Fixed Point Theory, 19(2018), No. 2, 631-642
DOI: 10.24193/fpt-ro.2018.2.50
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

PERIODIC SOLUTIONS OF A CLASS OF NONLINEAR PLANAR SYSTEMS

ZAITAO LIANG

School of Mathematics and Big Data,
Anhui University of Science and Technology, Huainan 232001,
Anhui, China
Department of Mathematics, College of Science,
Hohai University, Nanjing 210098, China
E-mail: liangzaitao@sina.cn

Abstract. In this paper, by the use of a generalized version of the Poincaré-Birkhoff fixed point theorem due to Franks, we prove the existence of at least two geometrically distinct periodic solutions for a class of nonlinear planar systems, and at least one of them is unstable.
Key Words and Phrases: Nonlinear planar system, exact symplectic map, boundary twist condition, Poincaré-Birkhoff fixed point theorem.
2010 Mathematics Subject Classification: 34C25, 37C25.

1. Introduction

It is well known that the classical forced pendulum equation

$$x'' + a\sin x = p(t),$$

has at least two geometrically distinct T-periodic solutions, where a is a positive real number and p is a T-periodic C^1 function with mean value zero over a period. Here we say that two T-periodic solutions are geometrically distinct if they do not differ by a multiple of 2π. The existence of at least one periodic solution was proved by Hamel [21], Dancer [7] and Willem [34], while the second periodic solution was found by Mawhin and Willem [29] using mountain pass arguments. In 1988, Franks [11] obtained a generalized version of Poincaré-Birkhoff fixed point theorem so that he could deal with the equation

$$x'' + \frac{\partial V}{\partial x} = p(t),$$

where $V(x)$ is periodic in x, and he proved that such a pendulum-type equation has at least two geometrically distinct periodic solutions. Recently, similar results were obtained in [27] for the forced relativistic pendulum equation

$$\left(\frac{x'}{\sqrt{1-(x')^2}}\right)' + a\sin x = p(t).$$

The aim of this paper is to generalize the above results to the following nonlinear planar system

$$\begin{cases} x' = f(y) + e_1(t), \\ y' = g(x) + e_2(t), \end{cases} \tag{1.1}$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$ is a reversible function satisfying the locally Lipschitz condition, $g \in C^1(\mathbb{R}/T_1\mathbb{Z}, \mathbb{R})$, $e_1, e_2 \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, by the use of the version of the Poincaré-Birkhoff fixed point theorem proved by Franks in [11]. Moreover, by the statement on the instability proved by Marò in [27], we prove that at least one of the two geometrically distinct periodic solutions of the system (1.1) is unstable.

The Poincaré-Birkhoff fixed point theorem was originally conjectured by Poincaré [30] in 1912 when he studied the restricted three body problems, and was first proved by Birkhoff [1, 2] in 1913. In broad terms, it states the existence of at least two fixed points of an area preserving homeomorphism of the planar annulus which keeps both boundary circles invariant and twists them in opposite directions. A serious problem occurs when we are trying to apply the Poincaré-Birkhoff theorem. In fact, the hypothesis of the invariance of the strip is hard to verify and not always guaranteed. To overcome this difficulty, Ding [8, 9], Franks [11, 12], Campos et al. [5], Le Calvez and Wang [24], Martins and Ureña [28], Margheri et al. [26] and Rebelo [31] offered new versions of the Poincaré-Birkhoff fixed point theorem in which the boundary invariance assumption was removed. During the past few decades, as a powerful tool, such a theorem and its modified versions have been widely applied in the search of periodic solutions of differential equations in a variety of situations, see [3, 4, 10, 13, 14, 15, 18, 22, 32, 33, 35, 36] and the references therein.

In particular, Poincaré-Birkhoff fixed point theorem and its modified versions have also been applied to study the existence of periodic solutions for planar systems [16, 17, 19, 20]. During the last two decades, the existence and the dynamical behaviors of periodic solutions of nonlinear planar systems have been studied in the literature. For example, in [23], by using a topological method, the existence of periodic solutions of a periodically perturbed planar autonomous system was studied when the unperturbed autonomous system has a limit cycle. By looking for the fixed points of the successor map, Liu et al. [25] studied the coexistence of unbounded solutions and periodic solutions for a class of planar systems with asymmetric nonlinearities. Based on the computation of the corresponding Birkhoff normal forms, Chu et al. [6] gave a sufficient condition for the stability of the equilibrium of a nonlinear planar system. Compared with the methods used in above papers, the method used in this paper, a generalized version of the Poincaré-Birkhoff fixed point theorem due to Franks [11], has some advantages. For example, we can not only show the existence of periodic solutions, but also give the dynamical behavior (such as instability) of periodic solutions.

The rest of this paper is organized as follows. In Section 2, we recall the Poincaré-Birkhoff fixed point theorem [11, 12]. In Section 3, we prove the existence and instability of periodic solutions for the system (1.1). In Section 4, we study the existence and instability of the periodic and subharmonic solutions with winding number for the system (1.1).

2. Preliminaries

Consider two strips $A = \mathbb{R} \times [-\alpha, \alpha]$ and $B = \mathbb{R} \times [-\beta, \beta]$, where $\beta > \alpha > 0$. We will work with a C^k-diffeomorphism $f : A \to B$ defined by

$$f(\theta, r) = (Q(\theta, r), P(\theta, r)),$$

where $Q, P : \mathbb{R}^2 \to \mathbb{R}$ are functions of class C^k satisfying the periodicity conditions

$$Q(\theta + 1, r) = Q(\theta, r) + 1, \quad P(\theta + 1, r) = P(\theta, r).$$

Such generalized periodicity conditions tell us that the map is the lift to $\mathbb{R}^2$ of the corresponding map $\bar{f} : \bar{A} \to \bar{B}$, where $\bar{A} = \mathbb{R}/\mathbb{Z} \times [-\alpha, \alpha]$ and $\bar{B} = \mathbb{R}/\mathbb{Z} \times [-\beta, \beta]$. After the identification $\theta + 1 = \theta$, the domain of f can be interpreted as an annulus or a cylinder. We shall think that it is a cylinder with vertical coordinator r and the variable θ as an angle. We say that f is isotopic to the inclusion, if there exists a function $H : A \times [0, 1] \to B$ such that for every $\lambda \in [0, 1]$, $H_\lambda(x) = H(\lambda, x)$ is a homeomorphism with $H_0(x) = f(x)$ and $H_1(x) = x$. The class of the maps satisfying the above characteristics will be indicated by $\varepsilon^k(A)$.

We say that $f \in \varepsilon^1(A)$ is exact symplectic if there exists a smooth function $V = V(\theta, r)$ such that

$$\mathrm{d}V = P\mathrm{d}Q - r\mathrm{d}\theta \text{ and } V(\theta + 1, r) = V(\theta, r). \tag{2.1}$$

Let us state the version of the Poincaré-Birkhoff fixed point theorem which we will use in the proof of our main result.

Theorem 2.1. [11, 27] *Let $f : A \to B$ be an exact symplectic diffeomorphism belonging to $\varepsilon^2(A)$ such that $f(A) \subset \mathrm{int}(B)$. Suppose that there exists $\epsilon > 0$ such that*

$$\begin{aligned} &Q(\theta, \alpha) - \theta > \epsilon, &&\forall\, \theta \in [0, 1), \\ &Q(\theta, -\alpha) - \theta < -\epsilon, &&\forall\, \theta \in [0, 1). \end{aligned} \tag{2.2}$$

Then f has at least two distinct fixed points p_1 and p_2 in A such that $p_1 - p_2 \neq (k, 0)$ for every $k \in \mathbb{Z}$. Moreover, at least one of the fixed points is unstable if f is analytic.

Hypothesis (2.2) is called the boundary twist condition. Theorem 2.1 is a slight modified version of Poincaré-Birkhoff fixed point theorem proved by Franks in [11, 12] and the statement on the instability was proved by Marò in [27]. Here we say that a fixed point p_1 of a one-to-one map $f : U \subset \mathbb{R}^N \to \mathbb{R}^N$ is said to be stable in the sense of Lyapunov if for every neighbourhood U_{p_1} of p_1 there exists another neighbourhood $U^* \subset U_{p_1}$ such that, for each $n > 0$, $f^n(U^*)$ is well defined and $f^n(U^*) \subset U_{p_1}$.

3. Existence and instability of periodic solutions

In order to avoid complicated changes of variable and adjustments, throughout the paper, we suppose that $T_1 = 1$. For the sake of convenience, we list the following assumptions:

(A_1) there exists a constant $\rho > 0$ such that

$$\begin{cases} f(v) + \max\limits_{t\in[0,T]} e_1(t) < 0, \text{ for } v < -\rho, \\ f(v) + \min\limits_{t\in[0,T]} e_1(t) > 0, \text{ for } v > \rho. \end{cases} \tag{3.1}$$

(A_2) g and e_2 have mean value equal to zero, i.e.,

$$\int_0^1 g(s)\mathrm{d}s = 0 \quad \text{and} \quad \frac{1}{T}\int_0^T e_2(t)\mathrm{d}t = 0.$$

(A_3) there exist two constants $c_1 \geq 0$ and $c_2 \geq 0$ such that

$$|f(y)| \leq c_1|y| + c_2, \text{ for every } y \in \mathbb{R}.$$

Before stating the main result of this section, we first prove several preliminary results. For the system (1.1), we can get the Hamiltonian

$$H(x, y, t) = \int_0^y f(s)\mathrm{d}s + e_1(t)y - \int_0^x g(s)\mathrm{d}s - e_2(t)x,$$

and the system (1.1) can be seen as

$$\begin{cases} x' = \frac{\partial H}{\partial y} = f(y) + e_1(t), \\ y' = -\frac{\partial H}{\partial x} = g(x) + e_2(t). \end{cases} \tag{3.2}$$

Let us denote by $(x, y) = (x(t, \theta, r), y(t, \theta, r))$ the solution of the system (3.2) satisfying the initial condition

$$\begin{cases} x(0) = \theta, \\ y(0) = r. \end{cases} \tag{3.3}$$

By the assumption (A_3), $g \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $e_1, e_2 \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, it is easy to see that there exist two constants $d_1 \geq 0$ and $d_2 \geq 0$ such that

$$|\nabla H(x, y, t)| \leq d_1\sqrt{x^2 + y^2} + d_2, \text{ for a.e } t \in [0, T] \text{ and every } (x, y) \in \mathbb{R}^2,$$

which is a typical sufficient condition of the following result, see [20].

Lemma 3.1. *Assume that condition* (A_3) *holds. Then the solution* (x, y) *of the Cauchy problem* (3.2)-(3.3) *is unique and globally defined.*

Now, we can define

$$S(\theta, r) = (Q(\theta, r), P(\theta, r)) = (x(T, \theta, r), y(T, \theta, r))$$

as the Poincaré map associated to the system (3.2). Clearly, the fixed points of the Poincaré map S correspond to the T-periodic solutions of the system (3.2). It follows from 1-periodicity of the function g with respect to x and Lemma 3.1 that

$$\begin{cases} x(t, \theta + 1, r) = x(t, \theta, r) + 1, \\ y(t, \theta + 1, r) = y(t, \theta, r). \end{cases} \tag{3.4}$$

Thus

$$\begin{cases} Q(\theta + 1, r) = Q(\theta, r) + 1, \\ P(\theta + 1, r) = P(\theta, r), \end{cases}$$

which implies that the Poincaré map S is defined on the cylinder.

From the theorem of differentiability with respect to the initial condition and taking into account that all the partial derivatives of $H(x,y,t)$ with respect to the variables (x,y) are of order equal to 2 are continuous in the variables (x,y,t), we know that $S \in C^2(A)$. Moreover, it follows from the solution (x,y) is unique and globally defined that S is a diffeomorphism of A. The isotopy to the identity is given by the flow

$$\begin{aligned}\varphi_\lambda(\theta,r) &= \varphi((1-\lambda)T,\theta,r)\\ &= (x((1-\lambda)T,\theta,r), y((1-\lambda)T,\theta,r)),\ \ \lambda\in[0,1].\end{aligned}$$

Notice that $\varphi_0(\theta,r)=S(\theta,r)$, $\varphi_1(\theta,r)=(\theta,r)$ and this isotopy is valid on the cylinder. Now we can assert that the Poincaré map $S\in\varepsilon^2(A)$.

Firstly, we prove the Poincaré map S is exact symplectic.

Lemma 3.2. *Assumed that* (A_1) *and* (A_2) *hold. Then the Poincaré map S associated to the system* (1.1) *is exact symplectic.*

Proof. Consider the C^1 function

$$V(\theta,r)=\int_0^T\big[\tilde F(x'(t,\theta,r)-e_1(t))+G(x(t,\theta,r))+e_2(t)x(t,\theta,r)\big]\mathrm{d}t$$

where $'=\frac{\mathrm{d}}{\mathrm{d}t}$, $\tilde F(\xi)=\int_0^\xi f^{-1}(s)\mathrm{d}s$, and $G(x)=\int_0^x g(s)\mathrm{d}s$. Obviously, by the assumption (A_2), we know that $G\in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R})$. Then by the 1-periodicity of the function g with respect to x and (3.4), we have

$$\begin{aligned}V(\theta+1,r)=&\int_0^T\big[\tilde F(x'(t,\theta+1,r)-e_1(t))+G(x(t,\theta+1,r))\big]\mathrm{d}t\\ &+\int_0^T e_2(t)x(t,\theta+1,r)\mathrm{d}t\\ =&\int_0^T\big[\tilde F(x'(t,\theta,r)-e_1(t))+G(x(t,\theta,r))\big]\mathrm{d}t\\ &+\int_0^T e_2(t)\big[x(t,\theta,r)+1\big]\mathrm{d}t\\ =&V(\theta,r)+\int_0^T e_2(t)\mathrm{d}t.\end{aligned}$$

Then, by (A_2), we have

$$V(\theta+1,r)=V(\theta,r).$$

Let us compute the partial derivatives of $V(\theta,r)$

$$\begin{aligned}V_\theta(\theta,r)=&\int_0^T\left[f^{-1}(x'(t,\theta,r)-e_1(t))\frac{\partial x'(t,\theta,r)}{\partial\theta}+g(x(t,\theta,r))\frac{\partial x(t,\theta,r)}{\partial\theta}\right]\mathrm{d}t\\ &+\int_0^T e_2(t)\frac{\partial x(t,\theta,r)}{\partial\theta}\mathrm{d}t.\end{aligned}$$

It follows from the second equation of the system (1.1) that

$$V_\theta(\theta,r)=\int_0^T\left[f^{-1}(x'(t,\theta,r)-e_1(t))\frac{\partial x'(t,\theta,r)}{\partial\theta}+y'(t,\theta,r)\frac{\partial x(t,\theta,r)}{\partial\theta}\right]\mathrm{d}t.\quad(3.5)$$

Integrating by parts and using the first equation of the system (1.1), we get

$$\begin{aligned}\int_0^T y'(t,\theta,r)\frac{\partial x(t,\theta,r)}{\partial\theta}\mathrm{d}t =&\left(y(t,\theta,r)\frac{\partial x(t,\theta,r)}{\partial\theta}\right)\Big|_0^T-\int_0^T y(t,\theta,r)\frac{\partial x'(t,\theta,r)}{\partial\theta}\mathrm{d}t\\ =&y(T,\theta,r)\frac{\partial x(T,\theta,r)}{\partial\theta}-y(0,\theta,r)\frac{\partial x(0,\theta,r)}{\partial\theta}\\ &-\int_0^T f^{-1}(x'(t,\theta,r)-e_1(t))\frac{\partial x'}{\partial\theta}\mathrm{d}t.\end{aligned}$$

Substituting the above equality into (3.5) gives

$$V_\theta(\theta,r)=y(T,\theta,r)\frac{\partial x(T,\theta,r)}{\partial\theta}-y(0,\theta,r)\frac{\partial x(0,\theta,r)}{\partial\theta}. \tag{3.6}$$

Analogously, we obtain that

$$V_r(\theta,r)=y(T,\theta,r)\frac{\partial x(T,\theta,r)}{\partial r}-y(0,\theta,r)\frac{\partial x(0,\theta,r)}{\partial r}. \tag{3.7}$$

By (3.6) and (3.7), we have

$$\begin{aligned}dV=&V_\theta\mathrm{d}\theta+V_r\mathrm{d}r\\ =&y(T,\theta,r)\left[\frac{\partial x(T,\theta,r)}{\partial\theta}\mathrm{d}\theta+\frac{\partial x(T,\theta,r)}{\partial r}\mathrm{d}r\right]\\ &-y(0,\theta,r)\left[\frac{\partial x(0,\theta,r)}{\partial\theta}\mathrm{d}\theta+\frac{\partial x(0,\theta,r)}{\partial r}\mathrm{d}r\right]\\ =&y(T,\theta,r)\mathrm{d}x(T,\theta,r)-y(0,\theta,r)\mathrm{d}x(0,\theta,r)\\ =&P\mathrm{d}Q-r\mathrm{d}\theta,\end{aligned}$$

which means that the function $V(\theta,r)$ satisfies (2.1). Thus, the Poincaré map S associated to the system (1.1) is exact symplectic. □

Secondly, we prove that the Poincaré map S satisfies the boundary twist condition.

Lemma 3.3. *Assumed that* (A$_1$) *holds. Then there exist constants* $\rho_*>0$ *and* $\epsilon>0$ *such that*

$$\begin{cases} Q(\theta,\rho_*)-\theta>\epsilon, & \forall\ \theta\in[0,1),\\ Q(\theta,-\rho_*)-\theta<-\epsilon, & \forall\ \theta\in[0,1).\end{cases}$$

Proof. Integrating the second equation of the system (1.1) from 0 to $t\in[0,T]$, we have

$$\begin{aligned}y(t)=&r+\int_0^t[g(x(s))+e(s)]\mathrm{d}s\\ \geq& r-T(\|g\|_\infty+\|e\|_\infty),\ \forall\ t\in[0,T].\end{aligned}$$

We can find a constant $\rho_0\geq T(\|g\|_\infty+\|e\|_\infty)+\rho>0$ such that if $r>\rho_0$ then $y(t)>\rho$, $\forall\ t\in[0,T]$. It follows from (3.1) that

$$x'(t)=f(y(t))+e_1(t)>0,\ \forall\ t\in[0,T],$$

which means that x is increasing for $t \in [0, T]$. So we can choose constant $\rho_* = \rho_0 + 1$, then we have

$$Q(\theta, \rho_*) - \theta = x(T, \theta, \rho_*) - x(0, \theta, \rho_*) > 0, \ \forall \ \theta \in [0, 1).$$

By a standard compactness argument, we can conclude that there exists $\epsilon > 0$ such that

$$Q(\theta, \rho_*) - \theta > \epsilon, \ \forall \ \theta \in [0, 1).$$

Analogously, we can conclude that

$$Q(\theta, -\rho_*) - \theta < -\epsilon, \ \forall \ \theta \in [0, 1). \qquad \square$$

Finally, we are in a position to state and prove the main result of this section.

Theorem 3.4. *Assume that* (A_1), (A_2) *and* (A_3) *hold. Then the system* (1.1) *has at least two geometrically distinct* T*-periodic solutions. Moreover, at least one of them is unstable if* f *and* g *are analytic.*

Proof. We will apply Theorem 2.1. Let us choose $A = \mathbb{R} \times [-\rho_*, \rho_*]$. Since the solutions of the system (1.1) are globally defined, one can find a larger B such that $S(A) \subset \mathrm{int}B$.

Furthermore, if f and g are analytic, we can get that the right hand side of the system (1.1) is analytic with respect to the variables (x, y). Therefore, it follows from the analytic dependence on initial conditions that the Poincaré map S is also analytic.

Note that we have proved Lemma 3.1, Lemma 3.2 and Lemma 3.3, then all conditions of Theorem 2.1 are satisfied. Thus we get that the Poincaré map S has at least two geometrically distinct fixed points, and one of them is unstable if f and g are analytic. That is, the system (1.1) has at least two geometrically distinct T-periodic solutions. Moreover, at least one of them is unstable if f and g are analytic. $\square$

As an example, let us consider the system

$$\begin{cases} x' = \frac{y}{(1+y^2)^{1/3}} + \left(\frac{1}{5}\right)^{\frac{1}{3}} \cos t + \left(\frac{1}{5}\right)^{\frac{1}{3}}, \\ y' = \sin 2\pi x + \cos t, \end{cases} \tag{3.8}$$

which can be regarded as a system of the form (1.1), where $T = 2\pi$, $T_1 = 1$,

$$f(y) = \frac{y}{(1 + y^2)^{1/3}}, \qquad g(x) = \sin 2\pi x,$$

$$e_1(t) = \left(\frac{1}{5}\right)^{\frac{1}{3}} \cos t + \left(\frac{1}{5}\right)^{\frac{1}{3}}, \quad e_2(t) = \cos t.$$

It is not difficult to verify that assumptions (A_1), (A_2) and (A_3) hold if we choose constants

$$\rho > 2, \ c_1 = 1, \ c_2 = 0.$$

Then Theorem 3.4 guarantees that the system (3.8) has at least two geometrically distinct 2π-periodic solutions, and at least one of them is unstable.

4. Periodic solutions with winding number

In this section, we study the existence of the so-called T-periodic solutions with winding number for the system (1.1), i.e. solutions (x, y) such that

$$(x(t+T), y(t+T)) = (x(t), y(t)) + (N, 0),\ N \in \mathbb{Z},\ \text{for all } t \in \mathbb{R}.$$

Such solutions are also called running solutions. Obviously, when $N = 0$, we recover the usual periodicity.

Let (x, y) be a T-periodic solution with winding number N of the system (1.1). Taking the change of variables

$$\begin{cases} u(t) = x(t) - \frac{N}{T}t, \\ v(t) = y(t), \end{cases}$$

notice that

$$\begin{aligned} u(t+T) &= x(t+T) - \frac{N}{T}(t+T) \\ &= x(t) - \frac{N}{T}t \\ &= u(t), \end{aligned}$$

which implies that the T-periodic solutions with winding number N of the system (1.1) correspond to the usual T-periodic solutions of the system

$$\begin{cases} u' = f(v) + e_1(t) - \frac{N}{T}, \\ v' = g(u + \frac{N}{T}t) + e_2(t). \end{cases} \tag{4.1}$$

Now we introduce a new assumption:

(A_4) There exist two integers N_1, N_2 with $N_1 \leq N_2$, and a constant $\rho_1 > 0$ such that

$$\begin{cases} f(v) + \max\limits_{t\in[0,T]} e_1(t) < \frac{N_1}{T},\ \text{for } v < -\rho_1, \\ f(v) + \min\limits_{t\in[0,T]} e_1(t) > \frac{N_2}{T},\ \text{for } v > \rho_1. \end{cases}$$

Proceeding as in the proof of Theorem 3.4, we can prove that the system (4.1) has at least two geometrically distinct T-periodic solutions. Moreover, at least one of them is unstable if f and g are analytic. Which means that the following conclusions hold.

Theorem 4.1. *Assumed that* (A_2), (A_3) *and* (A_4) *hold. Then for every integer* $N \in \{N_1, N_1+1, \ldots, N_2\}$, *the system* (1.1) *has at least two geometrically distinct T-periodic solutions with winding number N. Moreover, at least one of them is unstable if f and g are analytic.*

We could also look for subharmonic solutions with winding number for the system (1.1), i.e. solutions (x, y) satisfying, for some positive integer k,

$$(x(t+kT), y(t+kT)) = (x(t), y(t)) + (N, 0),\ N \in \mathbb{Z},\ \text{for all } t \in \mathbb{R}. \tag{4.2}$$

To this aim, we introduce the following assumption:

(A_5) For some positive integer k, there exist two integers N_1, N_2 with $N_1 \leq N_2$, and a constant $\rho_2 > 0$ such that

$$\begin{cases} f(v) + \max\limits_{t\in[0,T]} e_1(t) < \frac{N_1}{kT}, \text{ for } v < -\rho_2, \\ f(v) + \min\limits_{t\in[0,T]} e_1(t) > \frac{N_2}{kT}, \text{ for } v > \rho_2. \end{cases}$$

Theorem 4.2. *Assumed that* (A_2), (A_3) *and* (A_5) *hold. Then for every nonzero integer* $N \in \{N_1, N_1+1, \ldots, N_2\}$*, with* $|N|$ *and* k *are relatively prime, the system* (1.1) *has at least two geometrically distinct* k*-order subharmonic solutions with winding number* N*, satisfying* (4.2) *and* kT *is the minimal period. Moreover, at least one of them is unstable if* f *and* g *are analytic.*

Proof. By Theorem 4.1, with T replaced by kT, we get that the system (1.1) has at least two geometrically distinct solutions satisfying (4.2). Moreover, at least one of them is unstable if f and g are analytic. It remains to verify that kT is the minimal period of the subharmonic solutions with winding number N of the system (1.1).

The following is a standard procedure, one can consult for instance the arguments exposed in [10, 13, 20]. Assume by contradiction that lT is the minimal period, where $l \in \{1, 2, ..., k-1\}$, which means that there exists a nonzero integer j such that

$$x(t + lT) = x(t) + j, \ \forall\, t \in \mathbb{R}. \tag{4.3}$$

Obviously, there exist two positive integers n_1 and n_2 such that

$$n_1 l = n_2 k. \tag{4.4}$$

By (4.2), we have

$$x(t + n_2 kT) = x(t) + n_2 N, \ \forall\, T \in \mathbb{R}.$$

By (4.3), we have

$$x(t + n_1 lT) = x(t) + n_1 j, \ \forall\, t \in \mathbb{R}.$$

It follows from the above two equalities and the uniqueness of the solution (x, y) that

$$n_2 N = n_1 j,$$

i.e.,

$$\frac{n_2}{n_1} = \frac{j}{N}.$$

From (4.4), we know that

$$\frac{n_2}{n_1} = \frac{l}{k}.$$

Then we have

$$\frac{N}{k} = \frac{j}{l},$$

which is impossible because N and $|k|$ are relatively prime and j is a nonzero integer and $l \in \{1, 2, ..., k-1\}$. □

Acknowledgments. The author would like to express his great thanks to Professor Jifeng Chu and the referees for their valuable suggestions. This work was supported by the Project of Innovation in Scientific Research for Graduate Students of Jiangsu Province (No.KYZZ15_0155).

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Received: March 16, 2016; Accepted: January 19, 2017.