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STRONG CONVERGENCE OF PROJECTED REFLECTED GRADIENT METHODS FOR VARIATIONAL INEQUALITIES

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Abstract. The purpose of this work is to revisit the numerical approach to classical variational inequality problems, with monotone and Lipschitz continuous mapping, by means of a regularized dynamical method. A main feature of the method is that it formally requires only one projection step onto the feasible set and only one evaluation of the involved mapping (at each iteration), combined with some viscosity-like regularization process. A strong convergence theorem is established in a general setting that allows the use of varying step-sizes without any requirement of additional projections. We also point out that the considered method in absence of regularization does not generate a Fejer-monotone monotone sequence. So a new analysis is developed for this purpose.
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1. Introduction

Throughout this paper H is a real Hilbert space endowed with inner product and induced norm denoted by $\langle \cdot,\cdot \rangle$ and $| \cdot |$, respectively, C is a closed convex subset of H. Consider the following classical variational inequality problem (first introduced by Stampacchia in [24]):

$$\text{find } u \in C \text{ such that } \langle Au, v-u\rangle \geq 0 \quad \forall v \in C, \tag{1.1}$$

where $A : H \to H$ is assumed to be monotone and L-Lipschitz continuous over H (for some positive value L), namely

$$\langle Ax - Ay, x-y\rangle \geq 0 \quad \forall (x,y) \in H^2 \quad \text{(monotonicity)}, \tag{1.2a}$$

$$|Ax - Ay| \leq L|x-y| \quad \forall (x,y) \in H^2 \quad (L\text{-Lipschitz continuity}). \tag{1.2b}$$

As a standing assumption we assume that the solution set of (1.1), denoted by S, is nonempty.

It is well-known that (1.1) encompasses many significant real-world problems arising in mechanics, economics and so on (see, e.g., [1, 2, 4, 21] and the references therein). This problem has recently attracted considerable attention and numerous related algorithmic solutions have been developed (through projection techniques) under the classical assumption (1.2); see, e.g., [15, 31].

Definition 1.1. The metric projection $P_C : H \to C$ is the operator defined for all $x \in H$ by $P_C x := \text{argmin}_{z\in C}|z - x|$.

Projection-type methods are very useful and natural tools for solving (1.1) since this latter can be equivalently rewritten as the following fixed point problem: find $u \in C$ such that $u = P_C(u - \lambda Au)$, where λ is any positive real number.

Let us recall that the oldest strategy for solving (1.1) under the classical assumption (1.2) is the so-called extra-gradient method (introduced by Korpelevich [13]) which consists of the following two-step projection procedure:

$$\overline{x}_n = P_C(x_n - \lambda_n Ax_n), \quad x_{n+1} = P_C(x_n - \lambda_n A\overline{x}_n), \tag{1.3}$$

where (λ_n) is a positive sequence that ensures the weak convergence of the method for the classical step-size requirement

$$(\lambda_n) \subset [\bar{\mu}, \bar{\nu}] \text{ for some values } \bar{\mu}, \bar{\nu} \in \left(0, \tfrac{1}{L}\right). \tag{1.4}$$

Afterwards the extra-gradient method were refined through several extensions involving Armijo-type rules (see, e.g., Khobotov [12] Marcotte [20], Sun [25], Iusem [9], Tseng [29]) and outer approximation techniques (see, e.g., Solodov and Svaiter [23]). These methods (by Iusem and Svaiter [11] and Solodov and Svaiter [23]) were able to drop the Lipschitz continuity condition together with a more effective Armijo-type line search even for a pseudo-monotone mapping A (also see Iusem and Pérez [10] for extension to nonsmooth cases of A). However the proposed methods always involve a projection onto C (at least) at each iteration together with an addition projection onto either C or onto its intersection with some hyperplane. These methods involves several evaluations of the operator A at each iteration (including the computation the trial values for the predictor step-sizes).

Then attempting to enhance the complexity of theses numerical approaches, by reducing the number of evaluations of the operators P_C and A, can be interesting in situations where the projection on C is hard to compute, but also relative to huge-scale problems (from control optimal) in which computing a value of A is expensive.

Note that it has been already investigated modified extra-gradient methods with only one projection onto C per iteration. As a special case of a general algorithm that can be applied to our problem we mention the following one-step projection method proposed by Tseng [30]:

$$y_n = P_C(x_n - \lambda_n Ax_n), \quad x_{n+1} = y_n + \lambda_n(Ax_n - Ay_n). \tag{1.5}$$

This method formally involve one projection step but its convergence was established by using an Armijo-Goldstein-type stepsize rule for which the trial values of λ_n require some projections onto C. Other examples are given by modified extra-gradient methods with only one projection onto C per iteration together with a cheaper projection step onto some hyperplane (see, e.g., Censor, Gibali and Reich [5], Malitsky and Semenov [19]). The algorithm by Malitsky and Semenov [19] was proposed so as to reduce the complexity of the existing modified extragradient-type methods. It can be noticed on every iteration of the method discussed in [19] that not only one projection on C is performed, but also only one value of A is computed. However, the convergence of these methods were stated under a similar condition to (1.4).

In this paper we focus our attention on a new numerical approach to problem (1.1) based on the following projected reflected gradient method recently proposed by Malitsky [18]:

$$y_n = 2x_n - x_{n-1}, \quad x_{n+1} = P_C(x_n - \lambda_n A y_n), \tag{1.6}$$

with positive step-sizes (λ_n). This latter process formally involves only one projection step and one evaluation of A per iteration, while its convergence was mainly established in the special case of constant step-sizes $\lambda_n = \lambda$ with $\lambda \in \left(0, \frac{\sqrt{2}-1}{L}\right)$.

Our purpose here is to revisit the method in [18] through a more general framework (with the same interesting features) that combines varying step-sizes and some viscosity-like procedure. This latter can be regarded as a regularization process which is supposed to induce the convergence in norm of the iterates. Another advantage of this procedure is to allows us to select a particular solution of (1.1). Specifically, we provide precise conditions for convergence without any additional requirement of projection for evaluating the step-sizes.

2. The considered algorithm and its related convergence results

2.1. **A dynamical projected gradient method.** In order to compute a solution of (1.1) we investigate the following regularized variant of (1.6).

Algorithm 2.1:

(**Step 0**) Take $\delta \in (0, 1]$, $\lambda_{-1} \in (0, \infty)$, select any x_{-1} and x_0 of C, and consider a mapping Q and $(\alpha_n) \subset [0, \infty)$ such that:

(C1) $Q : C \to C$ is a strict contraction of modulus $\rho \in [0, 1)$, i.e., $|Qx - Qy| \leq \rho|x - y|$ for all $(x, y) \in C^2$,

(C2) $\alpha_n \in (0, 1]$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_n \alpha_n = \infty$,

(C3) $\alpha_{n-1}/\alpha_n \leq \tau$ (for some positive value τ).

(**Step 1**) Set $\theta_n = \frac{\lambda_n}{\delta\lambda_{n-1}}$ and compute (for $n \geq 0$):

$$q_n = \alpha_n Q x_n + (1 - \alpha_n)x_n, \tag{2.1a}$$

$$\bar{y}_n = x_n + \theta_n(x_n - x_{n-1}), \tag{2.1b}$$

$$x_{n+1} = P_C(q_n - \lambda_n A\bar{y}_n). \tag{2.1c}$$

(**Step 2**) Let $n \leftarrow n + 1$ and goto Step 1.

For the sake of simplicity in this paper we will sometimes use the following notation: $\dot{x}_{n+1} = x_{n+1} - x_n$ and $F = I - Q$.

Remark 2.1. Algorithm 2.1 with $\alpha_n = 0$ and $\theta_n = 1$ (given by $\delta = 1$ and $\lambda_n = \lambda$ for some positive λ) reduces to (1.6). So (2.1) can be regarded as a generalized variant of (1.6). However (2.1) will be shown to be strongly convergent for $\delta \in (0, 1)$ and other appropriate conditions on the involved parameters. Note that a preliminary work regarding weak convergence results was done in [16] by considering the particular case of (2.1) with $\alpha_n = 0$. It is also interesting to point out that this latter method does not generate a Fejer-monotone sequence. The techniques of analysis used in this work are somewhat different from the classical ones.

2.2. **Step-sizes rules and main convergence results.** This paper establishes the convergence in norm of the sequence given by (2.1) relative to convenient choices of the involved step-sizes. The strong limit attained by (x_n) is the unique element x_* of S (the solution set of (1.1)) verifying

$$x_* = (P_S \circ Q)x_*, \tag{2.2}$$

where P_S denotes the metric projection onto S, which equivalently solves the following (hierarchical) variational inequality problem:

$$\text{find } x_* \in S \text{ such that } \langle (I-Q)x_*, v - x_* \rangle \geq 0 \quad \forall v \in S. \tag{2.3}$$

Remark 2.2. It is worthwhile recalling that S (the solution set of (1.1) is closed and convex whenever A is assumed to be monotone ([8]).

Algorithm 2.1 will be first discussed relative to a general framework regarding the choice of the step-sizes (λ_n). Two special cases of our general setting will be also investigated. The first case is related to pre-defined step-sizes (namely, the sequence (λ_n) is known in advance). This latter situation encompasses typical choices of parameters (such as constant step-sizes) but also varying step-sizes. The second case includes some additional line-search procedure so as to determine convenient choices of the step-sizes.

2.2.1. *General step-sizes rules.* For the sake of simplicity, by considering the sequence $(\bar{y}_n)$ generated by Algorithm 2.1, we introduce the set of indexes J and the sequence (k_n) defined by

$$J = \{ n \in \mathbb{N} \ \mid \ \bar{y}_n - \bar{y}_{n-1} \neq 0 \}, \tag{2.4a}$$

$$k_n = \begin{cases} \frac{|A\bar{y}_n - A\bar{y}_{n-1}|}{|\bar{y}_n - \bar{y}_{n-1}|}, & \text{if } n \in J, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4b}$$

Given any element $\bar{y}_{-1} \in H$ (for computing k_0) and positive values λ_{-2} and λ_{-1} (for computing λ_0), we assume that the following general step-sizes rules with $\delta \in (0,1)$ and $\epsilon \in (0,1)$ are satisfied in Algorithm 2.1 (for all $n \geq 0$):

$$\lambda_n k_n \leq \epsilon\delta(\sqrt{2}-1), \tag{2.5a}$$

$$\lambda_n \leq \lambda_{n-1} \left(\delta + \frac{\lambda_{n-1}}{\lambda_{n-2}} \right)^{1/2}, \tag{2.5b}$$

$$\lambda_n \in [\bar{\mu}, \bar{\nu}] \quad \text{(for some positive values } \bar{\mu} \text{ and } \bar{\nu}). \tag{2.5c}$$

The main convergence result of this section is given below and it establishes the convergence of Algorithm 2.1 in the above general setting of parameters.

Theorem 2.1. *Let (x_n) be the sequence generated by Algorithm 2.1 under condition (1.2) together with (C1)-(C3) and parameters verifying (2.5) with $\delta \in (0,1)$ and $\epsilon \in (0,1)$. Then (x_n) converges strongly to the unique solution x_* of (2.3).*

Theorem 2.1 will be proved in Section 5.1.

2.2.2. *Convergence with specific step-size rules.* Two specific situations discussed in [16] and covered by condition (2.5) can be applied to Algorithm 2.1. The first one is related to the case when some upper bound of L (the Lipschitz constant of A) is known while the second case is concerned with a line search procedure that excludes the knowledge of any estimate of L.

A) Classical step-size rules:

Given $\delta \in (0,1)$ in Algorithm 2.1 , we choose parameters $(\lambda_n)_{n\geq -2}$ such that:

$$1-\delta < \frac{\lambda_{-1}}{\lambda_{-2}}, \tag{2.6a}$$

$$1-\delta < \frac{\lambda_n}{\lambda_{n-1}} < r_n, \quad \text{where } r_n = \left(\delta + \frac{\lambda_{n-1}}{\lambda_{n-2}}\right)^{1/2} \quad (\text{for } n \geq 0), \tag{2.6b}$$

$$(\lambda_n) \subset [\bar{\mu}, \bar{\nu}] \quad \text{for positive values } \bar{\mu}, \bar{\nu} \in \left(0, \tfrac{\delta(\sqrt{2}-1)}{L}\right). \tag{2.6c}$$

Remark 2.3. It can be observed that $r_n > 1$ whenever $\frac{\lambda_{n-1}}{\lambda_{n-2}} > 1-\delta$. So it is easily checked by induction that conditions (2.6a) and (2.6b) altogether make sense besides $r_n > 1$ (for $n \geq 0$). Consequently this latter procedure gives us the possibility (at each iteration n) of choosing λ_n such that $\lambda_n \geq \lambda_{n-1}$ or $\lambda_n \leq \lambda_{n-1}$ so as to ensure the last condition (2.6c).

Theorem 2.2. *Let (x_n) be the sequence generated by Algorithm 2.1 under conditions (1.2), (C1)-(C3) and (2.6) with $\delta \in (0,1)$. Then (x_n) converges strongly to the unique solution x_* of (2.3).*

Proof. Theorem 2.2 is a straightforward consequence of Theorem 2.1 by observing (from the L-Lipschitz continuity of A) that (2.6c) yields (2.5a) (for some $\epsilon \in (0,1)$) as well as (2.5c). □

B) Line-search procedure PRGS:

Given $\delta \in (0,1]$, $\epsilon \in (0,1)$, $y_{-1} \in H$ and two elements $\lambda_{-1}, \ \lambda_{-2} \in (0, \bar{\nu}]$, where $\bar{\nu}$ is any positive value, we define the step-size λ_n ($n \geq 0$) relative to some other parameter $\gamma \in (0,1)$ as follows:

(i1) For any integer i, we set

$$t_{n,i} := \gamma^i r_n \text{ where } r_n = \lambda_{n-1}\left(\delta + \tfrac{\lambda_{n-1}}{\lambda_{n-2}}\right)^{1/2},$$
$$\bar{y}_{n,i} = x_n + \tfrac{t_{n,i}}{\delta \lambda_{n-1}}(x_n - x_{n-1}),$$
$$k_{n,i} = \begin{cases} \frac{|A\bar{y}_{n,i} - A\bar{y}_{n-1}|}{|\bar{y}_{n,i} - \bar{y}_{n-1}|}, & \text{if } \bar{y}_{n,i} - \bar{y}_{n-1} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(i2) Next, we choose $\lambda_n = t_{n,i_n}$, where i_n is the smallest nonnegative integer i verifying the following conditions:

$$t_{n,i} \leq \bar{\nu}, \tag{2.7a}$$

$$t_{n,i} k_{n,i} \leq \epsilon\delta(\sqrt{2}-1). \tag{2.7b}$$

Let us prove that the above procedure makes sense.

Lemma 2.1. *If A satisfies the Lipschitz continuity condition (1.2b), then procedure PRGS is well-defined.*
Moreover (2.7) is satisfied whenever $t_{n,i} \le c$, where $c = \min\{\bar{\nu}, \quad \epsilon\delta\frac{(\sqrt{2}-1)}{L}\}$.

Proof. From the L-Lipschitz continuity of A we easily observe that (2.7b) is satisfied whenever $t_{n,i} \le \epsilon\delta\frac{(\sqrt{2}-1)}{L}$. So it can be noticed that (2.7) holds for any small enough positive value $t_{n,i}$ such that $t_{n,i} \le c$. □

Lemma 2.2. *If A satisfies the Lipschitz continuity condition (1.2b), then the sequence (λ_n) generated by procedure PRGS is bounded away from zero.*

The proof of Lemma 2.2 follows the same lines as for the case $\alpha_n = 0$ discussed in [16] but it is given (for the sake of completeness) in the last section of this paper.

Theorem 2.3. *Let (x_n) be the sequence generated by Algorithm 2.1 under condition (1.2), (C1)-(C3) together with (λ_n) given by Procedure PRGS. Then (x_n) converges strongly to the unique solution x_* of (2.3).*

Proof. Theorem 2.3 is a straightforward consequence of Theorem 2.1 and Lemmas 2.1 and 2.2. □

3. Estimates and preliminaries

In this section we give a series of preliminary estimates that will be used for the convergence analysis of Algorithm 2.1.

3.1. **Preliminaries.** To begin with, we recall some classical results that can be also found in [18].

Remark 3.1. For any $(u, v, w, w_1) \in H^4$ and for any $c \in (0, +\infty)$ we have

$$\langle u, v\rangle = -(1/2)|u - v|^2 + (1/2)|u|^2 + (1/2)|v|^2; \tag{3.1a}$$

$$2|u||v| \le c|u|^2 + \frac{1}{c}|v|^2; \tag{3.1b}$$

$$|u - v|^2 \le (1 + c)|u - w|^2 + \left(1 + \frac{1}{c}\right)|u - w|^2; \tag{3.1c}$$

$$2|u - v||w| \le \left((1 + \sqrt{2})|u - w_1|^2 + |w_1 - v|^2 + \sqrt{2}|w|^2\right). \tag{3.1d}$$

Note that (3.1b) is nothing but the Peter-Paul inequality, (3.1c) is immediate from (3.1b), while (3.1d) is deduced from the following two inequalities obtained from (3.1b) and (3.1c), respectively:

$$2|u - v||w| \le \left(\frac{1}{\sqrt{2}}|u - v|^2 + \sqrt{2}|w|\right),$$

$$|u - v|^2 \le (2 + \sqrt{2})|u - w_1|^2 + \sqrt{2}|w_1 - v|^2.$$

Now, we recall some properties of the metric projection from H onto C.

Remark 3.2. The operator $P_C : H \to C$ is nonexpansive and satisfies the following classical inequalities (see, e.g., [28]):

$$\langle x - P_C x, P_C x - y\rangle \geq 0 \quad \text{for any } (x,y) \in H \times C, \tag{3.2a}$$

$$|x-y|^2 \geq |x - P_C x|^2 + |y - P_C x|^2 \quad \text{for any } (x,y) \in H \times C. \tag{3.2b}$$

3.2. **General estimates on the numerical method.** Let us establish some estimates related to sequences $((z_n, \bar{y}_n, q_n, x_n)) \subset H^2 \times C^2$ and $((\lambda_n, \theta_n, \alpha_n)) \subset (0, +\infty)^3$ such that

$$q_n = x_n - \alpha_n F x_n, \tag{3.3a}$$

$$\bar{y}_n = x_n + \theta_n (x_n - x_{n-1}), \tag{3.3b}$$

$$z_n = q_n - \lambda_n A \bar{y}_n, \tag{3.3c}$$

$$x_{n+1} = P_C(z_n). \tag{3.3d}$$

where $F = I - Q$ is given by (C1).

To that end we follow a similar methodology as in [18].

Lemma 3.1. *Let $(z_n, \bar{y}_n) \subset H^2$ and $(q_n, x_n) \subset C^2$ verify (3.3). Then, for any $u \in C$, we have the following inequality*

$$\begin{aligned} &|x_{n+1} - u|^2 - |q_n - u|^2 + |x_{n+1} - q_n|^2 \\ &\quad \leq -\langle A\bar{y}_n - Au, \bar{y}_n - u\rangle \\ &\quad +2\lambda_n \langle A\bar{y}_n - A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle \\ &\quad +2\lambda_n \langle A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle - 2\lambda_n \langle Au, \bar{y}_n - u\rangle. \end{aligned} \tag{3.4}$$

Proof. From (3.2b) and taking $u \in C$ we have

$$|P_C(z_n) - u|^2 \leq |z_n - u|^2 - |z_n - P_C(z_n)|^2,$$

and so, by $x_{n+1} = P_C(z_n)$ and $z_n = q_n - \lambda_n A\bar{y}_n$, we equivalently obtain

$$|x_{n+1} - u|^2 \leq |q_n - u - \lambda_n A\bar{y}_n|^2 - |x_n - x_{n+1} - \lambda_n A\bar{y}_n|^2.$$

Simplifying the above inequality yields

$$|x_{n+1} - u|^2 \leq |q_n - u|^2 - |x_{n+1} - q_n|^2 + 2\lambda_n \langle A\bar{y}_n, u - x_{n+1}\rangle. \tag{3.5}$$

Regarding the last term in the right-side of (3.5) we have

$$\begin{aligned} \langle A\bar{y}_n, u - x_{n+1}\rangle &= \langle A\bar{y}_n, u - \bar{y}_n\rangle + \langle A\bar{y}_n, \bar{y}_n - x_{n+1}\rangle \\ &= \langle A\bar{y}_n - Au, u - \bar{y}_n\rangle + \langle Au, u - \bar{y}_n\rangle \\ &\quad + \langle A\bar{y}_n - A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle + \langle A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle. \end{aligned}$$

Combining this last inequality with (3.5) yields the desired result. □

Now we focus on estimating separately each of the last three terms in the right-hand side of inequality (3.4).

Lemma 3.2. *For any sequences $(z_n, \bar{y}_n) \subset H^2$ and $(q_n, x_n) \subset C^2$ verifying (3.3), we have*

$$\begin{aligned} &2\lambda_{n-1} \langle A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle \\ &\quad \leq \frac{1}{\theta_n}\left(|\dot{x}_{n+1}|^2 - |x_n - \bar{y}_n|^2 - |x_{n+1} - \bar{y}_n|^2\right) \\ &\quad +2\alpha_{n-1}\langle F x_{n-1}, \dot{x}_{n+1}\rangle - 2\alpha_{n-1}\theta_n \langle F x_{n-1}, \dot{x}_n\rangle, \end{aligned} \tag{3.6}$$

where $\dot{x}_j = x_j - x_{j-1}$ (for any integer j).

Proof. From $x_n = P_C(z_{n-1})$ and $(x_n) \subset C$, by (3.2a) we have

$$\langle x_n - z_{n-1}, x_n - x_{n+1}\rangle \le 0 \text{ and } \langle x_n - z_{n-1}, \theta_n(x_n - x_{n-1})\rangle \le 0.$$

So by $z_{n-1} = q_{n-1} - \lambda_{n-1}A\overline{y}_{n-1}$ we deduce that

$$\langle x_n - q_{n-1} + \lambda_{n-1}A\overline{y}_{n-1}, \overline{y}_n - x_{n+1}\rangle \le 0,$$

or equivalently, using the notation $\dot{x}_n = x_n - x_{n-1}$, we get

$$\begin{aligned}\lambda_{n-1}&\langle A\overline{y}_{n-1}, \overline{y}_n - x_{n+1}\rangle \\ &\le \langle x_n - q_{n-1}, x_{n+1} - \overline{y}_n\rangle, \\ &= \langle \dot{x}_n, x_{n+1} - \overline{y}_n\rangle + \langle x_{n-1} - q_{n-1}, x_{n+1} - \overline{y}_n\rangle.\end{aligned} \tag{3.7}$$

Let us consider separately the two terms in the right side of the previous inequality. Regarding the first term, by $\bar{y}_n - x_n = \theta_n\dot{x}_n$ (from (3.3d)) and by (3.1a) we have

$$\begin{aligned}2\langle \dot{x}_n, x_{n+1} - \overline{y}_n\rangle &\le -(2/\theta_n)\langle x_n - \overline{y}_n, x_{n+1} - \overline{y}_n\rangle \\ &= (1/\theta_n)\left(|\dot{x}_{n+1}|^2 - |x_n - \overline{y}_n|^2 - |x_{n+1} - \overline{y}_n|^2\right).\end{aligned}$$

Regarding the second term, by $x_{n-1} - q_{n-1} = \alpha_{n-1}Fx_{n-1}$ (from (3.3a)) and using the definition of $\overline{y}_n$ we have

$$\langle x_{n-1} - q_{n-1}, x_{n+1} - \overline{y}_n\rangle = \alpha_{n-1}\langle Fx_{n-1}, \dot{x}_{n+1} - \theta_n\dot{x}_n\rangle.$$

Combining the last three results entails (3.6). □

The following result is independent of the considered method.

Lemma 3.3. *For any sequences $(\bar{y}_n, x_n) \subset H^2$ and $(\lambda_n) \subset [0,\infty)$ we have*

$$\begin{aligned}2\lambda_n&\langle A\bar{y}_n - A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle \\ &\le (k_n\lambda_n)\left((1+\sqrt{2})|\bar{y}_n - x_n|^2 + |x_n - \bar{y}_{n-1}|^2 + \sqrt{2}|x_{n+1} - \bar{y}_n|^2\right),\end{aligned}$$

where (k_n) is defined in (2.4).

Proof. From the definition of k_n we obviously have

$$2\lambda_n\langle A\bar{y}_n - A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle \le 2(k_n\lambda_n)|\bar{y}_n - \bar{y}_{n-1}| \times |x_{n+1} - \bar{y}_n|.$$

Thus the desired result follows immediately from (3.1d). □

Lemma 3.4. *For any sequences $(z_n, \bar{y}_n) \subset H^2$ and $(q_n, x_n) \subset C^2$ verifying (3.3) with (1.2a) (monotonicity of A) and for any $u \in C$, we have*

$$\begin{aligned}|x_{n+1} - u|^2 &- |q_n - u|^2 \\ &\le \left(-1 + \tfrac{\lambda_n}{\lambda_{n-1}\theta_n}\right)|\dot{x}_{n+1}|^2 \\ &-2\lambda_n(1+\theta_n)G_n + 2\lambda_n\theta_nG_{n-1} \\ &-a_n|\bar{y}_n - x_n|^2 \\ &-b_n|x_{n+1} - \bar{y}_n|^2 + (k_n\lambda_n)|x_n - \bar{y}_{n-1}|^2 \\ &+2\alpha_{n-1}\left(\tfrac{\lambda_n}{\lambda_{n-1}}\right)\langle Fx_{n-1}, \dot{x}_{n+1}\rangle - 2\alpha_{n-1}\theta_n\left(\tfrac{\lambda_n}{\lambda_{n-1}}\right)\langle Fx_{n-1}, \dot{x}_n\rangle,\end{aligned} \tag{3.8}$$

where $a_n = \frac{\lambda_n}{\lambda_{n-1}\theta_n} - (k_n\lambda_n)(1+\sqrt{2})$, $b_n = \frac{\lambda_n}{\lambda_{n-1}\theta_n} - (k_n\lambda_n)\sqrt{2}$ and $G_n = \langle Au, x_n - u\rangle$, (k_n) being defined in (2.4), while $\dot{x}_j = x_j - x_{j-1}$ (for any integer j).

Proof. From Lemma 3.1 and invoking the monotonicity of A, we clearly have

$$\begin{aligned} &|x_{n+1}-u|^2 - |q_n-u|^2 + |x_{n+1}-q_n|^2 \\ &\quad \le 2\lambda_n\langle A\bar{y}_n - A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle \\ &\quad\quad +2\lambda_n\langle A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle - 2\lambda_n\langle Au, \bar{y}_n - u\rangle. \end{aligned} \tag{3.9}$$

Moreover, by Lemma 3.3 we have

$$\begin{aligned} &2\lambda_n\langle A\bar{y}_n - A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle \\ &\quad \le (k_n\lambda_n)\left((1+\sqrt{2})|\bar{y}_n - x_n|^2 + |x_n - \bar{y}_{n-1}|^2 + \sqrt{2}|x_{n+1}-\bar{y}_n|^2\right), \end{aligned}$$

while Lemma 3.2 gives us

$$\begin{aligned} &2\langle A\bar{y}_{n-1}, \bar{y}_n - x_{n+1}\rangle \\ &\quad \le \tfrac{1}{\lambda_{n-1}\theta_n}\left(|\dot{x}_{n+1}|^2 - |x_n - \bar{y}_n|^2 - |x_{n+1}-\bar{y}_n|^2\right) \\ &\quad\quad +2\tfrac{\alpha_{n-1}}{\lambda_{n-1}}\langle Fx_{n-1}, \dot{x}_{n+1}\rangle - 2\tfrac{\alpha_{n-1}}{\lambda_{n-1}}\theta_n\langle Fx_{n-1}, \dot{x}_n\rangle. \end{aligned}$$

Combining the previous two inequalities with (3.9) amounts to

$$\begin{aligned} &|x_{n+1}-u|^2 - |q_n-u|^2 + |\dot{x}_{n+1}|^2 \\ &\quad \le (k_n\lambda_n)\left((1+\sqrt{2})|\bar{y}_n - x_n|^2 + |x_n - \bar{y}_{n-1}|^2 + \sqrt{2}|x_{n+1}-\bar{y}_n|^2\right) \\ &\quad\quad + \left(\tfrac{\lambda_n}{\theta_n\lambda_{n-1}}\right)\left(|\dot{x}_{n+1}|^2 - |x_n-\bar{y}_n|^2 - |x_{n+1}-\bar{y}_n|^2\right) \\ &\quad\quad +2\alpha_{n-1}\left(\tfrac{\lambda_n}{\lambda_{n-1}}\right)\langle Fx_{n-1}, \dot{x}_{n+1}\rangle - 2\alpha_{n-1}\theta_n\left(\tfrac{\lambda_n}{\lambda_{n-1}}\right)\langle Fx_{n-1}, \dot{x}_n\rangle \\ &\quad\quad -2\lambda_n\langle Au, \bar{y}_n - u\rangle, \end{aligned} \tag{3.10}$$

namely

$$\begin{aligned} &|x_{n+1}-u|^2 - |q_n-u|^2 + |\dot{x}_{n+1}|^2 \\ &\quad \le \tfrac{\lambda_n}{\lambda_{n-1}\theta_n}|\dot{x}_{n+1}|^2 - a_n|\bar{y}_n - x_n|^2 - b_n|x_{n+1}-\bar{y}_n|^2 + (k_n\lambda_n)|x_n - \bar{y}_{n-1}|^2 \\ &\quad\quad +2\alpha_{n-1}\left(\tfrac{\lambda_n}{\lambda_{n-1}}\right)\langle Fx_{n-1}, \dot{x}_{n+1}\rangle - 2\alpha_{n-1}\theta_n\left(\tfrac{\lambda_n}{\lambda_{n-1}}\right)\langle Fx_{n-1}, \dot{x}_n\rangle \\ &\quad\quad -2\lambda_n\langle Au, \bar{y}_n - u\rangle, \end{aligned}$$

where $a_n = \frac{\lambda_n}{\lambda_{n-1}\theta_n} - (k_n\lambda_n)(1+\sqrt{2})$ and $b_n = \frac{\lambda_n}{\lambda_{n-1}\theta_n} - (k_n\lambda_n)\sqrt{2}$. The desired inequality follows by noticing that

$$\langle Au, \bar{y}_n - u\rangle = (1+\theta_n)\langle Au, x_n - u\rangle - \theta_n\langle Au, x_{n-1}-u\rangle,$$

which completes the proof. □

4. Convergence analysis

4.1. **Projection part of the method.** The main estimate of this section is stated under the following conditions on the parameters (for any $n \ge 0$):

$$\theta_n = \frac{\lambda_n}{\lambda_{n-1}\delta}, \tag{4.1a}$$

$$\epsilon\delta(\sqrt{2}-1) - (k_n\lambda_n) \ge 0, \tag{4.1b}$$

$$\lambda_n\theta_n \le \lambda_{n-1}(1+\theta_{n-1}), \tag{4.1c}$$

where δ and ϵ are positive values.

Remark 4.1. Let us observe for $\theta_n = \frac{\lambda_n}{\lambda_{n-1}\delta}$ (namely (4.1a)) that condition (4.1c) is equivalent to $\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^2 \leq \delta + \left(\frac{\lambda_{n-1}}{\lambda_{n-2}}\right)$, which corresponds to (2.5b).

Lemma 4.1. *Let (x_n), $(\bar{y}_n)$ and (q_n) be generated by Algorithm 2.1 under conditions (1.2a) and (2.5) with $\delta \in (0,1)$ and $\epsilon \in (0,1)$. Then, for any $u \in S$, there exist two positive values σ and ν such that*

$$\begin{aligned} &|x_{n+1} - u|^2 - |q_n - u|^2 \\ &\quad \leq -\sigma|\dot{x}_{n+1}|^2 - \Gamma_n + \Gamma_{n-1} - \nu|\bar{y}_n - x_n|^2 \\ &\quad\quad +(2\delta)\alpha_{n-1}\theta_n\langle Fx_{n-1}, \dot{x}_{n+1}\rangle - (2\delta)\alpha_{n-1}\theta_n^2\langle Fx_{n-1}, \dot{x}_n\rangle, \end{aligned} \tag{4.2}$$

where (Γ_n) is defined by

$$\Gamma_n = (k_{n+1}\lambda_{n+1})\,|x_{n+1} - \bar{y}_n|^2 + 2\lambda_{n+1}\theta_{n+1}\langle Au, x_n - u\rangle, \tag{4.3}$$

together with (k_n) given in (2.4), while $\dot{x}_j = x_j - x_{j-1}$ (for any integer j).

Proof. Inequality (4.2) is a straightforward consequence of Lemma 3.4. Indeed, given $u \in S$, we readily have $G_n := \langle Au, x_n - u\rangle \geq 0$ (since $(x_n) \subset C$ and A is assumed to be monotone). Moreover, by (4.1a), the quantities a_n and b_n in (3.8) reduce to $a_n = \left(\delta - (k_n\lambda_n)(1+\sqrt{2})\right)$, $b_n = \left(\delta - (k_n\lambda_n)\sqrt{2}\right)$, and so, by (4.1b), it is easily checked that $a_n \geq \nu$ (where $\nu = \epsilon(1-\delta)$) and $b_n \geq \lambda_{n+1}k_{n+1}$. Consequently, in light of (3.8) and (4.1), we obtain

$$\begin{aligned} &|x_{n+1} - u|^2 - |q_n - u|^2 \\ &\quad \leq (-1+\delta)\,|\dot{x}_{n+1}|^2 \\ &\quad -2\lambda_{n+1}\theta_{n+1}G_n + 2\lambda_n\theta_n G_{n-1} - \nu|\bar{y}_n - x_n|^2 \\ &\quad -(k_{n+1}\lambda_{n+1})|x_{n+1} - \bar{y}_n|^2 + (k_n\lambda_n)|x_n - \bar{y}_{n-1}|^2 \\ &\quad +(2\delta)\alpha_{n-1}\theta_n\langle Fx_{n-1}, \dot{x}_{n+1}\rangle - (2\delta)\alpha_{n-1}\theta_n^2\langle Fx_{n-1}, \dot{x}_n\rangle, \end{aligned} \tag{4.4}$$

which leads to the desired result with $\sigma = 1 - \delta$. □

4.2. **Viscosity part of the method.** The following estimates are related to the regularized part of the method.

Lemma 4.2. *Let (x_n) and (q_n) be generated by Algorithm 2.1 under condition (C1). Then the following statements are reached:*

$$|q_n - q|^2 \leq (1 - 2(1-\rho)\alpha_n)|x_n - q|^2 + \alpha_n\left(\alpha_n|Fx_n|^2 - 2\langle Fq, x_n - q\rangle\right), \tag{4.5a}$$

$$|Fx_n| \leq ((\rho+1)|x_n - q| + |Fq|), \tag{4.5b}$$

$$|Fx_n|^2 \leq 2\left((\rho+1)^2|x_n - q|^2 + |Fq|^2\right), \tag{4.5c}$$

where ρ is given by (C1), q is any element of H and $F = I - Q$.

Proof. Take $q \in H$ and let us prove each item separately:

1) From (2.1a), we have $q_n - x_n = -\alpha_n Fx_n$, and so we obtain

$$|q_n - x_n|^2 = (\alpha_n)^2|Fx_n|^2$$

as well as

$$2\langle q_n - x_n, x_n - q\rangle = 2\alpha_n\langle Fx_n, x_n - q\rangle.$$

Moreover, using (3.1a), we have

$$2\langle q_n - x_n, x_n - q\rangle = -|q_n - q|^2 + |x_n - q|^2 + |x_n - q_n|^2.$$

Then it follows that

$$-|q_n - q|^2 + |x_n - q|^2 + \alpha_n^2 |Fx_n|^2 = 2\alpha_n \langle Fx_n, x_n - q\rangle. \ .$$

In addition, by condition (C1) on the operator Q, we have

$$\begin{aligned}\langle Fx_n, x_n - q\rangle &= \langle Fx_n - Fq, x_n - q\rangle + \langle Fq, x_n - q\rangle \\ &\geq (1-\rho)|x_n - q|^2 + \langle Fq, x_n - q\rangle.\end{aligned}$$

So we are led to

$$-|q_n - q|^2 + (1 - 2\alpha_n(1-\rho))|x_n - q|^2 + |x_n - q_n|^2 \geq 2\alpha_n \langle Fq, x_n - q\rangle,$$

that is the desired inequality (4.5a).

2) Let us recall that $q_n - x_n = -\alpha_n F x_n$ and observe that $F = I - Q$ is $(1+\rho)$-Lipschitz continuous, since Q is assumed to be ρ-Lipschitz continuous (by (C1)). Hence, writing $|Fx_n| = |(Fx_n - Fq) + Fq|$, we immediately deduce (4.5b).

3) The latter inequality (4.5c) is obvious from (4.5b). □

4.3. **Boundedness of the iterates.** A preliminary estimate is needed for studying the asymptotic behavior of the sequences generated by the considered method.

Lemma 4.3. *Let (x_n) be generated by Algorithm 2.1 under conditions (1.2a), (C1) and (2.5) with $\delta \in (0,1)$ and $\epsilon \in (0,1)$. Then for any $u \in S$ and for some positive values ν_1, σ and η, we have*

$$\begin{aligned} S_{n+1} - S_n \leq & -\sigma|\dot{x}_{n+1}|^2 - \nu_1|\dot{x}_n|^2 \\ & +(2\delta)\alpha_{n-1}\theta_n|Fx_{n-1}||\dot{x}_{n+1}| + (2\delta)\alpha_{n-1}\theta_n^2|Fx_{n-1}||\dot{x}_n| \\ & -\eta\alpha_n|x_n - u|^2 + \alpha_n\left(\alpha_n|Fx_n|^2 - 2\langle Fu, x_n - u\rangle\right), \end{aligned} \tag{4.6}$$

where $F = I - Q$ and S_n is defined by

$$S_n = |x_n - u|^2 + (k_n\lambda_n)\,|x_n - \bar{y}_{n-1}|^2 + 2\lambda_n\theta_n\langle Au, x_{n-1} - u\rangle, \tag{4.7}$$

(k_n) being defined in (2.4), while $\dot{x}_j = x_j - x_{j-1}$ (for any integer j).

Proof. This result is immediate from (4.2) and (4.5a) together with $\eta := 2(1-\rho)$ and also noticing that (θ_n) is bounded from below under condition (2.5) (hence, by $\bar{y}_n - x_n = \theta_n \dot{x}_n$, we clearly have $\nu|\bar{y}_n - x_n|^2 \geq \nu_1|\dot{x}_n|^2$ for some positive ν_1). □

The next lemma can be found in [14] (Lemma 3.1) and its proof is given for the sake of completeness.

Lemma 4.4. [14] *Let (Γ_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $(\Gamma_{n_j})_{j\geq 0}$ of (Γ_n) such that*

(h1) $\Gamma_{n_j} < \Gamma_{n_j+1}$ *for all* $j \geq 0$.

Also consider the sequence of integers $(\tau(n))_{n\geq n_0}$ defined by

(h2) $\tau(n) = \max\{k \leq n \ \mid \ \Gamma_k < \Gamma_{k+1}\}$.

Then $(\tau(n))_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\tau(n) = \infty$, and, for all $n \geq n_0$, the following two estimates hold:

(r1) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$,

(r2) $\Gamma_n \leq \Gamma_{\tau(n)+1}$.

Proof. Clearly, by (h1), we can see that $(\tau(n))$ is a well-defined sequence, and the fact that it is nondecreasing is obvious as well as $\lim_{n\to\infty}\tau(n)=\infty$ and (r1). Let us prove (r2). It is easily observed that $\tau(n)\le n$. Consequently, we prove (r2) by distinguishing the three cases: (c1) $\tau(n)=n$; (c2) $\tau(n)=n-1$; (c3) $\tau(n)<n-1$. In the first case (i.e., $\tau(n)=n$), (r2) is immediately given by (r1). In the second case (i.e., $\tau(n)=n-1$), (r2) becomes obvious. In the third case (i.e., $\tau(n)\le n-2$), by (h2) and for any integer $n\ge n_0$, we easily observe that $\Gamma_j\ge\Gamma_{j+1}$ for $\tau(n)+1\le j\le n-1$, namely $\Gamma_{\tau(n)+1}\ge\Gamma_{\tau(n)+2}\ge ...\ge\Gamma_{n-1}\ge\Gamma_n$, which entails the desired result. □

At once we establish the boundedness of the iterates given by (2.1).

Lemma 4.5. *Suppose that $(\alpha_n)\subset(0,1]$, $\alpha_n\to 0$ (as $n\to\infty$) and that conditions (1.2), (C1), (C3) and (2.5) with $\delta\in(0,1)$ and $\epsilon\in(0,1)$ are satisfied. Then the sequences (x_n) and $(\bar y_n)$ generated by Algorithm 2.1 are bounded.*

Proof. Given $u\in S$, by (4.6), (C3) and since (θ_n) is bounded (according to (2.5)), we readily have

$$\begin{aligned}S_{n+1}-S_n\le{}&-\sigma|\dot x_{n+1}|^2-\nu_1|\dot x_n|^2+c_1\delta_n\\&-\eta\alpha_n|x_n-u|^2+\alpha_n\left(\alpha_n|Fx_n|^2+2|Fu||x_n-u|\right),\end{aligned}\tag{4.8}$$

where σ, ν_1, η and c_1 are positive values and δ_n is defined by

$$\delta_n=\alpha_{n-1}|Fx_{n-1}||\dot x_{n+1}|+\alpha_{n-1}|Fx_{n-1}||\dot x_n|.$$

From the L-Lipschitz continuity of A and using Young's inequality, we also have

$$|Fx_{n-1}|^2\le 2|Fx_n|^2+2L^2|\dot x_n|^2;$$

hence by the Peter-Paul inequality it is not difficult to see that there exists some positive value κ such that

$$c_1\alpha_{n-1}|Fx_{n-1}||\dot x_{n+1}|\le\frac{\sigma}{2}|\dot x_{n+1}|^2+\frac{\kappa}{2}\alpha_{n-1}^2(|Fx_n|^2+|\dot x_n|^2)$$

and

$$c_1\alpha_{n-1}|Fx_{n-1}||\dot x_n|\le\frac{\nu_1}{4}|\dot x_n|^2+\frac{\kappa}{2}\alpha_{n-1}^2(|Fx_n|^2+|\dot x_n|^2).$$

It follows that

$$\delta_n\le\frac{1}{c_1}\left(\frac{\sigma}{2}|\dot x_{n+1}|^2+\kappa\alpha_{n-1}^2|Fx_n|^2+\left(\frac{\nu_1}{4}+\kappa\alpha_{n-1}^2\right)|\dot x_n|^2\right).\tag{4.9}$$

So by this last result and (4.8) we obtain

$$\begin{aligned}S_{n+1}-S_n\le{}&-\tfrac{\sigma}{2}|\dot x_{n+1}|^2-\left(\tfrac{3\nu_1}{4}-\kappa\alpha_{n-1}^2\right)|\dot x_n|^2+\kappa\alpha_{n-1}^2|Fx_n|^2\\&-\eta\alpha_n|x_n-u|^2+\alpha_n\left(\alpha_n|Fx_n|^2+2|Fu||x_n-u|\right).\end{aligned}\tag{4.10}$$

Clearly, by $\alpha_n\to 0$ as $n\to\infty$ and for $n\ge n_0$ (where n_0 is some large enough integer), we have $\frac{3\nu_1}{4}-\kappa\alpha_{n-1}^2\ge\frac{\nu_1}{2}$, while in light of (4.5c) we additionally have

$$|Fx_n|^2\le 2\left((\rho+1)^2|x_n-u|^2+|Fu|^2\right).$$

Consequently, for $n\ge n_0$ and by (4.10), we observe that there exists some positive values κ_1 such that

$$\begin{aligned}S_{n+1}-S_n\le{}&-\tfrac{\sigma}{2}|\dot x_{n+1}|^2-\tfrac{\nu_1}{2}|\dot x_n|^2+\kappa_1\alpha_{n-1}^2(|x_n-u|^2+1)\\&-\eta\alpha_n|x_n-u|^2+\kappa_1\alpha_n\left(\alpha_n|x_n-u|^2+\alpha_n+|x_n-u|\right).\end{aligned}\tag{4.11}$$

Therefore, assuming that $\alpha_{n-1}/\alpha_n \leq \gamma$ for some positive constant γ (by condition (C3)) we deduce that

$$\begin{aligned} S_{n+1} - S_n \leq & -\tfrac{\sigma}{2}|\dot{x}_{n+1}|^2 - \tfrac{\nu_1}{2}|\dot{x}_n|^2 \\ & +\alpha_n \left(-\left(\eta - (\gamma^2+1)\kappa_1\alpha_n\right)|x_n - u|^2 + \kappa_1|x_n - u| + \kappa_1\alpha_n(\gamma^2+1)\right). \end{aligned} \tag{4.12}$$

Then it is a simple matter to see that, for $n \geq n_1$ (where n_1 is some large enough integer), we have $\eta - (\gamma^2+1)\kappa_1\alpha_n \geq \frac{\eta}{2}$ as well as the following estimate

$$\begin{aligned} S_{n+1} - S_n \leq & -\tfrac{\sigma}{2}|\dot{x}_{n+1}|^2 - \tfrac{\nu_1}{2}|\dot{x}_n|^2 \\ & +\alpha_n \left(-\tfrac{\eta}{2}|x_n - u|^2 + \kappa_1|x_n - u| + \kappa_1\alpha_n(\gamma^2+1)\right). \end{aligned} \tag{4.13}$$

Now we apply Lemma 4.4 (in light of (4.13)) so as to prove the boundedness of (x_n). The following two possibilities can be considered regarding the sequence (S_n):

- Either S_n is non-increasing, and so it is obvious that (x_n) is bounded.
- Or, by Lemma 4.4, there exists a subsequence (S_{n_k}) such that

$$S_n \leq S_{n_k+1}, \tag{4.14}$$

together with

$$\begin{aligned} 0 < S_{n_k+1} - S_{n_k} & \\ \leq & -\tfrac{\sigma}{2}|\dot{x}_{n_k+1}|^2 - \tfrac{\nu_1}{2}|\dot{x}_{n_k}|^2 \\ & +\alpha_{n_k}\left(-\tfrac{\eta}{2}|x_{n_k} - u|^2 + \kappa_1|x_{n_k} - u| + \kappa_1\alpha_{n_k}(\gamma^2+1)\right). \end{aligned} \tag{4.15}$$

Let us prove that (x_{n_k}) is a bounded sequence. Clearly, from this last inequality we have

$$\left(\eta - (\gamma^2+1)\kappa_1\alpha_{n_k}\right)|x_{n_k} - u|^2 \leq \kappa_1|x_{n_k} - u| + \kappa_1\alpha_{n_k}(\gamma^2+1);$$

hence recalling that $\alpha_{n_k} \to 0$ as $k \to \infty$ (by (C2)), we immediately deduce that (x_{n_k}) is bounded.

Now we prove that (S_{n_k+1}) is a bounded sequence. Towards that end, by taking into account the definition of S_{n+1}, namely

$$S_{n+1} = |x_{n+1} - u|^2 + (k_{n+1}\lambda_{n+1})|x_{n+1} - \bar{y}_n|^2 + 2\lambda_{n+1}\theta_{n+1}\langle Au, x_n - u\rangle, \tag{4.16}$$

we realize that we just need to establish the boundedness of the sequences $(\bar{y}_{n_k})$ and (x_{n_k+1}). Let us observe that (4.15) yields

$$\begin{aligned} 0 < S_{n_k+1} - S_{n_k} & \\ \leq \alpha_{n_k}\left(-\tfrac{\eta}{2}|x_{n_k} - u|^2 + \kappa_1|x_{n_k} - u| + \kappa_1\alpha_{n_k}(\gamma^2+1)\right), \end{aligned} \tag{4.17}$$

which amounts to $\lim_{k\to\infty}(S_{n_k+1} - S_{n_k}) = 0$ and so, again using (4.15) we deduced that

$$\lim_{k\to\infty}|\dot{x}_{n_k+1}| = \lim_{k\to\infty}|\dot{x}_{n_k}| = 0.$$

Hence, it follows that (x_{n_k+1}) and $(\bar{y}_{n_k})$ are bounded sequences. As a consequence, by (4.14) we conclude that (S_n) is bounded and so are (x_n) and $(\bar{y}_n)$. □

4.4. **Optimality of weak cluster points.** The next lemma gives us a sufficient condition for the optimality of the weak cluster points of sequences generated by Algorithm 2.1.

Lemma 4.6. *Let* $((x_n, q_n, \bar{y}_n)) \subset C^2 \times H$ *satisfy* $x_{n+1} = P_C(q_n - \lambda_n A\bar{y}_n)$, *where* $(\lambda_n) \subset [\bar{\nu}, \infty)$ *(for some positive value* $\bar{\nu}$*) and* $A : H \to H$ *is monotone and Lipschitz continuous over* H*. Assume in addition that there exists an increasing sequence of indexes* (n_k) *such that:*

$$(x_{n_k}) \text{ converges weakly to some } u \text{ of } C, \tag{4.18a}$$

$$\lim_{k\to\infty} |x_{n_k+1} - q_{n_k}| = \lim_{k\to\infty} |\bar{y}_{n_k} - x_{n_k+1}| = 0, \tag{4.18b}$$

$$\lim_{k\to\infty} |x_{n_k+1} - x_{n_k}| = 0. \tag{4.18c}$$

Then u *belongs to* S *(the solution set of* (1.1)*).*

Proof. Let $q \in C$. Clearly, from (3.2a) and $x_{n+1} = P_C(z_n)$ with $z_n = q_n - \lambda_n A\bar{y}_n$ we have $\langle x_{n+1} - z_n, q - x_{n+1}\rangle \geq 0$, namely

$$0 \leq \langle x_{n+1} - q_n + \lambda_n A\bar{y}_n, q - x_{n+1}\rangle,$$

or equivalently,

$$0 \leq \langle x_{n+1} - q_n, q - x_{n+1}\rangle + \lambda_n\langle A\bar{y}_n, q - \bar{y}_n\rangle + \lambda_n\langle A\bar{y}_n, \bar{y}_n - x_{n+1}\rangle.$$

Hence, by monotonicity of A we obtain

$$0 \leq \langle x_{n+1} - q_n, q - x_{n+1}\rangle + \lambda_n\langle Aq, q - \bar{y}_n\rangle + \lambda_n\langle A\bar{y}_n, \bar{y}_n - x_{n+1}\rangle$$

that is

$$0 \leq \langle \frac{1}{\lambda_n}(x_{n+1} - q_n), q - x_{n+1}\rangle + \langle Aq, q - \bar{y}_n\rangle + \langle A\bar{y}_n, \bar{y}_n - x_{n+1}\rangle.$$

Moreover, by (4.18c) and $(x_{n_k}) \rightharpoonup u$ weakly, we also have $(x_{n_k+1}) \rightharpoonup u$ weakly. So it is obvious that (x_{n_k+1}) is bounded since it is assumed to be weakly convergent. Then it is immediate from (4.18b) that $(\bar{y}_{n_k})$ is also bounded and that it converges weakly to u as $k \to +\infty$. Hence $(A\bar{y}_{n_k})$ is bounded (by Lipschitz continuity of A) while (λ_n) is assumed to be bounded away from zero. Consequently, passing to the limit in the last inequality (with indexes n_k) entails that u solves the Minty's variational inequality:

find $u \in C$ such that $\langle Aq, q - u\rangle \geq 0$ (for any $q \in C$).

This latter problem is well-known to be equivalent to (1.1) under the considered assumptions. This ensures that $u \in S$. □

4.5. **Some key results for viscosity methods.** In this section we provide a result (Lemma 4.8) that will be useful for proving the convergence of the viscosity method under consideration. The following preliminary lemma is needed for this purpose. This lemma can be found in [17] and its proof is given for the sake of completeness.

Lemma 4.7. *Let* $\{a_n\}$ *be a sequence of nonnegative number such that*

(h) $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n r_n$,

where $\{r_n\} \subset (-\infty, \infty)$ *is bounded above and* $\{\gamma_n\} \subset [0,1]$ *satisfies* $\sum_n \gamma_n = \infty$. *Then it holds that*

(r) $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} r_n$.

Proof. Let $p \in N^*$ and set $\sigma_p = \sup_{j\geq p} r_j$. Then by (h) and for $n \geq p$ we immediately have $a_{n+1} - a_n + \gamma_n(a_n - \sigma_p) \leq 0$, hence denoting $b_n = a_n - \sigma_p$ we equivalently obtain $b_{n+1} \leq (1-\gamma_n)b_n$, which by induction yields

$$b_{n+1} \leq \left(\Pi_{k=p}^{n}(1-\gamma_k)\right) b_p. \tag{4.19}$$

Clearly, we deduce that $b_{n+1} \leq |b_p|$, so that (b_n) is bounded above and so is (a_n). Moreover, assuming that $\sum_{n\geq 0}\gamma_n = \infty$ (hence $\lim_{n\to\infty}\Pi_{k=p}^{n}(1-\gamma_k) = 0$), and passing to the limit as $n \to \infty$ in (4.19), we get $\limsup_{n\to\infty} b_n = \limsup_{n\to\infty} b_{n+1} \leq 0$, which is equivalent to $\limsup_{n\to\infty} a_n \leq \sigma_p$, so that $p \to \infty$ yields (r). □

The next result can be regarded as a new tool for proving the convergence of many viscosity-type methods.

Lemma 4.8. *Let* (S_n) *be a sequence of nonnegative real numbers such that*

(h) $S_{n+1} \leq (1-\gamma_n)S_n - P_n - \gamma_n R_n$,

where $(P_n) \subset [0, +\infty)$, $\{R_n\} \subset (-\infty, \infty)$ *is a bounded sequence and* $\{\gamma_n\} \subset [0,1]$ *satisfies* $\sum_n \gamma_n = \infty$. *Then there exist* $\beta \geq 0$ *and some increasing sequence of indexes* (n_k) *verifying the following statements:*

(a) $\limsup_{n\to+\infty} S_n \leq -\lim_{k\to+\infty} R_{n_k}$,

(b) $P_{n_k} \leq \beta\gamma_{n_k}$,

(c) $\lim_{k\to\infty}(S_{n_k+1} - S_{n_k}) = 0$ *(if* $\gamma_n \to 0$ *as* $n \to \infty$*).*

Proof. The proof can be divided into the following two cases (A) and (B):

(A) First of all, we prove that (a) and (b) hold (in general) for some increasing sequence of indexes (n_k). Observe that (h) can be rewritten as

$$S_{n+1} \leq (1-\gamma_n)S_n - \gamma_n H_n,$$

where $H_n = (1/\gamma_n)P_n + R_n$. In addition, H_n is bounded from below (since R_n is assumed to be bounded and (P_n) is assumed to be nonnegative). Consequently, from Lemma 4.7 we deduce that

(f1) $\limsup_{n\to+\infty} S_n \leq \limsup_{n\to\infty}(-H_n) = -\liminf_{n\to\infty} H_n$.

So $\liminf_{n\to\infty} H_n$ is a finite real number. Consequently, there exists some subsequence (H_{n_k}) of (H_n) such that

(f2) $\liminf_{n\to\infty} H_n = \lim_{k\to\infty} H_{n_k}$.

It follows that (H_{n_k}) is bounded (as it is convergent). So, from the definition of H_n and recalling that (R_n) is assumed to be bounded, we deduce that $((1/\gamma_{n_k})P_{n_k})$ is bounded. Consequently, there exists a convergent subsequence (again denoted (H_{n_k})) of (H_{n_k}) such that $((1/\gamma_{n_k})P_{n_k})$ remains bounded and (R_{n_k}) converges as $k \to \infty$. Hence we immediately have $P_{n_k} \leq \beta\gamma_{n_k}$ (for some positive constant β), while (f1) and (f2) entail that

$$\limsup_{n\to+\infty} S_n \leq -\lim_{k\to\infty} H_{n_k} \leq -\lim_{k\to\infty} R_{n_k}.$$

(B) Now, assuming that $\gamma_n \to 0$ (as $n \to \infty$), we prove that (a), (b) and (c) are satisfied relative to a same increasing sequence of indexes (m_k). Note that the result of Lemma 4.8 is obvious from (A) in the special case when (S_n) is non-increasing (because such a nonnegative sequence is convergent). So we assume that (S_n) does not decrease at infinity. Then by Lemma 4.4 we know that there exists an increasing sequence of indexes (m_k) verifying the following items (j1) and (j2):

(j1) $S_{m_k+1} - S_{m_k} \geq 0$,

(j2) $S_n \leq S_{m_k+1}$.

Then from (h) and (j1) we have

$$0 \leq S_{m_k+1} - S_{m_k} \leq -\gamma_{m_k} R_{m_k},$$

hence, recalling that (R_n) is bounded and that $\gamma_n \to 0$ (as $n \to \infty$), we obtain

(j3) $S_{m_k+1} - S_{m_k} \to 0$ as $k \to \infty$.

Let us notice from the boundedness of (R_n) that there exists a subsequence (n_k) of (m_k) verifying

(j4) $\liminf_{k\to\infty} R_{m_k} = \lim_{k\to\infty} R_{n_k}$.

Item (c) is then obvious from (j3). Again using (h) and (j1) we readily have

(j5) $(1/\gamma_{m_k})P_{m_k} + S_{m_k} \leq -R_{m_k}$.

It follows immediately that $(1/\gamma_{n_k})P_{n_k} \leq -R_{n_k}$, which entails (b). Now, using (j2) and (j3) gives us

$$\limsup_{n\to\infty} S_n \leq \limsup_{k\to\infty}(S_{m_k+1} - S_{m_k}) + \limsup_{k\to\infty} S_{m_k} = \limsup_{k\to\infty} S_{m_k}.$$

Consequently, observing that $S_{m_k} \leq -R_{m_k}$ (according to (j5)) and using (j4), we obtain

$$\limsup_{n\to\infty} S_n \leq \limsup_{k\to\infty}(-R_{m_k}) = -\lim_{k\to\infty} R_{n_k},$$

that is (a). This completes the proof. □

5. Proofs of Theorem 2.1 and Lemma 2.2

5.1. **Proof of Theorem 2.1.** Let x_* be the solution of (2.3). From Lemma 4.3, we know that there are positive values σ, ν_1 and η such that

$$\begin{aligned} S_{n+1} - S_n \leq &-\sigma|\dot{x}_{n+1}|^2 - \nu_1|\dot{x}_n|^2 \\ &+ (2\delta)\alpha_{n-1}\theta_n|Fx_{n-1}||\dot{x}_{n+1}| + (2\delta)\alpha_{n-1}\theta_n^2|Fx_{n-1}||\dot{x}_n| \\ &- \alpha_n\left(\eta|x_n - u|^2 - \alpha_n|Fx_n|^2 + 2\langle Fu, x_n - u\rangle\right), \end{aligned}$$

hence using condition (C3) yields

$$S_{n+1} - S_n \leq -P_n - \alpha_n\left(K_n + \eta|x_n - x_*|^2 + 2\langle Fx_*, x_n - x_*\rangle\right),$$

where σ, ν_1 and η are positive values and P_n, K_n and S_n are given by

$$P_n := \sigma|\dot{x}_{n+1}|^2 + \nu_1|\dot{x}_n|^2,$$
$$K_n := -(2\delta)\tau\theta_n\left(|Fx_{n-1}||\dot{x}_{n+1}| + \theta_n|Fx_{n-1}||\dot{x}_n|\right) - \alpha_n|Fx_n|^2,$$
$$S_n = |x_n - x_*|^2 + (k_n\lambda_n)\,|x_n - \bar{y}_{n-1}|^2 + 2\lambda_n\theta_n\langle Ax_*, x_{n-1} - x_*.$$

The inequality above can be rewritten as

$$S_{n+1} - (1 - \eta\alpha_n)S_n \leq -P_n - \alpha_n\eta R_n, \tag{5.1}$$

where R_n is given by

$$R_n := (1/\eta)\left(-\eta S_n + K_n + \eta|x_n - x_*|^2 + 2\langle Fx_*, x_n - x_*\rangle\right).$$

It is clear that (x_n) is a bounded sequence (according to Lemma 4.5). Thus (R_n) is also a bounded sequence (by (2.5) together with the definition of (θ_n) and the Lipschitz continuity of F). As a consequence, by the boundedness of (x_n) and applying Lemma 4.8, we deduce that there exist a positive constant β and some subsequence (S_{n_k}) such that

(k1) $\limsup_{n\to+\infty} S_n \leq -\lim_{k\to+\infty} R_{n_k}$,

(k2) $P_{n_k} \leq \beta\alpha_{n_k}$,

(k3) $\lim_{k\to+\infty}(S_{n_k+1} - S_{n_k}) = 0$,

(k4) $x_{n_k} \rightharpoonup u$ weakly as $k \to \infty$ (for some $u \in C$).

Let us prove that $u \in S$. Clearly, by (k2) and $\alpha_n \to 0$ (as $n \to \infty$) we have

$$\lim_{k\to\infty} |\dot{x}_{n_k+1}| = \lim_{k\to\infty} |\dot{x}_{n_k}| = 0. \tag{5.2}$$

It is then easily checked from $x_n - q_n = \alpha_n Fx_n$ and $\bar{y}_n = x_n + \theta_n\dot{x}_n$ (in light of the definitions of q_n and $\bar{y}_n$) that

$$\lim_{k\to\infty} |x_{n_k+1} - q_{n_k}| = \lim_{k\to\infty} |\bar{y}_{n_k} - x_{n_k+1}| = 0. \tag{5.3}$$

Hence by (k4) and invoking Lemma 4.6 we deduce that $u \in S$.

Now we focus on proving that $S_n \to 0$ as $n \to \infty$. Using (k1) and (k3) yields

$$\limsup_{n\to+\infty} S_n \leq -\lim_{k\to+\infty} F_{n_k}, \quad \text{where } F_n := R_n + (S_n - S_{n+1}). \tag{5.4}$$

From the definitions of F_n and R_n we also have

$$F_n = \frac{1}{\eta}\left(-\eta S_n + K_n + \eta|x_n - x_*|^2 + 2\langle Fx_*, x_n - x_*\rangle\right) + (S_n - S_{n+1})$$
$$= |x_n - x_*|^2 - S_{n+1} + \frac{1}{\eta}K_n + \frac{2}{\eta}\langle Fx_*, x_n - x_*\rangle,$$

so by the definition of S_n and introducing the following quantity

$$G_n := (k_{n+1}\lambda_{n+1})\,|x_{n+1} - \bar{y}_n|^2 + 2\lambda_{n+1}\theta_{n+1}\langle Ax_*, x_n - x_*\rangle,$$

we obtain

$$F_n = (|x_n - x_*|^2 - |x_{n+1} - x_*|^2) - G_n + (1/\eta)K_n + (2/\eta)\langle Fx_*, x_n - x_*\rangle. \tag{5.5}$$

In order to get the desired result we prove that $\lim_{k\to+\infty} F_{n_k} \geq 0$. In light of (5.3) we clearly have

$$\lim_{k\to\infty} (|x_{n_k} - x_*|^2 - |x_{n_k+1} - x_*|^2). \tag{5.6}$$

Recall that u and x_* belong to S, then, using the monotonicity of A gives us

$$0 \leq \langle Ax_*, u - x_*\rangle \leq \langle Au, u - x_*\rangle \leq 0,$$

so that

$$\langle Ax_*, u - x_*\rangle = 0.$$

Therefore, by (k4), it is a classical matter to check that

$$\lim_{k\to\infty} \langle Ax_*, x_{n_k} - x_*\rangle = \langle Ax_*, u - x_*\rangle = 0.$$

As a result, by (5.3) and by boundedness of the sequence $(k_n\lambda_n)$ and $(\lambda_n\theta_n)$ we obtain

$$\lim_{k\to\infty} G_{n_k} = 0.$$

It is also a simple matter to see from the definition of K_n that

$$\lim_{k\to\infty} K_{n_k} = 0.$$

Consequently, by (k4) in light of (5.5) and (2.3), we are led to

$$\lim_{k\to\infty} F_{n_k} = (2/\eta)\langle Fx_*, u - x_*\rangle \geq 0.$$

It follows immediately from (5.4) that $\limsup_{n\to\infty} S_n = 0$, which by the definition of S_n amounts to $\lim_{n\to\infty} |x_n - x_*| = 0$. □

5.2. **Analysis of the line search procedure PRGS.** In this section we focus on proving that the sequence (λ_n) generated by PRGS satisfies (2.5c).

5.2.1. *Preliminaries.* In order to describe the behavior of the step-sizes (λ_n) given by procedure PRGS, we provide the following two lemmas that was also used in [16].

The first lemma is stated as a classical result.

Lemma 5.1. *Let (u_n) and (b_n) be sequences of positive real numbers verifying*
$u_{n+1} = \nu(\delta + u_n)^{1/2}$*, for some positive values δ and ν.*
Then (u_n) converges as $n \to \infty$ to the unique positive value l_ν verifying $l_\nu^2 - \nu l_\nu - \nu\delta = 0$, namely $l_\nu = (1/2)(\nu + \sqrt{\nu^2 + 4\nu\delta})$. Moreover (u_n) is increasing if $u_0 \in [0, l_\nu)$ and non-increasing otherwise.

Lemma 5.2. *Let $\delta \in (0,\infty)$ and $h_n = \frac{\lambda_n}{\lambda_{n-1}}$, where (λ_n) is a sequence of positive real numbers. Suppose that (h_n) is bounded away from zero and assume for some integers m and N that the equality $h_n = (\delta + h_{n-1})^{1/2}$ is successively satisfied for $n = m, .., m + N$. Then, for some positive value C_1 (independent of m and N), we have $\lambda_n \geq C_1\lambda_{m-1}$ (for $n = m, .., m + N$).*

Proof. Let us suppose without loss of generality that N is some large enough integer. As (h_n) is assumed to be bounded below by some positive value δ_0, by Lemma 5.1 (also noticing that $l_1 > 1$) we observe that there exist $l \in (1, l_1)$ and some integer N_0 (that are independent of m and N) and such that $h_n \geq l$ for $n = m + N_0, .., m + N$, which by induction amounts to

(f1) $\lambda_n \geq l^{N-N_0} \lambda_{m+N_0}$ for $n = m + N_0, .., m + N$.

Moreover, recalling that $(h_n)_{m \leq n \leq N+m}$ is increasing (by Lemma 5.1), we have

(f2) $h_n \geq h_{m-1} \geq \delta_0$ for $n = m, .., N + m$,

so that

(f3) $\lambda_n \geq (\delta_0)^{n-m} \lambda_{m-1}$ for $n = m, .., N + m$.

In particular using (f3), we obtain

(f4) $\lambda_{m+N_0} \geq (\delta_0)^{N_0} \lambda_{m-1}$.

Now, combining (f1) and (f4) yields

(f5) $\lambda_n \geq l^{N-N_0} (\delta_0)^{N_0} \lambda_{m-1}$ for $n = m + N_0, .., N + m$.

Furthermore (f3) leads us to

(f6) $\lambda_n \geq (\min\{1, \delta_0\})^{N_0-1} \lambda_{m-1}$ for $n = m, .., N_0 + m - 1$.

The desired result follows from (f5) and (f6) and recalling that $l > 1$. □

5.2.2. *Proof of Lemma 2.2.* The proof follows that same lines as in [16] and it can be divided into the following steps:

(A) Let us prove that procedure PRGS generates a bounded sequence of integers (i_n). Indeed, by definition of λ_n and setting $h_n = \frac{\lambda_n}{\lambda_{n-1}}$, we have $h_n \leq (\delta + h_{n-1})^{1/2}$ for $n \geq 0$. Then it is obviously checked by invoking Lemma 5.1 that (h_n) is a bounded sequence. Therefore, the quantity $t_{n,i}$ in procedure PRGS satisfies $t_{n,i} \leq \mu \gamma^i$ for some positive value μ. Hence from Lemma 2.1 we observe that (2.7) is satisfied whenever $\gamma^i \leq (c/\mu)$, namely $i \geq \frac{\ln(c/\mu)}{|\ln \gamma|}$. It is immediately deduced that (i_n) is bounded.

(B) Clearly, by definition of λ_n and by $h_n = \frac{\lambda_n}{\lambda_{n-1}}$, we have $h_n = \gamma^{i_n} (\delta + h_{n-1})^{1/2}$. Hence it is readily checked from Lemma 5.1 that (h_n) is bounded below (since (i_n) is bounded) by some positive value denoted δ_0.

(C) Let us prove that there exists a subsequence $(i_{n_k})_k$ such that $i_{n_k} \geq 1$. Let us proceed by contradiction by assuming that there exits some integer n_0 such that $i_n = 0$ for $n \geq n_0$. Then from $h_n = \frac{\lambda_n}{\lambda_{n-1}}$ we obtain $h_n = (\delta + h_{n-1})^{1/2}$ for $n \geq n_0$, which entails (by Lemma 5.1) that (h_n) converges as $n \to \infty$ to the positive value $l_1 = (1/2)(1 + \sqrt{1 + 4\delta})$. Consequently, observing that $l_1 > 1$, we deduce that (λ_n) is not bounded, which contradicts the condition $(\lambda_n) \subset (0, \bar{\nu}]$ in procedure PRGS .

(D) Let (i_{n_k}) be the sequence introduced in (C). Clearly, by Lemma 2.1 and by $i_{n_k} \geq 1$, we know for any $k \geq 0$ that $t_{n_k, i_{n_k}-1} > c$ (thanks to the definition of i_n), or equivalently $\lambda_{n_k} > (1/\gamma)c$. Now assume for some $k \geq 0$ that there exists some positive integer d_k such that $i_n = 0$ for $n = i_{n_k+1}, ..., d_k$ and $i_{d_k+1} = 1$. Then by $h_n = \frac{\lambda_n}{\lambda_{n-1}}$ we obtain $h_n = (\delta + h_{n-1})^{1/2}$ for $n = i_{n_k+1}, ..., d_k$. So from (B) and Lemma 5.2, and for some positive constant C_0 (independent of k) we get $\lambda_n \geq C_0 \lambda_{i_{n_k}} \geq C_0(1/\gamma)c$ (for $n = i_{n_k+1}, ..., d_k$). So we immediately reach the desired conclusion. □

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