Fixed Point Theory, 19(2018), No. 2, 681-706
DOI: 10.24193/fpt-ro.2018.2.53
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

COMMON FIXED POINTS OF ASYMPTOTICALLY REGULAR SEMIGROUPS EQUIPPED WITH GENERALIZED LIPSCHITZIAN CONDITIONS

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Abstract. We use the generalized Lipschitzian condition as employed by Imdad and Soliman [M. Imdad, A.H. Soliman, *On uniformly generalized Lipschitzian mappings*, Fixed Point Theory Appl., **2010**(2010), Article ID 692401] to extend the Lipschitzian condition which was used on the common fixed point theorems for one parameter semigroups of asymptotically regular mappings in Banach spaces by several authors before. Our results extend some relevant common fixed point theorems due to the works of Górnicki [J. Górnicki, *Fixed points of asymptotically regular semigroups in Banach spaces*, Rend. Circ. Mat. Palermo (2), **XLVI**(1997), 89-118] and Wiśnicki [A. Wiśnicki, *On the structure of fixed-point sets of asymptotically regular semigroups*, J. Math. Anal. Appl., **393**(2012), 177-184].
Key Words and Phrases: Asymptotically regular semigroup, generalized Lipschitzian semigroup, common fixed point, weakly convergent sequence coefficient, Opial condition, uniformly convex Banach space.
2010 Mathematics Subject Classification: 47H10, 47H20.

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be asymptotically regular if

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0 \tag{1.1}$$

for all $x \in X$. The concept of asymptotic regularity was introduced by Browder and Petryshyn [4]. We know that the asymptotic regularity conditon (1.1) has become an imposed condition on some fixed point results of semigroups of Lipschtzian mappings (see, e.g., [31, 14, 5]) and more general semigroups of mappings which are not necessarily Lipschitzians (see, e.g., [24, 34, 33]). Note that, the fixed point theorems for Lipschitzian-type mappings have many applications, for instances, in best approximation theory, operator equations, differential and integral equations, variational inequalities, and variational inclusions (see, Chapter 8 of [1] and references therein).

In 1976, Ishikawa [25] provided the well known example of asymptotically regular mapping in Banach spaces. He proved that if C is a bounded closed convex subset of a Banach space E, and $T : C \to C$ is nonexpansive, then the mapping $T_\lambda = (1-\lambda)I+\lambda T$ is asymptotically regular for all $\lambda \in (0, 1)$. Other examples of asymptotically regular mappings can also be found in [39, 2]. On the other hand, Lin [27] constructed an example of asymptotically regular and Lipschitzian mapping without fixed point in ℓ_2-spaces. Also, Maluta *et al.* [30] showed an existence of continuous asymptotically regular mappings without fixed point in normed spaces.

In 2012, Wiśnicki [36] studied the existence theorems of common fixed points for one parameter semigroups of asymptotically regular mappings equipped with the Lipschitzian condition in Banach spaces. His results extended some relevant results due to the work of Górnicki in [19, 20, 21] by investigating the structure of common fixed point sets of the semigroups. For the other common fixed point theorems for the semigroups of asymptotically regular mappings equipped with the Lipschitzian condition in Banach spaces, we refer the reader to [18, 10, 12].

Many authors have attempted to extend the class of Lipschitzian mappings (see, e.g., [32, 26, 24]). Recall from [24] that a mapping $T : X \to X$ defined on a metric space (X, d) is said to be generalized Lipschitzian if for any $n \geq 1$, there exists $k_n > 0$ such that

$$\begin{aligned} &d(T^n x, T^n y) \\ &\qquad \leq k_n \max\left\{d(x, y), \frac{1}{2}d(x, T^n x), \frac{1}{2}d(y, T^n y), \frac{1}{2}d(x, T^n y), \frac{1}{2}d(y, T^n x)\right\} \end{aligned} \tag{1.2}$$

for all $x, y \in X$. Note that, if $k_n < 1$, then T^n is a quasi-contraction mapping (see [8] for detail of the concept of this mapping). In [24], Imdad and Soliman studied the existence theorem of common fixed points for strongly continuous one parameter semigroups of continuous asymptotically regular and generalized Lipschitzian mappings in metric spaces.

In this paper, we continue the work of Wiśnicki [36] and Imdad and Soliman [24]. Precisely, we study the existence of common fixed points of one parameter semigroups of asymptotically regular mappings equipped with the generalized Lipschitzian condition (1.2) in Banach spaces. In Section 3, we establish the common fixed point theorem for the semigroups by utilizing the weakly convergent sequence coefficient and the Opial's modulus. In Section 4, we investigate the structure of common fixed point sets of the semigroups having the asymptotic nonexpansiveness in uniformly convex Banach spaces. Then, we establish the common fixed point theorems in the framework of uniformly convex Banach spaces, and moreover, p-uniformly convex Banach spaces. As the results, Theorem 3.2 extends partially Theorem 4.2 of [36], Theorem 4.5 extends Theorem 1 of [18], and Theorem 4.8 extends partially Theorem 4.6 in [36] for a wider class of semigroups of self-mappings.

2. Preliminaries

We begin this section by establishing the concept of generalized Lipschitzian semigroups and asymptotically regular semigroups in Banach spaces.

Let E be a Banach space and C be a nonempty subset of E. Let G be an unbounded subset of $[0,\infty)$ such that $t+s \in G$ for all $s,t \in G$ and $t-s \in G$ for all $t,s \in G$ with $t \geq s$ (e.g., $G=[0,\infty)$ or $G=\mathbb{N}\cup\{0\}$). A family of mappings $\mathcal{T}=\{T_t : t \in G\}$ from C into itself is said to be a (one parameter) semigroup on C if for any $s,t \in G$ and $x \in C$ we have $T_{s+t}x = T_sT_tx$ and $T_0x = x$. In this paper, we do not necessarily assume that $\mathcal{T}$ is a strongly continuous semigroup. (We note that, many literature in fixed point theory treats $\mathcal{T}$ as a strongly continuous semigroup (see, e.g., [35, 18, 5, 24, 22, 33])). A semigroup $\mathcal{T}$ is said to be generalized Lipschitzian if for any $t \in G$, there exists $k_t > 0$ such that

$$\|T_tx - T_ty\| \leq k_t \max\left\{\|x-y\|, \frac{1}{2}\|x-T_tx\|, \frac{1}{2}\|y-T_ty\|, \frac{1}{2}\|x-T_ty\|, \frac{1}{2}\|y-T_tx\|\right\} \quad (2.1)$$

for all $x,y \in C$.

The infimum of constants k_t in (2.1) is called generalized Lipschitz constant and is denoted by $\varrho(T_t)$. Of course, if a semigroup $\mathcal{T}=\{T_t : t \in G\}$ of mappings on C is generalized Lipschitzian, then $\varrho(T_t) < \infty$ for all $t \in G$. But, if $\mathcal{T}$ is not generalized Lipschitzian, then there exists a mapping $T_{t_0} \in \mathcal{T}$ such that there is no constant k_{t_0} for which the inequality (2.1) holds for all $x,y \in C$. For this case, we define $\varrho(T_{t_0}) = \infty$.

A semigroup $\mathcal{T}=\{T_t : t \in G\}$ of mappings on C is said to be asymptotically regular if

$$\lim_{t\to\infty} \|T_{h+t}x - T_tx\| = 0$$

for all $x \in C$ and $h \in G$.

We show that there exists a semigroup of generalized Lipschitzian and asymptotically regular mappings which are not lipschizians.

Example 2.1. Let E be the real line $\mathbb{R}$, $C=[-L,L]$, where $1 < L < \infty$, and $G=[0,\infty)$. Let $\lambda \in (1,L)$ be fixed. Define a family $\mathcal{T}=\{T_t : t \in G\}$ of mappings on C as follows.

$$T_tx = \begin{cases} -\lambda^{1-t} & \text{if } x \in [-L,-\lambda), \\ \lambda^{-t}x & \text{if } x \in [-\lambda,\lambda], \\ 0 & \text{if } x \in (\lambda,L], \end{cases}$$

for all $t > 0$, and

$$T_0x = x \quad \text{for all } x \in C.$$

Firstly, it is easy to see that $\mathcal{T}$ is a semigroup. Moreover, since T_t is not continuous at λ for all $t > 0$, then $\mathcal{T}$ is not Lipschitzian semigroup. Here, the exact Lipschitz constant for T_t is ∞ for all $t > 0$.

Let $t \geq 0$ and $x,y \in [-L,L]$ be fixed. If $x,y \in [-L,-\lambda) \cup (\lambda,L]$, then

$$|T_tx - T_ty| = 0.$$

For the case of $x,y \in [-\lambda,\lambda]$ we have

$$|T_tx - T_ty| = \lambda^{-t}|x-y|.$$

If $x \in [-\lambda, \lambda]$ and $y \in [-L, -\lambda)$, then

$$|T_t x - T_t y| = \lambda^{-t}(x - (-\lambda)) < \lambda^{-t}|x - y|.$$

On the other hand, we note that, if $x \in [-\lambda, \lambda]$ and $y \in (\lambda, L]$ then

$$|T_t x - T_t y| = \lambda^{-t}|x| = 2\lambda^{-t}\left(\frac{1}{2}|x - T_t y|\right).$$

From the above observation, we obtain

$$|T_t x - T_t y| \le 2\lambda^{-t} \max\left\{|x - y|, \frac{1}{2}|x - T_t y|, \frac{1}{2}|y - T_t x|\right\}.$$

Hence, $\mathcal{T}$ is a generalized Lipschitzian semigroup with

$$\liminf_{t\to\infty} \varrho(T_t) = 0.$$

Next, take an arbitrary $h \ge 0$. Since

$$\begin{aligned}\lim_{t\to\infty} |T_{h+t}x - T_t x| &= \lim_{t\to\infty} \left|\lambda^{-h-t}x - \lambda^{-t}x\right| \\ &= \lim_{t\to\infty} \lambda^{-t}\left|\lambda^{-h}x - x\right| = 0 \text{ for all } x \in [-\lambda, \lambda],\end{aligned}$$

$$\lim_{t\to\infty} |T_{h+t}x - T_t x| = \lim_{t\to\infty} \lambda^{1-t}\left|\lambda^{-h} - 1\right| = 0 \text{ for all } x \in [-L, -\lambda),$$

and

$$\lim_{t\to\infty} |T_{h+t}x - T_t x| = \lim_{t\to\infty} |0 - 0| = 0 \text{ for all } x \in (\lambda, L],$$

then $\mathcal{T}$ is an asymptotically regular semigroup.

We shall recall some basic properties and important results concerning normal structure in Banach spaces which have essential role in metric fixed point theory.

The normal structure coefficient $N(E)$ of a Banach space E is the real number defined by (see [7])

$$N(E) = \inf\left\{\frac{diam(C)}{r(C)}\right\},$$

where the infimum is taken over all bounded closed convex subsets of E with $diam(C) > 0$. Here, $diam(C) = \sup\{\|x - y\| : x, y \in C\}$ is the diameter of C and $r(C) = \inf_{x\in C}\sup_{y\in C}\|x - y\|$ is the Chebyshev radius of C relative to itself. It is known that, if $N(E) > 1$ then E is reflexive.

Starting this point, we assume a Banach space E always does not have the Schur property, that is, there is a weakly convergent sequence which is not norm convergent. The weakly convergent sequence coefficient (or Bynum's coefficient) is defined by (see [7])

$$WCS(E) = \inf\left\{\frac{diam_a(x_n)}{r_a(x_n)}\right\},$$

where the infimum is taken over all weakly (not strongly) convergent sequences $\{x_n\}$ in E. Here,

$$r_a(x_n) = \inf\left\{\limsup_{n\to\infty}\|x_n - y\| : y \in \overline{conv}(\{x_n : n \ge 1\})\right\}$$

($\overline{conv}$ is the closed convex hull) and

$$diam_a(x_n) = \inf_{n\geq 1} \sup\{\|x_l - x_m\| : l, m \geq n\}.$$

By using the equivalent definition of the weakly convergent sequence coefficient (see, page 162 of [1]), it is easy to see that $1 \leq WCS(E) \leq 2$. Particularly, if E is a reflexive Banach space then (see, Theorem 1 of [7])

$$1 \leq N(E) \leq WCS(E) \leq 2.$$

To prove our main results, we shall use the other equivalence definition for the weakly convergent sequence coefficient.

Lemma 2.2. [13] *Let E be a Banach space which does not have the Schur property. Then,*

$$WCS(E) = \inf\left\{D[\{x_n\}] : x_n \rightharpoonup 0 \, and \, \|x_n\| \to 1\right\},$$

where

$$D[\{x_n\}] = \limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - x_m\|.$$

A general formula for $WCS(E)$ in an arbitrary Banach space E is unknown. But, we see from Proposition 8.2 of [3] and Theorem 3.3.(ii) of [28] that for a Hilbert space H, $WCS(H) = \sqrt{2}$. Also, it has been calculated in [7] that for an ℓ_p-space, $1 < p < \infty$, $WCS(\ell_p) = 2^{\frac{1}{p}}$. Therefore, for an ℓ_p-space, $1 < p < 2$, we have

$$N(\ell_p) = 2^{\frac{p-1}{p}} < 2^{\frac{1}{p}} = WCS(\ell_p).$$

For other values of the weakly convergent sequence coefficient in some Banach spaces, we refer the reader to [29] and references therein.

A Banach space E is said to satisfy the Opial condition, if whenever a sequence $\{x_n\}$ in E converges weakly to $x \in E$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in E\backslash\{x\}$ (see [31]). It is well known that any Hilbert space, also an ℓ_p-space, $1 < p < \infty$, satisfy the Opial condition. But, the Lebesgue space $L_p[0, 2\pi]$, $1 < p < \infty$, $p \neq 2$ fails to satisfy the Opial condition (see [31]).

Recall from [28] that the Opial's modulus of a Banach space E is the function $r_E : [0, \infty) \to \mathbb{R}$ given by

$$r_E(c) = \inf\left\{\liminf_{n\to\infty} \|x_n + x\| - 1\right\},$$

where the infimum is taken over all $x \in E$ with $\|x\| \geq c$ and all sequences $\{x_n\}$ in E such that $x_n \rightharpoonup 0$ and $\liminf_{n\to\infty} \|x_n\| \geq 1$. Note that, the function r_E is continuous and increasing. Related to the weakly convergent sequence coefficient, we know that (see, Corollary 3.2.(i) of [28])

$$1 + r_E(1) \leq WCS(E) \tag{2.2}$$

for any Banach space E. Also, (see, Theorem 2.1 of [38])

$$c - 1 \leq r_E(c) \leq c \tag{2.3}$$

for all $c \geq 0$.

Recall the notion of asymptotic center due to Edelstein (see [15]). Let C be a nonempty subset of E and $\{x_n\}$ be a bounded sequence in E. The asymptotic center of $\{x_n\}$ with respect to C is defined as the set

$$\mathcal{A}(C,\{x_n\}) = \left\{ x \in C : \limsup_{n\to\infty} \|x_n - x\| = \inf_{y\in C} \limsup_{n\to\infty} \|x_n - y\| \right\}.$$

We know that, if C is weakly compact then $\mathcal{A}(C,\{x_n\})$ is nonempty. Moreover, if C is also convex then $\mathcal{A}(C,\{x_n\})$ is nonempty, closed, and convex.

3. The common fixed point theorem

The following lemma is crucial in proving our main results.

Lemma 3.1. *Let C be a nonempty subset of a Banach space E. Suppose that*

$$\mathcal{T} = \{T_t : t \in G\}$$

is an asymptotically regular and generalized Lipschitzian semigroup on C such that

$$\lim_{n\to\infty} \varrho(T_{s_n}) < 2,$$

where $T_{s_n}u \to v$ as $n \to \infty$ for some $u, v \in C$ and $\{s_n\}$ is an increasing sequence in G such that $\lim_{n\to\infty} s_n = \infty$. Then, $T_t v = v$ for all $t \in G$.

Proof. By the assumption, we can choose a subsequence $\{t_n\}$ of $\{s_n\}$ such that $\sup_{r\geq 1} \varrho(T_{t_r}) = \mu < 2$. Let $r \geq 1$ and $\varepsilon > 0$ be fixed. Then, there exists $n_0 = n_0(r,\varepsilon) \geq 1$ such that

$$\|T_{t_n}u - v\| < \varepsilon \quad \text{and} \quad \|T_{t_r+t_n}u - T_{t_n}u\| < \varepsilon$$

for all $n \geq n_0$. Therefore, for any $n \geq n_0$ we have

$$\begin{aligned}
&\|T_{t_r}v - v\| \leq \|T_{t_r}v - T_{t_r+t_n}u\| + \|T_{t_r+t_n}u - T_{t_n}u\| + \|T_{t_n}u - v\| \\
&\leq \varrho(T_{t_r}) \max\left\{ \|v - T_{t_n}u\|, \frac{1}{2}\|v - T_{t_r}v\|, \frac{1}{2}\|T_{t_n}u - T_{t_r+t_n}u\|, \frac{1}{2}\|v - T_{t_r+t_n}u\|, \right. \\
&\quad \left. \frac{1}{2}\|T_{t_n}u - T_{t_r}v\| \right\} + 2\varepsilon \\
&\leq \mu \max\left\{ \varepsilon, \frac{1}{2}\|v - T_{t_r}v\|, \frac{\varepsilon}{2}, \frac{1}{2}(\|v - T_{t_n}u\| + \|T_{t_n}u - T_{t_r+t_n}u\|), \right. \\
&\quad \left. \frac{1}{2}(\|T_{t_n}u - v\| + \|v - T_{t_r}v\|) \right\} + 2\varepsilon \\
&\leq \mu \max\left\{ \varepsilon, \frac{1}{2}(\varepsilon + \varepsilon), \frac{1}{2}(\varepsilon + \|v - T_{t_r}v\|) \right\} + 2\varepsilon \\
&\leq \mu \left(\varepsilon + \frac{\varepsilon}{2} + \frac{1}{2}\|v - T_{t_r}v\| \right) + 2\varepsilon.
\end{aligned}$$

It follows that

$$0 \leq \|v - T_{t_r}v\| \leq \left(\frac{2}{2-\mu} \right) \left(\frac{3}{2}\mu + 2 \right) \varepsilon \to 0 \,\text{as}\, \varepsilon \downarrow 0.$$

Hence, $T_{t_r}v = v$ for all $r \geq 1$. Then, the asymptotic regularity of $\mathcal{T}$ yields

$$\|T_t v - v\| = \lim_{r\to\infty} \|T_t T_{t_r} v - T_{t_r} v\| = 0$$

for all $t \in G$. □

We now establish the common fixed point theorem for asymptotically regular semigroups equipped with the generalized Lipschitzian condition in Banach spaces which, in turn, as a partial extension of Theorem 4.2 of [36] for a wider class of semigroups of self-mappings, by replacing the exact Lipschitz constant with generalized Lipschitz constant.

Theorem 3.2. *Let C be a nonempty weakly compact subset of a Banach space E and $\mathcal{T} = \{T_t : t \in G\}$ be an asymptotically regular and generalized Lipschitzian semigroup on C. Assume that:*

(a) $\liminf_{t\to\infty} \varrho(T_t) < \sqrt{WCS(E)}$ *or,*
(b) $\liminf_{t\to\infty} \varrho(T_t) < 1 + r_E(1)$.

Then, there exists $z \in C$ such that $T_t z = z$ for all $t \in G$.

Proof. Choose an increasing sequence $\{s_n\}$ in G such that $\lim_{n\to\infty} s_n = \infty$ and

$$\lim_{n\to\infty} \varrho(T_{s_n}) = \liminf_{t\to\infty} \varrho(T_t) = \varrho(\mathcal{T}).$$

We consider two possible cases.
Case 1. $\varrho(\mathcal{T}) < 1$. Take an arbitrary $x \in C$. Then, the weak compactness of C allow us to ensure

$$\mathcal{A}(C, \{T_{s_n}x\}) = \left\{ z \in C : \limsup_{n\to\infty} \|T_{s_n}x - z\| = \inf_{y\in C} \limsup_{n\to\infty} \|T_{s_n}x - y\| \right\} \neq \emptyset.$$

Fix $z \in \mathcal{A}(C, \{T_{s_n}x\})$. We shall show that z is the common fixed point of $\mathcal{T}$.
Firstly, for each $r \geq 1$, using the asymptotic regularity of $\mathcal{T}$ we have

$$\begin{aligned}
&\limsup_{n\to\infty} \|T_{s_n}x - T_{s_r}z\| \leq \limsup_{n\to\infty} (\|T_{s_n}x - T_{s_r+s_n}x\| + \|T_{s_r+s_n}x - T_{s_r}z\|) \\
&\leq \limsup_{n\to\infty} \varrho(T_{s_r}) \max\left\{ \|T_{s_n}x - z\|, \frac{1}{2}\|T_{s_n}x - T_{s_r+s_n}x\|, \frac{1}{2}\|z - T_{s_r}z\|, \right. \\
&\quad \left. \frac{1}{2}\|T_{s_n}x - T_{s_r}z\|, \frac{1}{2}\|z - T_{s_r+s_n}x\| \right\} \\
&\leq \varrho(T_{s_r}) \limsup_{n\to\infty} \max\left\{ \|T_{s_n}x - z\|, \frac{1}{2}(\|z - T_{s_n}x\| + \|T_{s_n}x - T_{s_r}z\|), \right. \\
&\quad \left. \frac{1}{2}(\|z - T_{s_n}x\| + \|T_{s_n}x - T_{s_r+s_n}x\|) \right\}
\end{aligned}$$

$$= \varrho(T_{s_r}) \max \Big\{ \limsup_{n\to\infty} \|T_{s_n}x - z\|, \frac{1}{2} \limsup_{n\to\infty} (\|z - T_{s_n}x\| + \|T_{s_n}x - T_{s_r}z\|),$$
$$\frac{1}{2} \limsup_{n\to\infty} (\|z - T_{s_n}x\| + \|T_{s_n}x - T_{s_r+s_n}\|) \Big\}$$
$$\leq \varrho(T_{s_r}) \max \Big\{ \limsup_{n\to\infty} \|T_{s_n}x - z\|,$$
$$\frac{1}{2} \limsup_{n\to\infty} \|T_{s_n}x - z\| + \frac{1}{2} \limsup_{n\to\infty} \|T_{s_n}x - T_{s_r}z\| \Big\}. \tag{3.1}$$

Moreover, by taking the limit superior as $r \to \infty$ into (3.1) we have

$$\limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{s_n}x - T_{s_r}z\| \leq \varrho(\mathcal{T}) \max \Big\{ \limsup_{n\to\infty} \|T_{s_n}x - z\|,$$
$$\frac{1}{2} \limsup_{n\to\infty} \|T_{s_n}x - z\| + \frac{1}{2} \limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{s_n}x - T_{s_r}z\| \Big\}. \tag{3.2}$$

If

$$\max \Big\{ \limsup_{n\to\infty} \|T_{s_n}x - z\|, \frac{1}{2} \limsup_{n\to\infty} \|T_{s_n}x - z\| + \frac{1}{2} \limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{s_n}x - T_{s_r}z\| \Big\}$$
$$= \frac{1}{2} \limsup_{n\to\infty} \|T_{s_n}x - z\| + \frac{1}{2} \limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{s_n}x - T_{s_r}z\|,$$

then from (3.2) we have

$$\limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{s_n}x - T_{s_r}z\| \leq \frac{\varrho(\mathcal{T})}{2 - \varrho(\mathcal{T})} \limsup_{n\to\infty} \|T_{s_n}x - z\|$$
$$\leq \varrho(\mathcal{T}) \limsup_{n\to\infty} \|T_{s_n}x - z\|.$$

Therefore, by the definition of the asymptotic center we obtain

$$\limsup_{n\to\infty} \|T_{s_n}x - z\| \leq \varrho(\mathcal{T}) \limsup_{n\to\infty} \|T_{s_n}x - z\|. \tag{3.3}$$

Similarly, if

$$\max \Big\{ \limsup_{n\to\infty} \|T_{s_n}x - z\|, \frac{1}{2} \limsup_{n\to\infty} \|T_{s_n}x - z\| + \frac{1}{2} \limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{s_n}x - T_{s_r}z\| \Big\}$$
$$= \limsup_{n\to\infty} \|T_{s_n}x - z\|,$$

we also obtain the inequality (3.3). Since $\varrho(\mathcal{T}) < 1$, then from (3.3) we have

$$\limsup_{n\to\infty} \|T_{s_n}x - z\| = 0.$$

Hence, by Lemma 3.1 we obtain z as the common fixed point of $\mathcal{T}$.

Moreover, it is easy to see that the set of all common fixed points of $\mathcal{T}$ is a singleton. Indeed, if w is also the common fixed point of $\mathcal{T}$, then for any $t \in G$ we have

$$\begin{aligned} \|z-w\| &= \|T_t z - T_t w\| \\ &\le \varrho(T_t) \max\left\{\|z-w\|, \frac{1}{2}\|z-T_t z\|, \frac{1}{2}\|w-T_t w\|, \frac{1}{2}\|z-T_t w\|, \frac{1}{2}\|w-T_t z\|\right\} \\ &= \varrho(T_t)\|z-w\|. \end{aligned} \tag{3.4}$$

Taking the limit inferior as $t \to \infty$ into (3.4) we get

$$\|z-w\| \le \liminf_{t\to\infty} \varrho(T_t)\|z-w\| = \varrho(\mathcal{T})\|z-w\|.$$

Since $\varrho(\mathcal{T}) < 1$, we immediately have $z = w$.

Case 2. $\varrho(\mathcal{T}) \ge 1$. Firstly, since $WCS(E) \le 2$ then from the inequality (2.2) we have

$$\varrho(\mathcal{T}) < 2.$$

For any $x \in C$, by using Eberlein-Šmulian Theorem (see, page 18 of [9]), make it possible to select a subsequence $\{s_n(x)\}$ of $\{s_n\}$ such that $\{T_{s_n(x)}x\}$ is weakly convergent, say to $l(x)$ and

$$\lim_{n\to\infty} \|T_{s_n(x)}x - l(x)\|$$

exists. We construct a sequence $\{x_m\}_{m\in\mathbb{N}}$ in C and a sequence $\{s_n(x_m)\}_{n\in\mathbb{N}}$ in G inductively, in the following ways.

$$\begin{gathered} x = x_0 \in C \,\text{arbitrary}, \\ x_m = l(x_{m-1}) = w - \lim_{n\to\infty} T_{s_n(x_{m-1})}x_{m-1}, \\ \{s_n(x_m)\}_{n\in\mathbb{N}} \,\text{is a subsequence of}\, \{s_n(x_{m-1})\}_{n\in\mathbb{N}} \,\text{for all}\, m \in \mathbb{N}. \end{gathered}$$

Then, by using a diagonal argument we consider a subsequence $\{t_n\}$ of $\{s_n\}$ defined by

$$t_n = s_n(x_n) \text{ for } n \ge 1.$$

From the above observation, we have the sequence $\{T_{t_n}x_m\}$ is weakly convergent to x_{m+1} and

$$\lim_{n\to\infty} \|T_{t_n}x_m - x_{m+1}\|$$

exists for all $m \ge 0$.

For any $m \ge 0$ we write

$$d_m = \limsup_{n\to\infty} \|T_{t_n}x_m - x_{m+1}\| \quad \text{and} \quad D_m = \limsup_{n\to\infty} \|T_{t_n}x_m - x_m\|.$$

We shall show that there exists $\eta < 1$ such that $d_m \le \eta d_{m-1}$ for all $m \ge 1$.

Let $m \geq 1$ be fixed. Note that, for each $r \geq 1$, using the asymptotic regularity of $\mathcal{T}$ we have

$$\begin{aligned}
&\limsup_{n\to\infty} \|T_{t_n}x_{m-1} - T_{t_r}x_m\| \\
&\leq \limsup_{n\to\infty}(\|T_{t_n}x_{m-1} - T_{t_r+t_n}x_{m-1}\| + \|T_{t_r+t_n}x_{m-1} - T_{t_r}x_m\|) \\
&\leq \varrho(T_{t_r}) \limsup_{n\to\infty} \max\Big\{\|T_{t_n}x_{m-1} - x_m\|, \frac{1}{2}\|T_{t_n}x_{m-1} - T_{t_r+t_n}x_{m-1}\|, \\
&\quad \frac{1}{2}\|x_m - T_{t_r}x_m\|, \frac{1}{2}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|, \\
&\quad \frac{1}{2}(\|x_m - T_{t_n}x_{m-1}\| + \|T_{t_n}x_{m-1} - T_{t_r+t_n}x_{m-1}\|)\Big\} \\
&= \varrho(T_{t_r}) \max\Big\{d_{m-1}, 0, \frac{1}{2}\|x_m - T_{t_r}x_m\|, \frac{1}{2}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|, \\
&\quad \frac{1}{2}\limsup_{n\to\infty}(\|x_m - T_{t_n}x_{m-1}\| + \|T_{t_n}x_{m-1} - T_{t_r+t_n}x_{m-1}\|)\Big\} \\
&\leq \varrho(T_{t_r}) \max\Big\{d_{m-1}, \frac{1}{2}\|x_m - T_{t_r}x_m\|, \frac{1}{2}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|, \frac{1}{2}d_{m-1}\Big\} \\
&= \varrho(T_{t_r}) \max\Big\{d_{m-1}, \frac{1}{2}\|x_m - T_{t_r}x_m\|, \frac{1}{2}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|\Big\} \qquad (3.5)
\end{aligned}$$

Moreover, by taking the limit superior as $r \to \infty$ into (3.5) we have

$$\begin{aligned}
&\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\| \\
&\quad \leq \varrho(\mathcal{T}) \max\Big\{d_{m-1}, \frac{1}{2}D_m, \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|\Big\}. \qquad (3.6)
\end{aligned}$$

If

$$\begin{aligned}
&\max\Big\{d_{m-1}, \frac{1}{2}D_m, \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|\Big\} \\
&\quad = \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|,
\end{aligned}$$

then by using (3.6) we get $D_m = 0$. Hence, by Lemma 3.1 we have x_m as the common fixed point of $\mathcal{T}$. Therefore, we assume

$$\max\Big\{d_{m-1}, \frac{1}{2}D_m, \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|\Big\} = \max\Big\{d_{m-1}, \frac{1}{2}D_m\Big\}.$$

Then, by the weak lower semi-continuity of $\|.\|$ and (3.6) we have

$$\begin{aligned} D_m &= \limsup_{r\to\infty} \|T_{t_r}x_m - x_m\| \\ &\le \limsup_{r\to\infty}\limsup_{n\to\infty} \|T_{t_n}x_{m-1} - T_{t_r}x_m\| \\ &\le \varrho(\mathcal{T})\max\left\{d_{m-1}, \frac{1}{2}D_m, \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_{m-1} - T_{t_r}x_m\|\right\} \\ &= \varrho(\mathcal{T})\max\left\{d_{m-1}, \frac{1}{2}D_m\right\}. \end{aligned} \tag{3.7}$$

If $\max\left\{d_{m-1}, \frac{1}{2}D_m\right\} = \frac{1}{2}D_m$, then from (3.7) we see that $D_m = 0$, and thus Lemma 3.1 ensures x_m is the common fixed point of $\mathcal{T}$. Therefore, we also assume

$$\max\left\{d_{m-1}, \frac{1}{2}D_m\right\} = d_{m-1}.$$

In this case, from (3.7) we obtain

$$D_m \le \varrho(\mathcal{T})d_{m-1}. \tag{3.8}$$

Now, for the rest of the proof, we may assume $d_m > 0$. Since otherwise, Lemma 3.1 ensures x_{m+1} is the common fixed point of $\mathcal{T}$. Note that, by Lemma 2.2 we obtain

$$d_m \le \frac{1}{WCS(E)}\cdot\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_m - T_{t_r}x_m\|. \tag{3.9}$$

Here, the asymptotic regularity of $\mathcal{T}$ yields

$$\begin{aligned} &\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_m - T_{t_r}x_m\| \le \limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_r+t_n}x_m - T_{t_r}x_m\| \\ &\le \limsup_{r\to\infty}\limsup_{n\to\infty}\varrho(\mathcal{T}_{t_r})\max\Big\{\|T_{t_n}x_m - x_m\|, \frac{1}{2}\|T_{t_n}x_m - T_{t_r+t_n}x_m\|, \\ &\quad \frac{1}{2}\|x_m - T_{t_r}x_m\|, \frac{1}{2}\|T_{t_n}x_m - T_{t_r}x_m\|, \frac{1}{2}(\|x_m - T_{t_n}x_m\| + \|T_{t_n}x_m - T_{t_r+t_n}x_m\|)\Big\} \\ &\le \varrho(\mathcal{T})\max\left\{D_m, \frac{1}{2}D_m, \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_m - T_{t_r}x_m\|, \frac{1}{2}D_m\right\} \\ &= \varrho(\mathcal{T})\max\left\{D_m, \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_m - T_{t_r}x_m\|\right\} \end{aligned} \tag{3.10}$$

If

$$\begin{aligned} &\max\left\{D_m, \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_m - T_{t_r}x_m\|\right\} \\ &\quad = \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_m - T_{t_r}x_m\|, \end{aligned}$$

then from (3.10) we see that $D_m = 0$. Hence, Lemma 3.1 ensures x_m is the common fixed point of $\mathcal{T}$. Therefore, we assume

$$\max\left\{D_m, \frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_m - T_{t_r}x_m\|\right\} = D_m.$$

In this case, from (3.9) and (3.10) we obtain

$$d_m \leq \frac{\varrho(\mathcal{T})}{WCS(E)} D_m. \tag{3.11}$$

Part (a). Let

$$\theta = \frac{[\varrho(\mathcal{T})]^2}{WCS(E)}.$$

It is obvious that $\theta < 1$ by the hypothesis. Combining (3.8) and (3.11) to obtain

$$d_m \leq \theta d_{m-1}. \tag{3.12}$$

Part (b). Firstly, from the inequality (2.3) we see that the set $\{c \geq 0 : r_E(c) \leq \varrho(\mathcal{T})-1\}$ is not empty. For the rest of the proof, let $\rho = \sup\{c \geq 0 : r_E(c) \leq \varrho(\mathcal{T})-1\}$. By using the continuity of r_E at $c = 1$ and the assumption, we obtain $\rho < 1$. We shall assume $d_{m-1} > 0$. (Since otherwise, Lemma 3.1 ensures x_m is the common fixed point of $\mathcal{T}$).

Next, by following the argument in the proof of Theorem 5 of [21] (see also, Theorem 7.2 of [12]) with suitable modifications, we obtain

$$1 + r_E\left(\frac{\|T_{t_r}x_m - x_m\|}{d_{m-1}}\right) \leq \frac{1}{d_{m-1}} \liminf_{n\to\infty} \|T_{t_n}x_{m-1} - T_{t_r}x_m\|$$

for all $r \geq 1$. By taking the limit superior as $r \to \infty$ into the both sides, then by the continuity and the monotonically of r_E, we have from (3.6) that

$$\begin{aligned}
1 + r_E\left(\frac{D_m}{d_{m-1}}\right) &\leq \frac{\varrho(\mathcal{T})}{d_{m-1}} \limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{t_n}x_{m-1} - T_{t_r}x_m\| \\
&\leq \frac{\varrho(\mathcal{T})}{d_{m-1}} \max\left\{d_{m-1}, \frac{1}{2}D_m, \frac{1}{2} \limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{t_n}x_{m-1} - T_{t_r}x_m\|\right\} \\
&= \frac{\varrho(\mathcal{T})}{d_{m-1}} \max\left\{d_{m-1}, \frac{1}{2}D_m\right\} \\
&= \frac{\varrho(\mathcal{T})}{d_{m-1}} d_{m-1} = \varrho(\mathcal{T}).
\end{aligned}$$

By the definition of the constant ρ, it follows

$$D_m \leq \rho d_{m-1}. \tag{3.13}$$

Since $\varrho(\mathcal{T}) < 1 + r_E(1) \leq WCS(E)$, then by combining (3.13) and (3.11) we obtain

$$d_m \leq \rho d_{m-1}. \tag{3.14}$$

Now, let $\eta = \max\{\theta, \rho\}$. It is obvious that $\eta < 1$. Then, from (3.12) and (3.14) we have

$$d_m \leq \eta d_{m-1} \leq \cdots \leq \eta^m d_0 \to 0 \,\text{as}\, m \to \infty. \tag{3.15}$$

For each $m \geq 1$, one can see that

$$\begin{aligned}\|x_{m+1} - x_m\| &\leq \limsup_{n\to\infty}(\|x_{m+1} - T_{t_n}x_m\| + \|T_{t_n}x_m - x_m\|) \\ &\leq d_m + D_m \\ &\leq (\eta + \varrho(\mathcal{T}))d_{m-1}.\end{aligned}$$

So, it follows from (3.15) that $\{x_m\}$ is a Cauchy sequence in C. Let $z = \lim_{m\to\infty} x_m$. For each $n \geq 1$, observe that

$$\begin{aligned}&\|z - T_{t_n}z\| \\ &\leq \|z - x_m\| + \|x_m - T_{t_n}x_m\| + \varrho(T_{t_n})\Big\{\|x_m - z\|, \frac{1}{2}\|x_m - T_{t_n}x_m\|, \\ &\quad \frac{1}{2}\|z - T_{t_n}z\|, \frac{1}{2}\|x_m - T_{t_n}z\|, \frac{1}{2}\|z - T_{t_n}x_m\|\Big\} \\ &\leq \|z - x_m\| + \|x_m - T_{t_n}x_m\| + \varrho(T_{t_n})\Big\{\|x_m - z\|, \frac{1}{2}(\|x_m - z\| + \|z - T_{t_n}z\|) \\ &\quad \frac{1}{2}(\|x_m - z\| + \|x_m - T_{t_n}x_m\|)\Big\} \\ &\leq (1 + 2\varrho(T_{t_n}))\|z - x_m\| \\ &\quad + \left(1 + \frac{\varrho(T_{t_n})}{2}\right)\|x_m - T_{t_n}x_m\| + \frac{\varrho(T_{t_n})}{2}\|z - T_{t_n}z\|. \qquad (3.16)\end{aligned}$$

Then, by taking the limit superior as $n \to \infty$ into (3.16) we get

$$\limsup_{n\to\infty}\|z - T_{t_n}z\| \leq (1+2\varrho(\mathcal{T}))\|z - x_m\| + \left(1 + \frac{\varrho(\mathcal{T})}{2}\right)D_m + \frac{\varrho(\mathcal{T})}{2}\limsup_{n\to\infty}\|z - T_{t_n}z\|.$$

It follows from (3.8) and (3.15) that

$$\begin{aligned}0 &\leq \limsup_{n\to\infty}\|z - T_{t_n}z\| \\ &\leq \left(\frac{2}{2 - \varrho(\mathcal{T})}\right)\left((1 + 2\varrho(\mathcal{T}))\|z - x_m\| + \left(1 + \frac{\varrho(\mathcal{T})}{2}\right)\varrho(\mathcal{T})d_{m-1}\right) \\ &\leq \left(\frac{2}{2 - \varrho(\mathcal{T})}\right)\left((1 + 2\varrho(\mathcal{T}))\|z - x_m\| + \left(1 + \frac{\varrho(\mathcal{T})}{2}\right)\varrho(\mathcal{T})\eta^{m-1}d_0\right) \to 0\end{aligned}$$

as $m \to \infty$. Hence, by Lemma 3.1 we have $T_t z = z$ for all $t \in G$. □

Let E be an ℓ_p-space, $1 < p < \infty$. It is known that the Opial's modulus of E is given by (see, Corollary 3.2 of [28])

$$r_E(c) = (1 + c^p)^{\frac{1}{p}} - 1, \quad c \geq 0.$$

Thus, the following corollary is a partial extension of Corollary 4.4 of [36].

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of an ℓ_p-space, $1 < p < \infty$. Suppose that $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular and* *generalized Lipschitzian semigroup on C such that*

$$\liminf_{t\to\infty} \varrho(T_t) < 2^{\frac{1}{p}}.$$

Then, there exists $z \in C$ such that $T_t z = z$ for all $t \in G$.

Let p and q be real numbers with $p > 1$ and $q \geq 1$. Recall from [6] that Bynum's space $\ell_{p,q}$ is the space ℓ_p endowed with the norm $\|x\|_{p,q} = ((\|x^+\|_p)^p + (\|x^-\|_q)^q)^{\frac{1}{q}}$, where x^+ and x^- denote the positive and the negative part of x, respectively, and $\|.\|_p$ denotes the usual norm of ℓ_p. If $q > 1$, then (see, Theorem 2.2 of [38])

$$r_{\ell_{p,q}}(c) = \min\left\{(1+c^p)^{\frac{1}{p}} - 1, (1+c^q)^{\frac{1}{q}} - 1\right\}, \quad c \geq 0,$$

and $l_{p,q}$ is reflexive (see [6]). Thus, the following corollary is a partial extension of Corollary 4.5 of [36].

Corollary 3.4. *Let C be a nonempty bounded closed convex subset of a Bynum's space $\ell_{p,q}$, $1 < p, q < \infty$. Suppose that $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular and generalized Lipschitzian semigroup on C such that*

$$\liminf_{t\to\infty} \varrho(T_t) < \min\left\{2^{\frac{1}{p}}, 2^{\frac{1}{q}}\right\}.$$

Then, there exists $z \in C$ such that $T_t z = z$ for all $t \in G$.

4. The uniformly convex case

In this section, we examine the existence theorems for the case of uniformly convex Banach spaces. Recall that the modulus of convexity δ_E of a Banach space E is the function $\delta_E : [0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x-y\| \geq \varepsilon\right\}.$$

The characteristic ε_0 of convexity of E is defined by

$$\varepsilon_0 = \varepsilon_0(E) = \sup\{\varepsilon \in [0,2] : \delta_E(\varepsilon) = 0\}.$$

It is well known (see [17]) that the modulus of convexity δ_E of E has the following properties.

(a) δ_E is increasing on $[0,2]$, and moreover strictly increasing on $[\varepsilon_0, 2]$,
(b) δ_E is continuous on $[0,2)$ (but not necessary at $\varepsilon = 2$),
(c) $\delta_E(0) = 0$ and $\lim_{\varepsilon\to 2^-} \delta_E(\varepsilon) = 1 - \frac{\varepsilon_0}{2}$,
(d) $[\|a-x\| \leq r, \|a-y\| \leq r, \text{ and } \|x-y\| \geq \varepsilon] \Rightarrow \left\|a - \frac{x+y}{2}\right\| \leq r\left(1 - \delta_E(\frac{\varepsilon}{r})\right)$.

If E is a reflexive Banach space with the modulus of convexity δ_E, then (see, Theorem 3 of [7])

$$1 \leq \frac{1}{1-\delta_E(1)} \leq N(E).$$

A Banach space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon > 0$, which equivalently $\varepsilon_0 = 0$. We know that any uniformly convex Banach space E has $N(E) > 1$. A Hilbert space H is uniformly convex. This fact is a direct consequence of parallelogram identity. Also, an ℓ_p-space and L_p-space, $1 < p < \infty$, are uniformly convex. It is also known that $\delta_E(2) = 1$ for any uniformly convex Banach space E. This becomes an evident that δ_E is strictly increasing and continuous on $[0, 2]$.

For the first main result in this section, we examine the structure of common fixed point sets of asymptotically regular semigroups having the asymptotic nonexpansiveness in uniformly convex Banach spaces.

Theorem 4.1. *Let C be a nonempty weakly compact convex subset of a uniformly convex Banach space E. Suppose that $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular and generalized Lipschitzian semigroup on C with $k_t \geq 1$ for all $t \in G$ and $\lim_{t\to\infty} k_t = 1$, where k_t is the constant in* (2.1). *Then, $\mathcal{F}(\mathcal{T})$, i.e., the set of all common fixed points of $\mathcal{T}$ is nonempty, closed, and convex.*

Proof. From the assumption, we have

$$\liminf_{t\to\infty} \varrho(T_t) \leq \lim_{t\to\infty} k_t = 1.$$

Since $WCS(E) > 1$, then by Theorem 3.2 we obtain $\mathcal{F}(\mathcal{T})$ is nonempty.

We shall show that $\mathcal{F}(\mathcal{T})$ is closed. Let $\{x_t\}_{t\in G}$ be an arbitrary net in $\mathcal{F}(\mathcal{T})$ such that $x_t \to x$. Then, we can find an increasing sequence $\{s_n\}$ in G such that $\lim_{n\to\infty} s_n = \infty$ and $x_{s_n} \to x$. For any $n \geq 1$,

$$\begin{aligned}
\|T_{s_n}x - x_{s_n}\| &= \|T_{s_n}x - T_{s_n}x_{s_n}\| \\
&\leq k_{s_n} \max\left\{\|x - x_{s_n}\|, \frac{1}{2}\|x - T_{s_n}x\|, \frac{1}{2}\|x_{s_n} - T_{s_n}x_{s_n}\| \right. \\
&\quad \left. \frac{1}{2}\|x - T_{s_n}x_{s_n}\|, \frac{1}{2}\|x_{s_n} - T_{s_n}x\|\right\} \\
&= k_{s_n} \max\left\{\|x - x_{s_n}\|, \frac{1}{2}\|x - T_{s_n}x\|, \frac{1}{2}\|x - x_{s_n}\|, \frac{1}{2}\|x_{s_n} - T_{s_n}x\|\right\} \\
&\leq k_{s_n} \max\left\{\|x - x_{s_n}\|, \frac{1}{2}(\|x - x_{s_n}\| + \|x_{s_n} - T_{s_n}x\|)\right\} \\
&\leq \left(k_{s_n} + \frac{1}{2}\right)\|x - x_{s_n}\| + \frac{k_{s_n}}{2}\|T_{s_n}x - x_{s_n}\|. \qquad (4.1)
\end{aligned}$$

Taking the limit superior as $n \to \infty$ into (4.1) we get

$$\limsup_{n\to\infty} \|T_{s_n}x - x_{s_n}\| \leq \frac{1}{2} \limsup_{n\to\infty} \|T_{s_n}x - x_{s_n}\|.$$

Hence,

$$\limsup_{n\to\infty} \|T_{s_n}x - x_{s_n}\| = 0. \qquad (4.2)$$

Then by using the triangle inequality, we see from (4.2) and Lemma 3.1 that $x \in \mathcal{F}(\mathcal{T})$. It concludes that $\mathcal{F}(\mathcal{T})$ is closed.

Next, we show that $\mathcal{F}(\mathcal{T})$ is convex. Let $x, y \in \mathcal{F}(\mathcal{T})$ with $x \neq y$ and $z = \frac{x+y}{2} \in C$. Then, for each $t \in G$ we have

$$\begin{aligned}\|T_t z - x\| &= \|T_t z - T_t x\| \\ &\leq k_t \max\left\{\|z-x\|, \frac{1}{2}\|z - T_t z\|, \frac{1}{2}\|x - T_t x\|, \frac{1}{2}\|z - T_t x\|, \frac{1}{2}\|x - T_t z\|\right\} \\ &= \frac{k_t}{2}\max\left\{\|x-y\|, \|z - T_t z\|, \|x - T_t z\|\right\}. \qquad (4.3)\end{aligned}$$

It follows that

$$\|T_t z - x\| \leq \frac{k_t}{2} N_t, \qquad (4.4)$$

where

$$N_t = \max\{\|x-y\|, \|z - T_t z\|, \|x - T_t z\|, \|y - T_t z\|\}.$$

Analogously, we also have

$$\|T_t z - y\| \leq \frac{k_t}{2} N_t. \qquad (4.5)$$

Now, choose an increasing sequence $\{t_n\}$ in G such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{t\to\infty} k_t = \lim_{n\to\infty} k_{t_n} = 1,$$

where $k_{t_n} < 2$ for all $n \geq 1$.

We claim $\|x-y\| > \limsup_{n\to\infty} \|z - T_{t_n} z\|$. Indeed, otherwise,

$$\limsup_{n\to\infty} \|z - T_{t_n} z\| \neq 0.$$

Moreover, without loss of generality, we may assume that

$$\limsup_{n\to\infty} \|z - T_{t_n} z\| = \lim_{n\to\infty} \|z - T_{t_n} z\|.$$

From (4.3), we get for each $n \geq 1$,

$$\|T_{t_n} z - x\| \leq \frac{k_{t_n}}{2}\left(\max\{\|x-y\|, \|z - T_{t_n} z\|\} + \|x - T_{t_n} z\|\right).$$

Therefore,

$$\|T_{t_n} z - x\| \leq \frac{k_{t_n}}{2 - k_{t_n}} M_{t_n},$$

where

$$M_{t_n} = \max\{\|x-y\|, \|z - T_{t_n} z\|\}.$$

A similar way yields

$$\|T_{t_n} z - y\| \leq \frac{k_{t_n}}{2 - k_{t_n}} M_{t_n}.$$

Thus, by the property (d) we get

$$\|T_{t_n} z - z\| \leq \frac{k_{t_n}}{2 - k_{t_n}} M_{t_n}\left(1 - \delta_E\left(\frac{(2 - k_{t_n})\|x-y\|}{k_{t_n} M_{t_n}}\right)\right).$$

Taking the limit as $n \to \infty$ into both sides, we have from the continuity of δ_E that

$$\begin{aligned}
\lim_{n\to\infty} \|z - T_{t_n} z\| &\le \max\left\{\|x-y\|, \lim_{n\to\infty} \|z - T_{t_n} z\|\right\} \cdot \\
&\quad \left(1 - \delta_E\left(\frac{\|x-y\|}{\max\left\{\|x-y\|, \lim_{n\to\infty} \|z - T_{t_n} z\|\right\}}\right)\right) \\
&= \lim_{n\to\infty} \|z - T_{t_n} z\| \left(1 - \delta_E\left(\frac{\|x-y\|}{\lim_{n\to\infty} \|z - T_{t_n} z\|}\right)\right) \\
&< \lim_{n\to\infty} \|z - T_{t_n} z\|,
\end{aligned}$$

a contradiction.
Now, from (4.3) we obtain

$$\|T_{t_n} z - x\| \le \frac{k_{t_n}}{2} \left(\max\{\|x-y\|, \|z - T_{t_n} z\|\} + \|x - T_{t_n} z\|\right).$$

Taking the limit superior as $n \to \infty$ into both sides, we have

$$\begin{aligned}
\limsup_{n\to\infty} \|T_{t_n} z - x\| &\le \frac{1}{2}\left(\max\left\{\|x-y\|, \limsup_{n\to\infty} \|z - T_{t_n} z\|\right\} + \limsup_{n\to\infty} \|x - T_{t_n} z\|\right) \\
&= \frac{1}{2}\left(\|x-y\| + \limsup_{n\to\infty} \|x - T_{t_n} z\|\right).
\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \|T_{t_n} z - x\| \le \|x-y\|. \tag{4.6}$$

By a similar argument, we obtain

$$\limsup_{n\to\infty} \|T_{t_n} z - y\| \le \|x-y\|. \tag{4.7}$$

On the other hand, from (4.4) and (4.5), using the property (d) we get

$$\|T_{t_n} z - z\| \le \frac{k_{t_n}}{2} N_{t_n} \left(1 - \delta_E\left(\frac{2\|x-y\|}{k_{t_n} N_{t_n}}\right)\right).$$

Therefore, by taking the limit superior as $n \to \infty$ into the both sides, from (4.6), (4.7), and the continuity and the monotonically of δ_E we have

$$\begin{aligned}
\limsup_{n\to\infty} \|T_{t_n} z - z\| &\le \limsup_{n\to\infty} \left(\frac{k_{t_n}}{2} N_{t_n}\right) \left(1 - \delta_E\left(\liminf_{n\to\infty} \frac{2\|x-y\|}{k_{t_n} N_{t_n}}\right)\right) \\
&\le \frac{1}{2}\|x-y\| \left(1 - \delta_E\left(\frac{2\|x-y\|}{\limsup_{n\to\infty} (k_{t_n} N_{t_n})}\right)\right) \\
&\le \frac{1}{2}\|x-y\|(1 - \delta_E(2)).
\end{aligned}$$

It follows that $\limsup_{n\to\infty} \|T_{t_n} z - z\| = 0$. Hence, by Lemma 3.1 we have $z \in \mathcal{F}(\mathcal{T})$. It concludes $\mathcal{F}(\mathcal{T})$ is convex. □

Remark 4.2. We note that, if the generalized Lipschitzian semigroup $\mathcal{T} = \{T_t : t \in G\}$ in Theorem 4.1 is restricted to an asymptotically nonexpansive semigroup, that is, $\mathcal{T}$ satisfies the condition: for any $t \in G$, there exists $k_t \geq 1$ such that

$$\|T_t x - T_t y\| \leq k_t \|x - y\|$$

for all $x, y \in C$ and $\lim_{t\to\infty} k_t = 1$, then the assumption of the asymptotic regularity condition of $\mathcal{T}$ in Theorem 4.1 can be removed (see [16]).

Next, we shall establish the common fixed point theorem for asymptotically regular semigroups in the framework of uniformly convex Banach spaces.

The following lemma can be found in page 141 of [1].

Lemma 4.3. *Let E be a uniformly convex Banach space satisfying the Opial condition and C be a nonempty closed convex subset of E. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$, then $\{z\} = \mathcal{A}(C, \{x_n\})$.*

The following lemma is useful in proving our next result.

Lemma 4.4. *Let E be a uniformly convex Banach space and $a \in (0, 2]$. Then, the equation*

$$\beta_a \left(1 - \delta_E \left(\frac{a}{{\beta_a}^2}\right)\right) = 1$$

has a unique solution $\beta_a \in (1, 2)$.

Proof. Let us define the real-valued function h_a on $[1, 2]$ by

$$h_a(t) = \frac{1}{t} + \delta_E \left(\frac{a}{t^2}\right).$$

It is easy to see that h_a is strictly decreasing and continuous. Moreover, we see by Theorem 2.3.7.(a) in [1] that

$$h_a(1) = 1 + \delta_E(a) > 1 \quad \text{and} \quad h(2) = \frac{1}{2} + \delta_E \left(\frac{a}{4}\right) \leq \frac{1}{2} + \frac{a}{8} \leq \frac{3}{4} < 1.$$

It follows that there exists a unique $t_a \in (1, 2)$ such that $h_a(t_a) = 1$, that is,

$$t_a \left(1 - \delta_E \left(\frac{a}{{t_a}^2}\right)\right) = 1. \qquad \square$$

The result below extends Theorem 1 of [18] for a wider class of semigroups of self-mappings, by replacing the exact Lipschitz constant with generalized Lipschitz constant.

Theorem 4.5. *Let E be a uniformly convex Banach space satisfying the Opial condition and C be a nonempty weakly compact convex subset of E. Suppose that $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular and generalized Lipschitzian semigroup on C such that $\liminf_{t\to\infty} \varrho(T_t) < \beta$, where β is the unique solution of the equation*

$$\beta \left(1 - \delta_E \left(\frac{WCS(E)}{\beta^2}\right)\right) = 1. \tag{4.8}$$

(Note that, in a Hilbert space, $\beta = (3^{\frac{1}{2}} - 1)^{-\frac{1}{2}}$ and in an ℓ_p-space, $2 \leq p < \infty$, $\beta = \beta_p = \frac{1}{2}\left(2^{p-1} + (1 + 2^{3-p})^{\frac{1}{2}}\right)^{\frac{1}{p}}$ (see, Remark 1 of [18])). Then, there exists $z \in C$ such that $T_t z = z$ for all $t \in G$.

Proof. Choose an increasing sequence $\{s_n\}$ in G such that $\lim_{n\to\infty} s_n = \infty$ and

$$\lim_{n\to\infty} \varrho(T_{s_n}) = \liminf_{t\to\infty} \varrho(T_t) = \varrho(\mathcal{T}) < \beta.$$

Note that, by Lemma 4.4 we have the constant $\beta \in (1, 2)$ as the unique solution of the equation (4.8). If $\varrho(\mathcal{T}) < 1$, then by Theorem 3.2 we obtain the existence of common fixed points of $\mathcal{T}$. For the case $1 \leq \varrho(\mathcal{T}) < \beta$, without loss of generality, we may assume that $1 \leq \varrho(T_{s_n}) < \infty$ for all $n \geq 1$. By using a similiar argument as in the proof of Theorem 3.2, the weak sequential compactness of C allows us to construct a sequence $\{x_m\}_{m\in\mathbb{N}}$ in C in the following ways:

$$x = x_0 \in C \,\text{arbitrary}, \quad x_{m+1} = w - \lim_{n\to\infty} T_{t_n} x_m \,\text{for all}\, m \geq 0,$$

for some subsequence $\{t_n\}$ of $\{s_n\}$ such that

$$\lim_{n\to\infty} \|T_{t_n} x_m - x_{m+1}\| \,\text{exists for all}\, m \geq 0.$$

For any $m \geq 0$, we write

$$d_m = \limsup_{n\to\infty} \|T_{t_n} x_m - x_{m+1}\| \quad \text{and} \quad D_m = \limsup_{n\to\infty} \|T_{t_n} x_m - x_m\|.$$

Unlike the proof of Theorem 3.2, in this proof, we shall show that there exists $\eta < 1$ such that $D_m \leq \eta D_{m-1}$ for all $m \geq 1$.

Let $m \geq 1$ be fixed. Note that, by using a similar argument as in the proof of Theorem 3.2, for any $r \geq 1$ we obtain

$$\limsup_{n\to\infty} \|T_{t_n} x_{m-1} - T_{t_r} x_m\| \leq \varrho(T_{t_r}) M_{t_r}(x_{m-1}), \tag{4.9}$$

where

$$M_{t_r}(x_{m-1}) = \max\left\{d_{m-1}, \frac{1}{2}\|x_m - T_{t_r} x_m\|, \frac{1}{2}\limsup_{n\to\infty} \|T_{t_n} x_{m-1} - T_{t_r} x_m\|\right\}.$$

Then, by taking the limit superior as $r \to \infty$ into (4.9) we have

$$\begin{aligned} &\limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{t_r} x_{m-1} - T_{t_n} x_m\| \\ &\quad \leq \varrho(\mathcal{T}) \max\left\{d_{m-1}, \frac{1}{2} D_m, \frac{1}{2} \limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{t_r} x_{m-1} - T_{t_n} x_m\|\right\}. \end{aligned}$$

Therefore, as in the proof of Theorem 3.2, we may assume

$$\begin{aligned} &\max\left\{d_{m-1}, \frac{1}{2} D_m, \frac{1}{2} \limsup_{r\to\infty} \limsup_{n\to\infty} \|T_{t_r} x_{m-1} - T_{t_n} x_m\|\right\} \\ &\quad = \max\left\{d_{m-1}, \frac{1}{2} D_m\right\}. \end{aligned} \tag{4.10}$$

Moreover, by using the weak semi-continuity of $\|.\|$, we may also assume

$$\max\left\{d_{m-1}, \frac{1}{2}D_m\right\} = d_{m-1}. \tag{4.11}$$

Next, for the rest of the proof, we may assume $d_{m-1} > 0$. Since otherwise, Lemma 3.1 ensures x_m is the common fixed point of $\mathcal{T}$. Then, by using a similar argument as in the proof of Theorem 3.2 with a suitable modification, we obtain

$$d_{m-1} \leq \frac{\varrho(\mathcal{T})}{WCS(E)} D_{m-1}. \tag{4.12}$$

By using the convexity of C and Lemma 4.3, for each $r \geq 1$ we also have

$$d_{m-1} \leq \limsup_{n\to\infty} \left\| T_{t_n}x_{m-1} - \frac{x_m + T_{t_r}x_m}{2} \right\|. \tag{4.13}$$

On the other hand, by the definition of $M_{t_r}(x_{m-1})$ and (4.9),

$$\limsup_{n\to\infty} \|T_{t_n}x_{m-1} - x_m\| = d_{m-1} \leq M_{t_r}(x_{m-1}) \leq \varrho(T_{t_r})M_{t_r}(x_{m-1})$$

and

$$\limsup_{n\to\infty} \|T_{t_n}x_{m-1} - T_{t_r}x_m\| \leq \varrho(T_{t_r})M_{t_r}(x_{m-1}).$$

Therefore, for each $\varepsilon > 0$ we can choose $n_0 \geq 1$ such that

$$\|T_{t_n}x_{m-1} - x_m\| \leq \varrho(T_{t_r})M_{t_r}(x_{m-1}) + \varepsilon$$

and

$$\|T_{t_n}x_{m-1} - T_{t_r}x_m\| \leq \varrho(T_{t_r})M_{t_r}(x_{m-1}) + \varepsilon$$

for all $n \geq n_0$. Thus, from the property (d) we obtain

$$\left\| T_{t_n}x_{m-1} - \frac{x_m + T_{t_r}x_m}{2} \right\| \leq (\varrho(T_{t_r})M_{t_r}(x_{m-1}) + \varepsilon) \cdot \left(1 - \delta_E\left(\frac{\|T_{t_r}x_m - x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1}) + \varepsilon}\right)\right)$$

for all $n \geq n_0$. By using the continuity of δ_E, letting $\varepsilon \to 0$ into the both sides, we get

$$\limsup_{n\to\infty} \left\| T_{t_n}x_{m-1} - \frac{x_m + T_{t_r}x_m}{2} \right\| \leq \varrho(T_{t_r})M_{t_r}(x_{m-1}) \left(1 - \delta_E\left(\frac{\|T_{t_r}x_m - x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}\right)\right). \tag{4.14}$$

Therefore, by the monotonically and the continuity of δ_E, from (4.13) and (4.14) we have

$$\begin{aligned}
&d_{m-1}\\
&\quad\le \liminf_{r\to\infty}\left(\varrho(T_{t_r})M_{t_r}(x_{m-1})\left(1-\delta_E\left(\frac{\|T_{t_r}x_m-x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}\right)\right)\right)\\
&\quad\le \limsup_{r\to\infty}\left(\varrho(T_{t_r})M_{t_r}(x_{m-1})\right)\cdot\liminf_{r\to\infty}\left(1-\delta_E\left(\frac{\|T_{t_r}x_m-x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}\right)\right)\\
&\quad= \limsup_{r\to\infty}\left(\varrho(T_{t_r})M_{t_r}(x_{m-1})\right)\cdot\left(1-\delta_E\left(\limsup_{r\to\infty}\frac{\|T_{t_r}x_m-x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}\right)\right) \qquad (4.15)
\end{aligned}$$

Here, one can see from the definition of $M_{t_r}(x_{m-1})$, (4.10), and (4.11) that

$$\begin{aligned}
&\limsup_{r\to\infty}\|T_{t_r}x_m-x_m\|\\
&\quad\le \limsup_{r\to\infty}\frac{\|T_{t_r}x_m-x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}\cdot\limsup_{r\to\infty}\left(\varrho(T_{t_r})M_{t_r}(x_{m-1})\right)\\
&\quad\le \limsup_{r\to\infty}\frac{\|T_{t_r}x_m-x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}\\
&\qquad\cdot\varrho(\mathcal{T})\max\left\{d_{m-1},\frac{1}{2}D_m,\frac{1}{2}\limsup_{r\to\infty}\limsup_{n\to\infty}\|T_{t_n}x_{m-1}-T_{t_r}x_m\|\right\}\\
&\quad= \limsup_{r\to\infty}\frac{\|T_{t_r}x_m-x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}\cdot\varrho(\mathcal{T})\max\left\{d_{m-1},\frac{1}{2}D_m\right\}\\
&\quad= \limsup_{r\to\infty}\frac{\|T_{t_r}x_m-x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}\cdot\varrho(\mathcal{T})d_{m-1}.
\end{aligned}$$

It follows that

$$\frac{D_m}{\varrho(\mathcal{T})d_{m-1}}\le\limsup_{r\to\infty}\frac{\|T_{t_r}x_m-x_m\|}{\varrho(T_{t_r})M_{t_r}(x_{m-1})}. \qquad (4.16)$$

Thus by the monotonically of δ_E, we see from (4.15) and (4.16) that

$$d_{m-1}\le\varrho(\mathcal{T})d_{m-1}\cdot\left(1-\delta_E\left(\frac{D_m}{\varrho(\mathcal{T})d_{m-1}}\right)\right).$$

Hence,

$$\delta_E\left(\frac{D_m}{\varrho(\mathcal{T})d_{m-1}}\right)\le 1-\frac{1}{\varrho(\mathcal{T})}.$$

Therefore, using the monotonically and the continuity of δ_E, and (4.12) we have

$$D_m\le\varrho(\mathcal{T})d_{m-1}{\delta_E}^{-1}\left(1-\frac{1}{\varrho(\mathcal{T})}\right)\le\frac{[\varrho(\mathcal{T})]^2}{WCS(E)}{\delta_E}^{-1}\left(1-\frac{1}{\varrho(\mathcal{T})}\right)D_{m-1}.$$

Let

$$\eta=\frac{[\varrho(\mathcal{T})]^2}{WCS(E)}{\delta_E}^{-1}\left(1-\frac{1}{\varrho(\mathcal{T})}\right).$$

It is easy to see from the assumption that $\eta<1$. Consequently,

$$D_m\le\eta D_{m-1}\le\cdots\le\eta^m D_0\to 0\,\text{as}\,m\to\infty. \qquad (4.17)$$

For each $m \geq 1$, one can see that

$$\begin{aligned}\|x_{m+1} - x_m\| &\leq \limsup_{n\to\infty}(\|x_{m+1} - T_{t_n}x_m\| + \|T_{t_n}x_m - x_m\|)\\ &\leq d_m + D_m \leq 2D_m.\end{aligned}$$

So, it follows from (4.17) that $\{x_m\}$ is a Cauchy sequence in C. Let $z = \lim_{m\to\infty} x_m$. Then, by using a similar argument as in the proof of Theorem 3.2 we obtain

$$\limsup_{n\to\infty}\|z-T_{t_n}z\| \leq (1+2\varrho(\mathcal{T}))\|z-x_m\|+\left(1+\frac{\varrho(\mathcal{T})}{2}\right)D_m+\frac{\varrho(\mathcal{T})}{2}\limsup_{n\to\infty}\|z-T_{t_n}z\|.$$

Thus, from (4.17) we get

$$\begin{aligned}0 &\leq \limsup_{n\to\infty}\|z - T_{t_n}z\|\\ &\leq \left(\frac{2}{2-\varrho(\mathcal{T})}\right)\left((1+2\varrho(\mathcal{T}))\|z-x_m\| + \left(1+\frac{\varrho(\mathcal{T})}{2}\right)\eta^m D_0\right) \to 0 \,\text{as}\, m\to\infty.\end{aligned}$$

Hence, by Lemma 3.1 we have $T_t z = z$ for all $t \in G$. □

Remark 4.6. In [35], Tan and Xu proved that $\gamma < N(E)$ for any uniformly convex Banach space E, where $\gamma > 1$ is the unique solution of the equation

$$\gamma\left(1-\delta_E\left(\frac{1}{\gamma}\right)\right) = 1.$$

Based on this fact, we see that $\gamma < \beta$, where β is the unique solution of the equation (4.8). Indeed, otherwise, that is $\gamma \geq \beta$, since $\gamma < N(E) \leq WCS(E)$ we get $\beta < WCS(E)$. It follows that

$$\frac{1}{\gamma} \leq \frac{1}{\beta} < \frac{WCS(E)}{\beta^2}.$$

Therefore,

$$1 = \beta\left(1-\delta_E\left(\frac{WCS(E)}{\beta^2}\right)\right) < \beta\left(1-\delta_E\left(\frac{1}{\gamma}\right)\right) < \gamma\left(1-\delta_E\left(\frac{1}{\gamma}\right)\right) = 1,$$

a contradiction. Thus, we can conclude that Theorem 4.5 extends and improves partially Theorem 5 of [20] in spaces satisfying the Opial condition, by replacing the assumptions of asymptotically regular mapping and the exact Lipschitz constant with asymptotically regular semigroup and generalized Lipschitz constant, and the assumption of the constant γ with β, respectively.

Let us investigate the existence theorem for asymptotically regular semigroups in p-uniformly convex Banach spaces. Let $p > 1$ be a real number. Recall that a Banach space E is said to be p-uniformly convex if $\inf\{\delta_E(\varepsilon) : 0 < \varepsilon \leq 2\} > 0$. We note that a Hilbert space H is 2-uniformly convex (indeed, $\delta_H(\varepsilon) = 1 - (1-(\frac{\varepsilon}{2})^2)^{\frac{1}{2}} \geq \frac{1}{8}\varepsilon^2$). Also, an L_p-space and ℓ_p-space, $1 < p < \infty$, are $\max\{2,p\}$-uniformly convex, because (see [23])

$$\delta_{L_p}(\varepsilon) = \delta_{\ell_p}(\varepsilon) > \begin{cases}\frac{p-1}{8}\varepsilon^2 & \text{if } 1 < p < 2,\\ \frac{1}{p2^p} & \text{if } p \geq 2.\end{cases}$$

In [37], Xu proved that if a Banach space E is p-uniformly convex, then there exists $c_p > 0$ such that

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p W_p(\lambda)\|x-y\|^p \tag{4.18}$$

for all $x, y \in E$, $0 \leq \lambda \leq 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Motivated by the proof of Theorem 1 of [37], we improve the fact above to be as follows.

Lemma 4.7. *Let E be a p-uniformly convex Banach space for some $p > 1$. Then, c_p as the best possible constant appeared in the inequality* (4.18) *is an element of* $(0, 1]$.

Proof. It is enough to prove there is no $c > 1$ such that the inequality (4.18) holds. On the contrary, by the definition of δ_E and the inequality (4.18), it is easy to see

$$\delta_E(\varepsilon) \geq 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p c\right)^{\frac{1}{p}}$$

for any $\varepsilon \in (0, 2]$. Therefore,

$$1 - \left(\frac{\varepsilon}{2}\right)^p c \geq (1 - \delta_E(\varepsilon))^p \geq 0.$$

It follows that

$$0 > 1 - c = \inf\left\{1 - \left(\frac{\varepsilon}{2}\right)^p c : 0 < \varepsilon \leq 2\right\} \geq 0,$$

a contradiction. □

In the framework of p-uniformly convex Banach spaces, the following result extends partially Theorem 4.6 of [36] for a wider class of semigroups of self-mappings, by replacing the exact Lipschitz constant with generalized Lipschitz constant.

Theorem 4.8. *Let E be a p-uniformly convex Banach space satisfying the Opial condition for some $p > 1$, and let C be a nonempty weakly compact convex subset of E. Suppose that $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular and generalized Lipschitzian semigroup on C such that*

$$\liminf_{t\to\infty} \varrho(T_t) < \max\left\{(1+c_p)^{\frac{1}{p}}, \left(\frac{1}{2}\left(1 + (1 + 4c_p[WCS(E)]^p)^{\frac{1}{2}}\right)\right)^{\frac{1}{p}}\right\}.$$

Then, there exists $z \in C$ such that $T_t z = z$ for all $t \in G$.

Proof. By Lemma 4.7 we have $\liminf_{t\to\infty} \varrho(T_t) < 2$. Then by combining the method of the proof of Theorem 3.2 above and Theorem 4.6 of [36], the result follows. □

For an ℓ_p-space, $1 < p < \infty$, by Corollary 3.3 and Theorem 4.8 we have the following corollary which, in turn, as a partial extension of Corollary 9 of [21].

Corollary 4.9. *Let C be a nonempty bounded closed convex subset of an ℓ_p-space, $1 < p < \infty$. Suppose that $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular and generalized Lipschitzian semigroup on C such that*

$$\liminf_{t\to\infty} \varrho(T_t) < \max\left\{2^{\frac{1}{p}}, (1+c_p)^{\frac{1}{p}}, \left(\frac{1}{2}\left(1 + (1 + 8c_p)^{\frac{1}{2}}\right)\right)^{\frac{1}{p}}\right\}.$$

Then, there exists $z \in C$ such that $T_t z = z$ for all $t \in G$.

Since a Hilbert space H is 2-uniformly convex and the identity

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

holds for all $x, y \in H$ and $0 \leq \lambda \leq 1$, then we have the following corollary which, in turn, as a partial extension of Corollary 4.3 of [36].

Corollary 4.10. *Let C be a nonempty bounded closed convex subset of a Hilbert space H. Suppose that $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular and generalized Lipschitzian semigroup on C such that $\liminf_{t\to\infty} \varrho(T_t) < \sqrt{2}$. Then, there exists $z \in C$ such that $T_t z = z$ for all $t \in G$.*

Remark 4.11. Example 2.1 shows that Theorem 3.2, Theorem 4.5, and Theorem 4.8 are more general than Theorem 4.2 of [36], Theorem 1 of [18], and Theorem 4.6 of [36], respectively.

Remark 4.12. Most of our results above can only extend partially some common fixed points theorems for the semigroups in the corresponding papers. We still cannot see clearly whether the set of common fixed points of the semigroup studied in this paper can enjoy the retract condition as well as in the papers of Górnicki [20, 21, 22] and Wiśnicki [36], or not.

Acknowledgement. We would like to thank the referees for his/her comments on the manuscript.

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Received: February 21, 2016; Accepted: June 2, 2016.