fundamental components: order-completeness and order-continuity.** In order to introduce the announced fixed point result in such a way that the spirit of Kleene's fixed point theorem is preserved we need to introduce the appropriate notions of order-completeness and order-continuity in quasi-metric spaces. With this aim we present an auxiliary result that provides conditions that ensure when the limit of an increasing sequence is, on the one hand, an upper bound and, on the other hand, the least upper bound of it. Observe that in the aforesaid result we analyze the role played by the "monotonic-convergence" property used in Theorem 1.6 in the quasi-metric approach.

Proposition 2.1. *Let (X,d) be a quasi-metric space. If $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $(X, \leq_d)$, then the following assertions hold:*

(1) *If $(x_n)_{n\in\mathbb{N}}$ converges to x with respect to $\tau(d^{-1})$, then x is un upper bound of $(x_n)_{n\in\mathbb{N}}$.*
(2) *If x is an upper bound of $(x_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ converges to x with respect to $\tau(d)$, then x is the least upper bound of $(x_n)_{n\in\mathbb{N}}$.*

Proof. (1) Since the sequence $(x_n)_{n\in\mathbb{N}}$ is increasing in $(X, \leq_d)$ we have that $x_n \leq_d x_m$ for all $m, n \in \mathbb{N}$ such that $m \geq n$. Thus $d(x_n, x_m) = 0$ for all $m, n \in \mathbb{N}$ such that $m \geq n$. Next we show that $d(x_n, x) = 0$ for all $n \in \mathbb{N}$. Assume, for the purpose of contradiction, that there exists $n_0 \in \mathbb{N}$ such that $0 < d(x_{n_0}, x)$. Then, given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_1$, since the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x with respect to $\tau(d^{-1})$. Hence $0 < d(x_{n_0}, x) \leq d(x_{n_0}, x_n) + d(x_n, x) = d(x_n, x) < \varepsilon$ for all $n \geq \max\{n_0, n_1\}$. It follows that $0 < d(x_{n_0}, x) \leq 0$, which is impossible. Therefore $d(x_n, x) = 0$ for all $n \in \mathbb{N}$ and, thus, $x_n \leq_d x$ for all $n \in \mathbb{N}$. It follows that x is an upper bound of $(x_n)_{n\in\mathbb{N}}$.

(2) Assume that there exists $y \in X$ such that $x_n \leq_d y$ for all $n \in \mathbb{N}$. Then $d(x_n, y) = 0$ for all $n \in \mathbb{N}$. Whence we deduce that $d(x, y) \leq d(x, x_n) + d(x_n, y) = d(x, x_n)$ for all $n \in \mathbb{N}$. Since $(x_n)_{n\in\mathbb{N}}$ converges to x in $(X, \tau(d))$ we have that there exists $n_2 \in \mathbb{N}$ such that $d(x, x_n) < \varepsilon$ for all $n \geq n_2$. Thus we deduce that $d(x, y) = 0$, which implies that $x \leq_d y$. Whence we conclude that x is the least upper bound of $(x_n)_{n\in\mathbb{N}}$. □

In the light of the preceding result and taking into account the fact that Kleene's fixed point theorem requires any kind of order-completeness we introduce the $\tau(\leq_d)$-completeness. Thus, from now on, we will say that a quasi-metric space (X,d) is $\tau(\leq_d)$-complete provided that every increasing sequence is convergent with respect to $\tau(d^{-1})$. Of course, the $\tau(\leq_d)$-completeness will play a central role in achieving our purpose.

The next example gives an instance of a quasi-metric space (X,d) which is not $\tau(\leq_d)$-complete.

Example 2.2. Let $(\mathbb{R}, d_u)$ be the quasi-metric space where d_u is the quasi-metric defined by

$$d_u(x, y) = \max\{y - x, 0\}$$

for all $x, y \in \mathbb{R}$. It is clear that $d_u(x, y) = 0 \Leftrightarrow y \leq x$, where $\leq$ stands for the usual order in $\mathbb{R}$. Thus $x \leq_{d_u} y \Leftrightarrow y \leq x$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = -n$. It is obvious such a sequence is increasing with respect to $\leq_{d_u}$ but it is not convergent with respect to $\tau(d_u^{-1})$.

Examples 2.5, 2.6, 2.7, 2.8 and 2.11 provide instances of quasi-metric spaces that are $\tau(\leq_d)$-complete.

Finally, according to Kleene's fixed point theorem, we need to introduce a suitable notion of order-continuity in the context under consideration. Hence, from now on, we will say that a mapping from a quasi-metric space (X,d) into itself is orbitally-$\leq_d$-continuous if, given $z \in X$, then the sequence $(f^{n+1}(z))_{n\in\mathbb{N}}$ converges to $f(x)$ with respect to $\tau(d)$ whenever the sequence $(f^n(z))_{n\in\mathbb{N}}$ is increasing and x is an upper bound of it.

Examples 2.4, 2.5, 2.6 and 2.8 give instances of self-mappings in quasi-metric spaces that are orbitally-$\leq_d$-continuous mappings.

2.2. **The fixed point result.** Once we have found counterparts of the two main notions in Kleene's fixed point theorem, chain-completeness and $\leq$-continuity, in the quasi-metric context when the specialization partial order is considered, we are able to provide the promised result.

Theorem 2.3. *Let (X,d) be a $\tau(\leq_d)$-complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *f is monotone and orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$. Moreover, $Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*$.

Proof. Let $x_0 \in X$ such that $x_0 \leq_d f(x_0)$. Of course we can assume that $x_0 \neq f(x_0)$, since otherwise x_0 is a fixed point of f which holds that $x_0 \in \uparrow_{\leq_d} x_0$. Since $x_0 \leq_d f(x_0)$ and f is monotone we immediately obtain that

$$f^n(x_0) \leq_d f^m(x_0)$$

for all $m,n \in \mathbb{N}$ with $n \leq m$. Thus we have that $d(f^n(x_0), f^m(x_0)) = 0$ for all $m,n \in \mathbb{N}$ with $m \geq n$. It follows that $(f^n(x_0))_{n\in\mathbb{N}}$ is an increasing sequence in $(X,\leq_d)$. Since the quasi-metric space (X,d) is $\tau(\leq_d)$-complete there exists $z^* \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to z^* with respect to $\tau(d^{-1})$. By assertion (1) in Proposition 2.1, we obtain that z^* is an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$.

Since f is orbitally-$\leq_d$-continuous we immediately deduce that $(f^n(x_0))_{n\in\mathbb{N}}$ converges to $f(z^*)$ with respect to $\tau(d)$. Hence, by assertion (2) in Proposition 2.1, we deduce that $f(z^*)$ is the least upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$.

The fact that f is monotone provides that $f(f(z^*))$ is an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$. Moreover, the orbitally-$\leq_d$-continuity of f gives that $(f^n(x_0))_{n\in\mathbb{N}}$ converges to $f(f(z^*))$ with respect to $\tau(d)$. Again, assertion (2) in Proposition 2.1 yields that $f(f(z^*))$ is the least upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$. So $f(f(z^*)) = f(z^*)$.

Set $x^* = f(z^*)$. Then we obtain that $x^* \in Fix(f)$ and that $x^* \in \uparrow_{\leq_d} x_0$.

Now, we prove that $Fix(f) \cap \uparrow_{\leq_d} x_0 = x^*$. To this end, suppose that there exists $y^* \in Fix(f) \cap \uparrow_{\leq_d} x_0$. Then we have that $f^n(x_0) \leq_d y^*$ for all $n \in \mathbb{N}$. The fact that x^* is the least upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$ yields that $x^* \leq_d y^*$. The orbitally-$\leq_d$-continuity of the mapping f yields that $(f^{n+1}(x_0))_{n\in\mathbb{N}}$ converges to y^* with respect to $\tau(d)$. Whence, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(y^*, f^{n+1}(x_0)) < \varepsilon$$

for all $n \geq n_0$. Hence we deduce that

$$d(y^*, x^*) \leq d(y^*, f^{n_0+1}(x_0)) + d(f^{n_0+1}(x_0), x^*) = d(y^*, f^{n_0+1}(x_0)) < \varepsilon.$$

It follows that $d(y^*, x^*) = 0$ and, thus, that $y^* \leq_d x^*$. Therefore we conclude that $x^* = y^*$ as claimed. Finally, we notice that similar arguments to those given for the proof of $Fix(f) \cap \uparrow_{\leq_d} x_0 = x^*$ allow to show that $\uparrow_{\leq_d} x^* = x^*$. □

In the light of Theorem 2.3, it is worthy to point out that there are orbitally-$\leq_d$-continuous mappings which are not monotone (see Example 2.6) and, thus, assumption (2) in statement of Theorem 2.3 is not redundant.

The next example shows that the $\tau(\leq_d)$-completeness of the quasi-metric space cannot be deleted in Theorem 2.3 in order to guarantee the existence of fixed point.

Example 2.4. Consider the quasi-metric space $(\mathbb{R}, d_u)$ introduced in Example 2.2. Of course, as we have shown in the aforementioned example, $(\mathbb{R}, d_u)$ is not $\tau(d_u)$-complete. Next define the mapping f from $\mathbb{R}$ into itself by $f(x) = x - 1$. Clearly f is monotone and $0 \leq_{d_u} f(0) = -1$. Moreover, it is not hard to see that f is orbitally-$\leq_{d_u}$-continuous. However, f has no fixed points.

In the below example we show that the existence of an element x_0 such that $x_0 \leq_d f(x_0)$ cannot be omitted in the statement of Theorem 2.3 to ensure the existence of fixed point.

Example 2.5. Set $X = \{0, 1\}$. Consider the quasi-metric space (X, d_S), where the quasi-metric d_S is defined as follows:

$$d_S(x,y) = \begin{cases} y - x & \text{if } x \leq y \\ 1 & \text{if } x > y \end{cases}.$$

It is evident that $x \leq_{d_S} y \Leftrightarrow x = y$. It is clear that (X, d_S) is $\tau(\leq_{d_S})$-complete. Define the mapping f from X into itself by $f(0) = 1$ and $f(1) = 0$. Observe that the increasing sequences are exactly the constant ones. Then f is monotone and orbitally-$\leq_{d_S}$-continuous. In addition, $x \leq_{d_S} f(x)$ does not hold for any $x \in X$. It is obvious that f has no fixed points.

Next we show that the monotonicity of the mapping cannot be omitted in the statement of Theorem 2.3 in order to provide the existence of a fixed point.

Example 2.6. Let X be the set introduced in Example 2.5. Consider the quasi-metric d_u defined as in Example 2.2. Denote by $d_u|_X$ the restriction of d_u to the set X. Observe that the increasing sequences are eventually constant. Thus $(X, d_u|_X)$ is $\tau(d_u)$-complete. Consider the mapping f from X into itself introduced in Example 2.5. Of course $1 \leq_{d_u|_X} f(1) = 0$. It is obvious that f is orbitally-$\leq_d$-continuous. Nevertheless f is not monotone because $1 \leq_{d_u|_X} 0$ but $0 = f(1) \not\leq_{d_u|_X} f(0) = 1$. It is clear that f has no fixed points.

Next we show that the orbitally-$\leq_d$-continuity of the mapping cannot be omitted in the statement of Theorem 2.3 in order to provide the existence of a fixed point.

Example 2.7. Set $\mathbb{R}_1^+ = \mathbb{R}^+ \setminus \{1\}$. Define on $\mathbb{R}_1^+$ the quasi-metric d by

$$d_1(x,y) = \begin{cases} 0 & \text{if } x \geq y \\ 1 & \text{if } x < y \end{cases}.$$

It is a simple matter to see that $(\mathbb{R}_1^+, d_1)$ is $\tau(\leq_{d_1})$-complete, since every increasing sequence is convergent to 0 with respect to $\tau(d_1^{-1})$. Define the mapping f from $\mathbb{R}_1^+$ into itself defined by $f(x) = \frac{x+1}{2}$ for all $x \in \mathbb{R}_1^+$. Clearly f is monotone and $x \leq_{d_1} f(x)$ for all $x \in \mathbb{R}_1^+$ with $x > 1$. We see at once that f is not orbitally-$\leq_{d_1}$-continuous, which is clear form the fact that the increasing sequence $(f^{n+1}(2))_{n\in\mathbb{N}}$ does not converge to $f(0) = \frac{1}{2}$ with respect to $\tau(d_1)$ although 0 is an upper bound of it. Of course, f has no fixed points.

The next examples shows that Theorem 2.3 does not yield the uniqueness of the fixed point in general.

Example 2.8. Consider the $\tau(d_S)$-complete quasi-metric space (X, d_S) introduced in Example 2.5. Define the mapping f from X into itself by $f(x) = x$ for all $x \in X$. It is obvious that f is monotone and orbitally-$\leq_{d_S}$-continuous and that $x \leq_{d_S} f(x)$ for all $x \in X$. Clearly, $Fix(f) = X$.

In the light of the preceding example one can wonder when Theorem 2.3 guarantees the uniqueness of fixed point. The next result provides an answer to such a question.

Corollary 2.9. *Let (X, d) be a $\tau(\leq_d)$-complete quasi-metric space such that there exists a least element x_* of X with respect to $\leq_d$. If f is a monotone and orbitally-$\leq_d$-continuous mapping from X into itself, then there exists $x^* \in X$ such that $Fix(f) = x^*$ and $\uparrow_{\leq_d} x^* = x^*$.*

Proof. Since x_* is the least element of $(X, \leq_d)$ we have that $x_* \leq_d f(x_*)$. By Theorem 2.3 we deduce the existence of $x^* \in X$ such that $x^* \in Fix(f)$ and $Fix(f) \cap \uparrow_{\leq_d} x_* = \uparrow_{\leq_d} x^* = x^*$. Since $\uparrow_{\leq_d} x_* = X$ we deduce from the preceding equalities that $Fix(f) = x^*$. □

In the next example we show that the existence of a least element cannot be deleted in the statement of Corollary 2.9 in order to guarantee the uniqueness of the fixed point in the whole space.

Example 2.10. Let (X, d_S) be the $\tau(\leq_{d_S})$-complete quasi-metric space introduced in Example 2.5. Consider the mapping f from X into itself given by $f(x) = x$ for all $x \in X$. Then it is clear that $x \leq_{d_S} f(x)$ for all $x \in X$ and that f is monotone and orbitally-$\leq_{d_S}$-continuous. Moreover, it is obvious that there does not exist a least element of X with respect to $\leq_{d_S}$. Furthermore, $Fix(f) = X$.

We end the section with a reflection about the relationship between our new result (Theorem 2.3) and Kleene's fixed point theorem (Theorem 1.1).

Taking into account Kleene's fixed point theorem the next questions can be posted: Is $(X, \leq_d)$ a chain-complete partially ordered set whenever (X, d) is a $\tau(\leq_d)$-complete quasi-metric space? Are monotone and orbitally-$\leq_d$-continuous mappings defined on $\tau(\leq_d)$-complete quasi-metric spaces always $\leq_d$-continuous? Of course, if the answers to the preceding questions were positive, then Theorem 2.3 could be an immediate consequence of Kleene's fixed point theorem. Fortunately, the answers to our questions are negative as the next examples show.

Example 2.11. Consider the quasi-metric space $(\mathbb{R}_1^+, d_u)$, where d_u is the quasi-metric defined as in Example 2.2. It is clear that $(\mathbb{R}_1^+, d_u)$ is $\tau(\leq_{d_u})$-complete. Now, consider the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = 1 + \frac{1}{2n}$ for all $n \in \mathbb{N}$. A straightforward computation shows that the sequence $(x_n)_{n\in\mathbb{N}}$ is increasing in $(\mathbb{R}_1^+, \leq_{d_u})$ and, however, it does not have least upper bound. So $(\mathbb{R}_1^+, \leq_{d_u})$ is not chain-complete.

Example 2.12. Let $([0,1], d_1^{-1}|_{[0,1]})$ be the quasi-metric space where $d_1|_{[0,1]}$ denotes the restriction of the quasi-metric d_1 introduced in Example 2.7 to $[0,1]$. It is obvious that $([0,1], d_1^{-1}|_{[0,1]})$ is $\tau(d_1^{-1})$-complete. Define the mapping f from $[0,1]$ into itself by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{1}{2}[\\ x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

It is not hard to check that f is monotone and orbitally-$\leq_{d_1^{-1}}$-continuous mapping. However, f is not $\leq_{d_1^{-1}}$-continuous. Indeed, consider the increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1]$ given by $x_n = \frac{1}{2} - \frac{1}{2n}$ for all $n \in \mathbb{N}$. It is clear that $\frac{1}{2}$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$. Moreover, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is increasing and its least upper bound is $\frac{1}{4}$. Since $\frac{1}{2} = f(\frac{1}{2}) \neq \frac{1}{4}$ we conclude that f is not $\leq_{d_1^{-1}}$-continuous.

2.3. **A few consequences.** Next we provide several results which follow from Theorem 2.3. A few of them allow to retrieve as a particular case quasi-metric versions of Theorems 1.5 and 1.6 when different kind of quasi-metric completeness are considered.

The first two results of the batch take advantage of the relationship between a few notions of continuity in the quasi-metric approach. Specifically between the orbitally-$\leq_d$-continuous, the orbitally-continuity and the mixed-continuity.

Following [5], a mapping f from a metric space (X, d) into itself is orbitally-continuous provided that, given $x, y \in X$, the sequence $(f^{n+1}(x))_{n\mathbb{N}}$ converges to $f(y)$ with respect to $\tau(d)$ whenever $(f^n(x))_{n\mathbb{N}}$ converges to y with respect to $\tau(d)$. Of course, the preceding notion can be adapted literally to the quasi-metric context simply replacing in the definition, the metric space by a quasi-metric one, and the convergence with respect to the topology induced by the metric by the convergence with respect to the topology induced by the quasi-metric.

Since every monotone and orbitally-continuous mapping from a $\tau(\leq_d)$-complete quasi-metric space into itself is orbitally-$\leq_d$-continuous, Theorem 2.3 yields as a consequence the result below.

Corollary 2.13. *Let (X, d) be a $\tau(\leq_d)$-complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *f is monotone and orbitally-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$. Moreover,

$$Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*.$$

In the following, given a quasi-metric space (X, d), we will say that a mapping from X into itself is mixed-continuous provided that f is continuous from $(X, \tau(d^{-1}))$ into $(X, \tau(d))$.

Corollary 2.14. *Let (X, d) be an $\tau(\leq_d)$-complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *f is monotone and mixed-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$. Moreover,

$$Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*.$$

Proof. We only need to prove that every monotone and mixed-continuous mapping is always orbitally-$\leq_d$-continuous whenever the quasi-metric space (X, d) is $\tau(\leq_d)$-complete. To this end, consider the existence of $x_0 \in X$ such that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is increasing. Then the $\tau(\leq_d)$-completeness of the quasi-metric space guarantees the existence of $x \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to $\tau(d^{-1})$. By assertion (1) in Proposition 2.1 we deduce that x in an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$. Since f is mixed-continuous we obtain that $(f^{n+1}(x_0))_{n\in\mathbb{N}}$ converges to x with respect to $\tau(d)$. So f is orbitally-$\leq_d$-continuous. □

The following results exploits the existing relationship between $\tau(\leq_d)$-completeness and different types of "metric" completeness that arise in a natural way in the quasi-metric framework.

Following [14], a quasi-metric space (X, d) is right K-sequentially complete provided that every right K-Cauchy sequence is convergent with respect to $\tau(d)$, where a sequence $(x_n)_{n\in\mathbb{N}}$ is sad to be right K-Cauchy if, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m \geq n \geq n_0$.

Corollary 2.15. *Let (X, d) be a quasi-metric space such that (X, d^{-1}) is right K-sequentially complete and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *f is monotone and orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$. Moreover,

$$Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*.$$

Proof. We only need to prove that if (X, d^{-1}) is right K-sequentially complete, then (X, d) is $\tau(\leq_d)$-complete. Indeed, consider an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \leq_d)$. Then $d(x_n, x_{n+1}) = 0$ for all $n \in \mathbb{N}$. Whence we deduce that

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \ldots d(x_{m-1}, x_m) = 0$$

for all $m, n \in \mathbb{N}$ with $m \geq n$. So the sequence $(x_n)_{n\in\mathbb{N}}$ is right K-Cauchy in (X, d^{-1}). Since (X, d^{-1}) is right K-sequentially complete we deduce the existence of $x \in X$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x with respect to $\tau(d^{-1})$. Thus (X, d) is $\tau(\leq_d)$-complete. Now, by Theorem 2.3, the result follows. □

On account of [10], a quasi-metric space (X, d) is Smyth complete provided that every left K-Cauchy sequence is convergent with respect to $\tau(d^s)$, where, according to [14], a sequence $(x_n)_{n\in\mathbb{N}}$ is said to be left K-Cauchy if, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n \geq n_0$.

Corollary 2.16. *Let (X, d) be a Smyth complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *f is monotone and orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$. Moreover,

$$Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*.$$

Proof. It is clear that if (X, d) is Smyth complete quasi-metric space, then (X, d^{-1}) is right K-sequentially complete. Thus, by Corollary 2.15, we obtain the $\tau(\leq_d)$-completeness of (X, d). Therefore the thesis of the result follows from Theorem 2.3. □

According to [11], a quasi-metric space (X, d) is said to be weightable provided there exists a function $w : X \to \mathbb{R}^+$ such that

$$d(x, y) + w(x) = d(y, x) + w(y)$$

for all $x, y \in X$. Since every bicomplete weightable quasi-metric space is always Smyth complete (see, for instance, [9]) we can deduce from Corollary 2.16 the following one.

Corollary 2.17. *Let (X, d) be a bicomplete weightable quasi-metric space and and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *f is monotone and orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$. Moreover,

$$Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*.$$

In the next results we provide a little more information about the fixed point when the mapping under consideration holds any type of Banach contractive condition. Clearly, they are related to Theorem 1.4 and they are quasi-metric versions of Theorems 1.5 and 1.6.

Corollary 2.18. *Let (X, d) be a $\tau(\leq_d)$-complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *$d(f(x), f(y)) = 0$ for every $x, y \in X$ such that $d(x, y) = 0$.*
(3) *f is orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$. Moreover,

$$Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*.$$

Proof. Assumption (2) in the statement of the result is equivalent to the monotonicity of f and, thus, Theorem 2.3 gives the desired conclusion. □

It is interesting to note that the contractive condition (1.3) in Theorem 1.4 is recovered form assumption (2) in Corollary 2.18.

Corollary 2.19. *Let (X, d) be a $\tau(\leq_d)$-complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*

(2) *There exists* $k \in [0,1]$ *such that*

$$d(f(x), f(y)) \leq kd(x,y) \tag{2.1}$$

for all $x, y \in X$ *with* $x \leq_d y$.

(3) f *is orbitally-*$\leq_d$*-continuous.*

Then there exists $x^* \in X$ *such that* $x^* \in Fix(f)$ *and* $x^* \in \uparrow_{\leq_d} x_0$. *Moreover,*

$$Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*.$$

Proof. Assumption (2) in the statement provides the monotonicity of f. So, the desired conclusion follows from Theorem 2.3. □

The next result gives more information about the fixed point whenever the mapping holds a contractive condition as (1.5) in Theorems 1.5 and 1.6.

Corollary 2.20. *Let* (X, d) *be a* $\tau(\leq_d)$*-complete quasi-metric space and let* f *be a mapping from* X *into itself. Assume that the following assertions hold:*

(1) *There exists* $x_0 \in X$ *such that* $x_0 \leq_d f(x_0)$.

(2) *There exists* $k \in [0,1[$ *such that*

$$d(f(x), f(y)) \leq kd(x,y) \tag{2.2}$$

for all $x, y \in X$ *with* $y \leq_d x$.

(3) f *is monotone and orbitally-*$\leq_d$*-continuous.*

Then there exists $x^* \in X$ *such that* $x^* \in Fix(f)$ *and* $x^* \in \uparrow_{\leq_d} x_0$ *with* $\uparrow_{\leq_d} x^* = x^*$. *Moreover,* $Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = Fix(f) \cap \downarrow_{\leq_d} x^* = x^*$.

Proof. The existence of $x^* \in X$ such that $x^* \in Fix(f)$ and $Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = x^*$ is provided by Theorem 2.3. Next we prove that $Fix(f) \cap \downarrow_{\leq_d} x^* = x^*$. Assume that there exists $y^* \in Fix(f) \cap \downarrow_{\leq_d} x^*$; i.e., $y^* \in Fix(f)$, $y^* \leq_d x^*$. By the contractive condition (2.2),

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq kd(x^*, y^*);$$

and this, along with $k < 1$, yields $d(x^*, y^*) = 0$; i.e., $x^* \leq_d y^*$. Hence, combining with the above, $x^* = y^*$. Whence we conclude that $Fix(f) \cap \downarrow_{\leq_d} x^* = x^*$. □

The next example shows that we cannot delete the contractive condition (2.2) in statement of Corollary 2.20 in order to guarantee the uniqueness of the fixed point in the set $\downarrow_{\leq_d} x^*$. Furthermore, observe that Examples 2.6 and 2.22 show that assumption (2) in the statement of Corollary 2.20 is not redundant, i.e., there exist mappings satisfying the contractive condition (2.2) which are not either monotone or orbitally-$\leq_d$-continuous, respectively.

Example 2.21. Let $(\mathbb{R}^+, d_u)$ be the $\tau(\leq_{d_u})$-complete quasi-metric where d_u is the restriction of the quasi-metric introduced in Example 2.2 to $\mathbb{R}^+$. Consider the monotone and orbitally-$\leq_{d_u}$-continuous mapping f from $\mathbb{R}^+$ defined by

$$f(x) = \begin{cases} \frac{x+1}{2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Of course, $x_0 \leq_{d_u} f(x_0)$ for all $x_0 \in \mathbb{R}^+$ with $x_0 \notin]0,1[$. Nonetheless, f does not satisfy condition (2.2) in Corollary 2.20. Indeed, $\frac{1}{4} \leq_{d_u} 0$ but there does not exist

$k \in [0, 1[$ such that $d(f(0), f(\frac{1}{4})) \leq kd(0, \frac{1}{4})$, since $d(f(0), f(\frac{1}{4})) = \frac{5}{8}$ and $d(0, \frac{1}{4}) = \frac{1}{4}$. Observe that $Fix(f) = \{0, 1\}$ and $\downarrow_{\leq_{d_u}} 0 = \{0, 1\}$.

Example 2.22. Let $([0, 1], d_u^{-1})$ be the quasi-metric space, where d_u is the restriction of the quasi-metric introduced in Example 2.2 to $[0, 1]$. A straightforward computation shows that $([0, 1], d_u^{-1})$ is $\tau(\leq_{d_u^{-1}})$-complete. Define the mapping f from $\mathbb{R}^+$ into itself by $f(x) = \frac{x}{2}$ for all $x \in [0, 1]$. Then it is obvious that f is monotone. Moreover, f is not orbitally-$\leq_{d_u^{-1}}$-continuous because the sequence $(f^n(0))_{n \in \mathbb{N}}$ is increasing and 1 is an upper bound of it but $(f^{n+1}(0))_{n \in \mathbb{N}}$ does not converge to $f(1) = \frac{1}{2}$. It is easy to cheek that

$$d_u^{-1}(f(x), f(y)) \leq \frac{1}{2} d_u^{-1}(x, y)$$

for all $x, y \in [0, 1]$ such that $y \leq_{d_u^{-1}} x$.

The next result are quasi-metric versions of Theorem 1.5 and 1.6.

Taking into account that every quasi-metric space (X, d) whose conjugate quasi-metric space (X, d^{-1}) is right K-sequentially complete enjoys the $\tau(\leq_d)$-completeness we can deduce from Corollary 2.20 the next ones.

Corollary 2.23. *Let (X, d) be a quasi-metric space such that (X, d^{-1}) is right K-sequentially complete and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *There exists $k \in [0, 1[$ such that*

$$d(f(x), f(y)) \leq kd(x, y) \tag{2.3}$$

for all $x, y \in X$ with $y \leq_d x$.

(3) *f is monotone and orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$ with $\uparrow_{\leq_d} x^* = x^*$. Moreover, $Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = Fix(f) \cap \downarrow_{\leq_d} x^* = x^*$.

Since Smyth complete and weightable bicomplete quasi-metric spaces are such that their conjugate quasi-metric spaces are right K-sequentially complete we obtain from Corollary 2.23 the results below.

Corollary 2.24. *Let (X, d) be a Smyth complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *There exists $k \in [0, 1[$ such that*

$$d(f(x), f(y)) \leq kd(x, y) \tag{2.4}$$

for all $x, y \in X$ with $y \leq_d x$.

(3) *f is monotone and orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$ and $x^* \in \uparrow_{\leq_d} x_0$ with $\uparrow_{\leq_d} x^* = x^*$. Moreover, $Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = Fix(f) \cap \downarrow_{\leq_d} x^* = x^*$.

Corollary 2.25. *Let (X, d) be a weightable bicomplete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists* $x_0 \in X$ *such that* $x_0 \leq_d f(x_0)$.
(2) *There exists* $k \in [0,1[$ *such that*

$$d(f(x), f(y)) \leq kd(x, y) \tag{2.5}$$

for all $x, y \in X$ *with* $y \leq_d x$.
(3) f *is monotone and orbitally-*$\leq_d$*-continuous.*

Then there exists $x^* \in X$ *such that* $x^* \in Fix(f)$ *and* $x^* \in \uparrow_{\leq_d} x_0$ *with* $\uparrow_{\leq_d} x^* = x^*$. *Moreover,* $Fix(f) \cap \uparrow_{\leq_d} x_0 = \uparrow_{\leq_d} x^* = Fix(f) \cap \downarrow_{\leq_d} x^* = x^*$.

The next result gives the uniqueness of the fixed point in the whole space whenever the mapping holds the contractive condition (1.4) in Theorem 1.4.

Corollary 2.26. *Let* (X, d) *be an* $\tau(\leq_d)$*-complete quasi-metric space and let* f *be a mapping from* X *into itself. Assume that the following assertions hold:*
(1) *There exists* $x_0 \in X$ *such that* $x_0 \leq_d f(x_0)$.
(2) *There exists* $k \in [0,1[$ *such that*

$$d(f(x), f(y)) \leq kd(x, y) \tag{2.6}$$

for all $x, y \in X$.
(3) f *is orbitally-*$\leq_d$*-continuous.*

Then there exists $x^* \in X$ *such that* $Fix(f) = x^*$, $\uparrow_{\leq_d} x^* = x^*$ *and* $x^* \in \uparrow_{\leq_d} x_0$.

Proof. The contractive condition (2.6) yields the monotonicity of the mapping f. Hence the existence of $x^* \in X$ such that $x^* \in Fix(f)$, $x^* \in \uparrow_{\leq_d} x_0$ and $\uparrow_{\leq_d} x^* = x^*$ is provided by Theorem 2.3. It remains to prove that $Fix(f) = x^*$. With this aim, assume that there exists $y^* \in Fix(f)$. By the contractive condition (2.6),

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq kd(x^*, y^*);$$

and this, along with $k < 1$, yields $d(x^*, y^*) = 0$; i.e., $x^* \leq_d y^*$. Combining with $\uparrow_{\leq_d} x^* = x^*$, we get $x^* = y^*$. □

Example 2.21 shows that the uniqueness of the fixed point in the whole space is not guaranteed whenever the contractive condition (2.6) in statement of Corollary 2.26 is weakened. In addition, Example 2.22 shows that assumption (3) in the statement of Corollary 2.26 is not redundant in the sense that there exist mappings satisfying the contractive condition (2.6) which are not orbitally-$\leq_d$-continuous.

In the light of Corollary 2.26 a natural question can be posted. What is the relationship between the aforementioned result and the Banach fixed point theorem for bicomplete quasi-metric spaces (Theorem 1.3)?

Notice that if every $\tau(\leq_d)$-complete quasi-metric space was always bicomplete, then the contractive condition (2.6) in statement of Corollary 2.26 would provide the existence and uniqueness of the fixed point in the whole space. Fortunately, the next examples show that there exist $\tau(\leq_d)$-complete quasi-metric spaces that are not bicomplete.

Example 2.27. Let $([0,1[, d_u|_{[0,1[})$ be the quasi-metric space where $d_u|_{[0,1[}$ denotes the restriction of the quasi-metric d_u introduced in Example 2.2 to $[0,1[$. It is a simple matter to check that $([0,1[, d_u|_{[0,1[})$ is $\tau(\leq_{d_u})$-complete. Consider the sequence

$(x_n)_{n\in\mathbb{N}}$ in $[0,1[$ given by $x_n = 1 - \frac{1}{2n}$ for all $n \in \mathbb{N}$. It is easily seen that such a sequence is Cauchy and it is not convergent with respect to $\tau(d_u|^s_{[0,1[})$.

Similar to the case of Corollary 2.19 we can obtain, from Corollary 2.26, the following fixed point results that are related to Theorem 1.4.

Corollary 2.28. *Let (X,d) be a quasi-metric space such that (X,d^{-1}) is right K-sequentially complete and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *There exists $k \in [0,1[$ such that*

$$d(f(x), f(y)) \leq kd(x,y) \tag{2.7}$$

for all $x, y \in X$.
(3) *f is orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $Fix(f) = x^*$, $\uparrow_{\leq_d} x^* = x^*$ and $x^* \in \uparrow_{\leq_d} x_0$.

Corollary 2.29. *Let (X,d) be a Smyth complete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *There exists $k \in [0,1]$ such that*

$$d(f(x), f(y)) \leq kd(x,y) \tag{2.8}$$

for all $x, y \in X$.
(3) *f is orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $Fix(f) = x^*$, $\uparrow_{\leq_d} x^* = x^*$ and $x^* \in \uparrow_{\leq_d} x_0$.

Corollary 2.30. *Let (X,d) be a weigtable bicomplete quasi-metric space and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) *There exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.*
(2) *There exists $k \in [0,1[$ such that*

$$d(f(x), f(y)) \leq kd(x,y) \tag{2.9}$$

for all $x, y \in X$.
(3) *f is orbitally-$\leq_d$-continuous.*

Then there exists $x^* \in X$ such that $Fix(f) = x^*$, $\uparrow_{\leq_d} x^* = x^*$ and $x^* \in \uparrow_{\leq_d} x_0$.

Acknowledgements. This project was funded by the National Plan for Science, Technology and Innovation (MAARIFAH) - King Abdulaziz City for Science and Technology - the Kingdom of Saudi Arabia - award number (12-MAT051-03). The authors also, acknowledge with thanks Science and Technology Unit, King Abdulaziz University for technical support. The authors are grateful to an anonymous referee for his useful suggestions and comments.

References

[1] M.A. Alghamdi, N. Shahzad, O. Valero, *New results on the Baire partial quasi-metric space, fixed point theory and asymtptotic complexity analysis for recursive programs*, Fixed Point Theory Appl., **14**(2014), 18 pp.

[2] M.A. Alghamdi, N. Shahzad, O. Valero, *Fixed point theorems in generalized metric spaces with applications to computer science*, Fixed Point Theory Appl., **118**(2013), 20 pp.

[3] J. de Bakker, E. de Vink, *Control Flow Semantics*, The MIT Press, Cambridge, 1996.

[4] A. Baranga, *The contraction principle as a particular case of Kleen's fixed point theorem*, Discrete Math., **98**(1991), 75-79.

[5] L.J.B. Ćirić, *On contraction type mappings*, Math. Balkanica, **1**(1971), 52-57.

[6] B.A. Davey, H.A. Priestley, *Introduction to Lattices and Order,* Cambridge University Press, Cambridge, 1990.

[7] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, New York, 2002.

[8] P. Hitzler, A.K. Seda, *Generalized distance functions in the theory of computation*, Comput. J., **53**(2010), 443-464.

[9] H.-P.A. Künzi, *Nonsymmetric topology,* in: Topology, János Bolyai Math. Studies, Budapest, **4**(1995), 303-338.

[10] H.-P.A. Künzi, M.P. Schellekens, *On the Yoneda completion of a quasi-metric space*, Theoret. Comput. Sci., **278**(2002), 159-194.

[11] S.G. Matthews, *Partial metric topology,* Ann. New York Acad. Sci., **728**(1994), 183-197.

[12] J.J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,* Order, **22**(2005), 223-239.

[13] J.J. Nieto, R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations,* Acta Math. Sinica (English Series), **23**(2007), 2205-2212.

[14] I.L. Reilly, P.V. Subrahmanyam, M.K. Vamanamrthy, *Cauchy sequences in quasi-pseudo-metric spaces*, Monatshefte Math., **93**(1982), 127-140.

[15] J.J.M.M. Rutten, *Elements of generalized ultrametric domain theory*, Theoret. Comput. Sci., **170**(1996), 349-381.

[16] D.S. Scott, *Outline of a mathematical theory of computation,* in: Proc. 4th Annual Princeton Conference on Information Sciences and Systems, 1970, 169-176.

[17] D.S. Scott, *Domains for denotational semantics*, LNCS, **140**(1982), 577-613.

[18] N. Shahzad, O. Valero, *On 0-complete partial metric spaces and quantitative fixed point techniques in Denotational Semantics*, Abstr. Appl. Anal., **2013**(2013), Article ID 985095, 11 pages.

[19] N. Shahzad, O. Valero, M.A. Alghamdi, M.A. Alghamdi, *A fixed point theorem in partial quasi-metric spaces and an application to software engineering*, Appl. Math. Comput., **268**(2015), 1292-1301.

[20] M.B. Smyth, *Quasi-uniformities: reconciling domains with metric spaces*, LNCS, **298**(1988), 236-253.

[21] M.B. Smyth, *Totally bounded spaces and compact ordered spaces as domain of computation*, in: Topology and Category Theory in Computer Science, Oxford University Press, Oxford, 1991, 207-229.

[22] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl., **117**(1986), 100-127

[23] M. Turinici, *Fixed points for monotone iteratively local contractions*, Demonstr. Math., **19**(1986), 171-180.

[24] G. Winskel, *The Formal Semantics of Programming Langugages*, The MIT Press, Cambridge, 1993.

Received: June 4, 2015; Accepted: February 25, 2016.