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ON GENERALIZED METRIC SPACES AND GENERALIZED CONVEX CONTRACTIONS

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Abstract. In this paper, we study the generalized metric introduced by Branciari. We find an induced metric of the generalized metric, by which some new properties of the generalized metric are presented. As a main result, we generalize several generalized, unified and extended fixed point theorems on generalized convex contractions.
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1. Introduction

Fixed point theorems on metric space have been successfully applied and have become a major theoretical tool in many topics such as differential equations, convex minimization, and operator theory. However, there are two problems in some practical cases: (1) the conditions in the fixed point theorems are too strong to be guaranteed; (2) the backgrounds of some questions are not on metric spaces. As an attempt to overcome these issues, we consider approximate fixed points of an α-admissible mapping (cf. [6, 7, 10]) on a generalized metric space. There have been some approximate fixed point theorems (cf. [4, 11, 12, 13]) and many variant generalizations and extensions of metric space. One of these generalizations is discussed by Branciari [3] who introduced the concept of a generalized metric space by replacing the triangle inequality by a rectangular one. Recently, some authors are keen on working on the fixed point theory on such generalized metric space (cf. [1, 2, 3, 5, 8]). We first recall the notion of a generalized metric space.

Definition 1.1. [3] For a nonempty set X, let $d : X \times X \longrightarrow [0, \infty]$ be a map satisfying the following conditions: For all $x, y \in X$, two distinct $u, v \in X$ which is different from x and y,

$$\begin{array}{ll} (GMS1) & d(x,y) = 0 \text{ if and only if } x = y, \\ (GMS2) & d(x,y) = d(y,x), \\ (GMS3) & d(x,y) \le d(x,u) + d(u,v) + d(v,y). \end{array} \tag{1.1}$$

The map d is called a generalized metric, and the pair (X, d) is called a generalized metric space, abbreviated as GMS.

One shall notice that the GMS looks like a metric space, the only difference is (GMS3), however, which makes the GMS much weaker than a metric space and full of mysteries. For the sake of completeness, we present the concepts of convergence, Cauchy sequence, completeness and continuity on a GMS as follows.

(1) A sequence $\{x_n\}$ in a GMS (X, d) is GMS convergent to x if $d(x_n, x) \to 0$ as $n \to \infty$.
(2) A sequence $\{x_n\}$ in a GMS (X, d) is GMS Cauchy sequence if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.
(3) A GMS (X, d) is called complete if every GMS Cauchy sequence in X is GMS convergent.
(4) A map $T : (X, d) \to (X, d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

It is known that the topology of GMS has some disadvantages: a generalized metric is not necessary to be continuous; a GMS convergent sequence is not necessary to be GMS Cauchy sequence; a GMS is not necessary to be Hausdorff. Hence the uniqueness of limits cannot be guaranteed. Since the bad topological properties, comparing with the usual metric space, the proofs of fixed point theorems on GMS are difficult, complex and a little tedious. In this paper, to avoid these we give some new properties of GMS in Section 2, which present a much better understanding and can also simplify the numerous steps of proofs. As a main result, we generalize several generalized, unified and extended fixed point theorems on generalized convex contractions in Section 3.

Now we give some new examples to show that GMS is wider than the usual metric space. From these examples, we could see that GMS possess some sort of relationships with bigraph.

One class of examples can be stated as follows.

Example 1.1. Let X_1, X_2 be two nonempty disjoint sets, $X = X_1 \cup X_2$, and $d : X \times X \to [0, +\infty)$ be such that $d(x_i, y_i) = 1$ for all distinct $x_i, y_i \in X_i$, $i = 1, 2$, and $d(x, x) = 0$ for all $x \in X$. In order to make (X, d) a GMS, we shall add the definitions of $d(x_1, x_2)$ for $x_i \in X_i$, $i = 1, 2$. There are some specific examples:

(1) Let $C \in (0, 1]$ be a constant, and $d(x_1, x_2) = C$ for all $x_i \in X_i$, $i = 1, 2$. One can easily check that if $C \in (0, \frac{1}{2})$, then (X, d) is a GMS but not a metric space.

(2) Let $C \in (0, 1]$ be a constant, and $d(x_1, x_2) \in [C, 3C]$ for all $x_i \in X_i$, $i = 1, 2$. This is clearly a generalization of Example 1.1 (1).

(3) Set $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,n}, \ldots\}$ for $i = 1, 2$, and $d(x_{1,n}, x_{2,m}) = \frac{1}{\min\{n,m\}}$ for $n, m \in \mathbb{N}^+$. In addition, $\frac{1}{\min\{n,m\}}$ can be replaced by $a_{\min\{n,m\}}$, where $\{a_k\}$ is a decreasing sequence which converges to zero.

The other class of examples has the following form.

Example 1.2. Assume that (X_1, d_1) and (X_2, d_2) are two GMS satisfying $d_i(x_i, y_i) < 1$ for any $x_i, y_i \in X_i$, $i = 1, 2$. Then we can define a generalized metric d on $X = X_1 \sqcup X_2$ as $d|_{X_i} = d_i$, and $d(z_1, z_2) = 1$ for any $z_i \in X_i$.

This paper is organized as follows: Section 2 is devoted to some new properties of GMS, which will be used in the sequel. In Section 3 we concentrate on fixed point theory of α-admissible operator on GMS, including four nontrivial examples. Section 4 collects the proofs of Theorems 3.1, 3.2 and 3.3 in Section 3.

2. The pseudo metric induced by generalized metric

A *pseudo metric* is the generalization of usual metric in which the distance between two distinct points can be zero. For reader's convenience, we present the definition here.

Definition 2.1. For a nonempty set X, a non-negative real-valued function $\rho : X \times X \longrightarrow [0, \infty]$ is called *pseudo metric* if there hold for every $x, y, z \in X$,

$$\begin{array}{ll} (PMS1) & \rho(x,x) = 0, \\ (PMS2) & \rho(x,y) = \rho(y,x), \\ (PMS3) & \rho(x,y) \leq \rho(x,z) + \rho(z,y). \end{array} \tag{2.1}$$

Unlike a metric space, points in a pseudo metric space need not be distinguishable, i.e., one may have $\rho(x, y) = 0$ for distinct points x and y. We first show an interesting and useful result that a generalized metric can induce a pseudo metric.

Proposition 2.1. *For a generalized metric d on X, the function $\rho : X \times X \to [0, +\infty)$ given by $\rho(x, y) = \inf\limits_{z \in X} (d(x, z) + d(z, y))$ is a pseudo metric on X.*

Proof. We see that (PMS1) and (PMS2) are obvious. It is sufficient to show $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$ for any $a, b, c \in X$. Without loss of generality, we may assume $a \neq b$ and $b \neq c$. Then the proof can be divided into some cases as follows.

Case I: $\rho(a, b) = d(a, b)$, $\rho(b, c) = d(b, c)$.

Then $\rho(a, b) + \rho(b, c) = d(a, b) + d(b, c) \geq \rho(a, c)$ holds trivially.

Case II: $\rho(a, b) < d(a, b)$, $\rho(b, c) = d(b, c)$.

By the definition of $\rho(\cdot, \cdot)$, for any $\varepsilon > 0$, there exists $z_1 \in X$ such that $z_1 \neq a$, $z_1 \neq b$ and $\rho(a, b) > d(a, z_1) + d(z_1, b) - \varepsilon$. Thus,

$$\rho(a,b)+\rho(b,c) = \rho(a,b)+d(b,c) > d(a,z_1)+d(z_1,b)+d(b,c)-\varepsilon \geq d(a,c)-\varepsilon \geq \rho(a,c)-\varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we have $\rho(a, b) + \rho(b, c) \geq \rho(a, c)$.

Case III: $\rho(a, b) = d(a, b)$, $\rho(b, c) < d(b, c)$. This case is similar to Case II.

Case IV: $\rho(a, b) < d(a, b)$, $\rho(b, c) < d(b, c)$.

For any $\varepsilon > 0$, there exists $z_1, z_2 \in X$ such that $z_1 \neq a$, $z_1 \neq b$, $z_2 \neq b$, $z_2 \neq c$, $\rho(a,b) > d(a,z_1) + d(z_1,b) - \frac{\varepsilon}{2}$ and $\rho(b,c) > d(b,z_2) + d(z_2,c) - \frac{\varepsilon}{2}$. Thus, it follows from the arbitrariness of $\varepsilon > 0$ and the inequality

$$\begin{aligned}\rho(a,b) + \rho(b,c) &> d(a,z_1) + d(z_1,b) + d(b,z_2) + d(z_2,c) - \varepsilon \\ &\geq d(a,z_2) + d(z_2,c) - \varepsilon \geq \rho(a,c) - \varepsilon\end{aligned}$$

that $\rho(a,c) \leq \rho(a,b) + \rho(b,c)$. □

Moreover, if we assume *Wilson condition* for (X,d), then the induced pseudo metric $\rho(\cdot,\cdot)$ becomes a metric. Here the Wilson condition was proposed by Wilson [15] to weaken the triangle inequality of metric space.

(W) For each pair of distinct points $a, b \in X$, there is a number $r_{a,b} > 0$ such that for every $c \in X$, $r_{a,b} \leq d(a,c) + d(c,b)$.

Proposition 2.2. *Let (X,d) be a GMS satisfying (W). Then the mapping $\rho : X \times X \to [0,+\infty)$ given by $\rho(x,y) = \inf_{z\in X} (d(x,z) + d(z,y))$ is a metric on X.*

Proof. In fact, if $a \neq b$, then by (W), for any distinct points $a, b \in X$, there exists $r_{a,b} > 0$ such that for any $z \in X$, $d(a,z) + d(z,b) \geq r_{a,b}$. Thus, we have $\rho(a,b) = \inf_{z\in X}(d(a,z) + d(z,b)) \geq r_{a,b} > 0$. Together with Proposition 2.1, $\rho(\cdot,\cdot)$ must be a metric on X. □

By Proposition 2.2, in a GMS satisfying condition (W), we have $0 < r_{a,b} \leq \rho(a,b)$ for any distinct pair a, b. We usually put $r_{a,b} = \rho(a,b)$. Furthermore, we can also prove

$$\inf_{m\in\mathbb{N}^+} \inf_{z_1,\ldots,z_m\in X} (d(x,z_1) + d(z_1,z_2) + \ldots + d(z_{m-1},z_m) + d(z_m,y)) = \rho(x,y).$$

Lemma 2.1. *Let (X,d) be a GMS and $x_1, x_2, \ldots, x_{2n} \in X$ satisfy $x_1 \neq x_2$, $x_2 \neq x_3$, $\ldots$, $x_{2n-1} \neq x_{2n}$, where n is a given positive integer. Then we have*

$$d(x_1,x_{2n}) \leq d(x_1,x_2) + d(x_2,x_3) + \ldots + d(x_{2n-1},x_{2n}). \tag{2.2}$$

Proof. Obviously, (2.2) holds for $n = 1$. We then show (2.2) for $n = 2$, i.e.,

$$d(x_1,x_4) \leq d(x_1,x_2) + d(x_2,x_3) + d(x_3,x_4) \tag{2.3}$$

whenever $x_1 \neq x_2$, $x_2 \neq x_3$, $x_3 \neq x_4$. In fact, if we further assume that $x_1 \neq x_3$ and $x_2 \neq x_4$, (2.3) holds from the definition of generalized metric. If $x_1 = x_3$, then $d(x_1,x_4) = d(x_3,x_4) \leq d(x_1,x_2) + d(x_2,x_3) + d(x_3,x_4)$. Similarly, (2.3) also holds if $x_2 = x_4$.

Now, we assume that (2.2) holds for n ($n \geq 2$). Then for $n+1$, we have the following cases.

Case I: $x_1 \neq x_{2n}$.

$$\begin{aligned}d(x_1,x_{2n+2}) &\leq d(x_1,x_{2n}) + d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2}) \\ &\leq \sum_{k=1}^{2n-1} d(x_k,x_{k+1}) + d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2}) \\ &= d(x_1,x_2) + d(x_2,x_3) + \ldots + d(x_{2n+1},x_{2n+2}).\end{aligned}$$

Case II: $x_3 \neq x_{2n+2}$.

$$\begin{aligned} d(x_1, x_{2n+2}) &\leq d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_{2n+2}) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + \sum_{k=3}^{2n+1} d(x_k, x_{k+1}) \\ &= d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Case III: $x_1 = x_{2n}$ and $x_3 = x_{2n+2}$.

$$\begin{aligned} d(x_1, x_{2n+2}) = d(x_3, x_{2n}) &\leq \sum_{i=3}^{2n-1} d(x_i, x_{i+1}) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Therefore, by mathematical induction, we have completed the proof. □

Remark 2.1. On many proofs of fixed point theorems in GMS, for proving the Cauchy property of a sequence $\{x_n\}$, the authors often first suppose $x_n \neq x_{n+1}$ for every n and then prove '$x_n \neq x_m$ for all $n \neq m$'. Now, according to Lemma 2.1, this step can be simplified.

Let $\{x_i\}_{i=1}^m$ be a finite length sequence in X. For $i = 1, 2, \ldots, m-1$, we set

$$\chi_i = \begin{cases} 1, & \text{if } x_i \neq x_{i+1} \\ 0, & \text{if } x_i = x_{i+1} \end{cases}$$

and

$$\chi(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m-1} \chi_i.$$

Assume $\rho(x, y) = \inf_{z \in X} (d(z, x) + d(z, y))$ and $\rho^\circ(x, y) = \inf_{z \in X \setminus \{x,y\}} (d(x, z) + d(z, y))$ for $x, y \in X$, then we have

Theorem 2.1. *Let (X, d) be a GMS and $x_1, x_2, \ldots, x_m$ be given points in X. If*

$$d(x_1, x_m) > d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{m-1}, x_m), \tag{2.4}$$

then we have

(1) $d(x_1, x_m) > \rho(x_1, x_m)$;
(2) $\chi(x_1, \ldots, x_m)$ *is even;*
(3) $\rho(x_i, x_{i+1}) = d(x_i, x_{i+1})$, $i = 1, \ldots, m-1$;
(4) *For a given i, if $x_i \neq x_{i+1}$, then*

$$\rho^\circ(x_i, x_{i+1}) \geq d(x_i, x_{i+1}) + d(x_1, x_m) - \sum_{j=1}^{m-1} d(x_j, x_{j+1}).$$

Proof. (1) Since ρ is a pseudo metric, and $\rho(\cdot, \cdot) \leq d(\cdot, \cdot)$, we have

$$\begin{aligned} \rho(x_1, x_m) &\leq \rho(x_1, x_2) + \rho(x_2, x_3) + \ldots + \rho(x_{m-1}, x_m) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{m-1}, x_m) < d(x_1, x_m). \end{aligned}$$

(2) Without loss of generality, we may assume $x_i \neq x_{i+1}$, $i = 1, 2, \ldots, m-1$. Hence, combining with Lemma 2.1, it can be easily shown that $\chi(x_1, x_2, \ldots, x_m) = m - 1$ must be even.

(3) It is sufficient to prove $d(x_i, x_{i+1}) = \rho(x_i, x_{i+1})$ for $i = 1, \ldots, 2n$, if there holds

$$d(x_1, x_{2n+1}) > d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{2n}, x_{2n+1})$$

for $x_1 \neq x_2$, $x_2 \neq x_3$, $\ldots$, $x_{2n} \neq x_{2n+1}$. Suppose the contrary that there exists $i \in \{1, \ldots, 2n\}$ satisfying $d(x_i, x_{i+1}) > \rho(x_i, x_{i+1})$, then by the definition of $\rho(x_i, x_{i+1})$ there exists $z \in X \backslash \{x_i, x_{i+1}\}$ such that $d(x_i, z) + d(z, x_{i+1}) < d(x_i, x_{i+1})$. By Lemma 2.1, we have

$$\begin{aligned} d(x_1, x_{2n+1}) &\leq d(x_1, x_2) + \ldots + d(x_{i-1}, x_i) + d(x_i, z) + d(z, x_{i+1}) \\ &\quad + d(x_{i+1}, x_{i+2}) + \ldots + d(x_{2n}, x_{2n+1}) \\ &< d(x_1, x_2) + \ldots + d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \\ &\quad + d(x_{i+1}, x_{i+2}) + \ldots + d(x_{2n}, x_{2n+1}), \end{aligned}$$

which is a contradiction.

(4) In fact, for z different from x_i and x_{i+1}, we know from (2) that

$$\chi(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_m) \text{ is odd.}$$

Hence, we have

$$\begin{aligned} &d(x_1, x_2) + \ldots + d(x_{i-1}, x_i) + d(x_i, z) + d(z, x_{i+1}) \\ &\qquad + d(x_{i+1}, x_{i+2}) + \ldots + d(x_{m-1}, x_m) \geq d(x_1, x_m) \end{aligned}$$

and then

$$d(x_i, z) + d(z, x_{i+1}) \geq d(x_i, x_{i+1}) + d(x_1, x_m) - \sum_{j=1}^{m-1} d(x_j, x_{j+1}).$$

Finally, by the arbitrariness of z, we have

$$\rho^\circ(x_i, x_{i+1}) \geq d(x_i, x_{i+1}) + d(x_1, x_m) - \sum_{j=1}^{m-1} d(x_j, x_{j+1}). \qquad \square$$

3. Generalized convex contractions on GMS and metric space

For completeness, we first present some definitions appeared in literatures.

Definition 3.1 ([14])**.** Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an *α-admissible* mapping if

$$x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 3.2 ([13, 9])**.** Let (X, d) be a GMS, $T : X \to X$ be a mapping and $\varepsilon > 0$ be a given real number. A point $x_0 \in X$ is said to be an *ε-fixed point*(*approximate fixed point*) of T if $d(x_0, Tx_0) < \varepsilon$. We say that T has the *approximated fixed point property* if for any $\varepsilon > 0$, there exists an ε-fixed point of T, equivalently, $\inf\limits_{x \in X} d(x, Tx) = 0$.

It is easy to verify the following proposition.

Proposition 3.1 ([11])**.** *Let (X,d) be a GMS and $T : X \to X$ be a mapping such that there exists $x \in X$ satisfying $d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$, then T has the approximate fixed point property.*

Definition 3.3 ([6])**.** Let (X,d) be a GMS and $\alpha : X \times X \to [0,+\infty)$ be a mapping. The metric space X is said to be *α-complete* if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in X.

We say that T is an *α-continuous* mapping on (X,d) if for each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ there holds $Tx_n \to Tx$ as $n \to \infty$.

Definition 3.4 ([11])**.** Let X be a nonempty set and $\alpha : X \times X \to [0,+\infty)$ be a mapping. We say that X has the *property (H)* whenever for each $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Now we start our notions and results.

Definition 3.5. Let (X,d) be a GMS and $\alpha : X \times X \to [0,+\infty)$ be a mapping. The mapping $T : X \to X$ is called a *generalized convex contraction* with the based mapping α if there exists $\lambda \in [0,1)$ such that: $\alpha(x,y) \geq 1$ implies

$$d(T^2x, T^2y) \leq \lambda \max\{d(x,Tx), d(y,Ty), d(Tx,T^2x), d(Ty,T^2y), d(x,y), d(Tx,Ty)\}.$$

Theorem 3.1. *Let (X,d) be a GMS and T be a generalized convex contraction with the based mapping α.*

(1) Assume there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$, then T has the approximate fixed point property.

(2) In addition, if T is α-continuous and (X,d) is an α-complete metric space, then T has a fixed point.

(3) Furthermore, if we add the condition (H+) below, then T must has a unique fixed point.

(H+) For any x, y, there exists z such that $\alpha(x,z) \geq 1$, $\alpha(y,z) \geq 1$, $\alpha(Tz,z) \geq 1$, $\alpha(T^2z,z) \geq 1$.

Note that if we remove '$\alpha(x_0, T^2x_0) \geq 1$', then we see from the example below that Theorem 3.1 (2) will be false.

Example 3.1. Let $X = \{0, 1, 2, \ldots, n, \ldots\}$. Take $d : X \times X \to [0,+\infty)$ such that:

$$d(q,p) = d(p,q) = \begin{cases} 0 & \text{if } p = q, \\ 1 & \text{if } |p-q| \text{ is an even positive number}, \\ 2^{-\min\{p,q\}} & \text{if } |p-q| \text{ is an odd positive number.} \end{cases}$$

By Example 1.1 we can check that (X,d) is a complete generalized metric space.

Let $\alpha : X \times X \to [0,+\infty)$ be a map satisfying $\alpha(n, n+1) = \alpha(n+1, n) = 1$, $\alpha(n, n+k) = \alpha(n+k, n) = 0$ for any $n \in X$, $k = 2, \ldots$, and $T : X \to X$ be a map satisfying $T(n) = n+1$ for each $n \in X$. Then T has no fixed points, but T satisfy all the conditions except that $\alpha(x_0, T^2x_0) \geq 1$ and $\alpha(z, T^2z) \geq 1$ for any $x_0, z \in X$.

However, in a metric space, the conditions '$\alpha(x_0, T^2x_0) \geq 1$' and '$\alpha(T^2z, z) \geq 1$', can be removed. Similar to Theorem 3.1, on metric space we have

Theorem 3.2. *Let (X, d) be a metric space and T be a generalized convex contraction with the based mapping α. Assume there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has the approximate fixed point property.*

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point. Furthermore, if we add the below hypotheses, then T must has a unique fixed point.

(1) For any fixed points u and v, there exists z such that $\alpha(u, z) \geq 1$, $\alpha(v, z) \geq 1$ and $\alpha(Tz, z) \geq 1$.

(2) For any fixed points u and v, there exists z such that $\alpha(u, z) \geq 1$, $\alpha(v, z) \geq 1$, and $\lambda < \frac{1}{2}$ associated with T.

The example below shows that if we replace the conditions (1) or (2) by Hypothesis (H) then the uniqueness of fixed point may not hold.

Example 3.2. Consider a subset of $l^1(\mathbb{R})$, $X = \{0, v, z_1, z_2, \ldots\}$, with the induced metric $d(x, y) = \|x - y\|_1$, where

$$v = (\frac{1}{6}, 0, 0, \ldots),\ z_1 = (1, 0, 0, \ldots),\ z_2 = (0, 1, 0, \ldots), \ldots.$$

Let T be a mapping such that $Tv = v$, $T0 = 0$ and $Tz_i = z_{i+1}$, and $\alpha : X \times X \to \mathbb{R}$ be a symmetric mapping such that $\alpha(0, z_i) = \alpha(v, z_i) = 1$, $\alpha(x, y) = 0$ for other pair $(x, y) \in X \times X$, where $i \in \mathbb{N}^+$. Then it can be easily shown that

$$\begin{aligned} d(0, T^{n+2}z) &= 1 \leq \frac{4}{3} \\ &= \frac{2}{3} \max\{d(T^n z, T^{n+1}z), d(T^{n+1}z, T^{n+2}z), d(0, T^n z), d(0, T^{n+1}z)\}, \\ d(v, T^{n+2}z) &\leq \frac{7}{6} \leq \frac{4}{3} \\ &= \frac{2}{3} \max\{d(T^n z, T^{n+1}z), d(T^{n+1}z, T^{n+2}z), d(v, T^n z), d(v, T^{n+1}z)\}. \end{aligned}$$

Now, we can see that T has two distinct fixed points.

Theorem 3.3. *Given a metric space (X, d), let two mappings $\alpha : X \times X \to R$ and $T : X \to X$ satisfy the property: $\alpha(x, y) \geq 1$ implies*

$$\begin{aligned} d(T^2x, T^2y) \leq{}& a_1 d(x, Tx) + a_2 d(Tx, T^2x) + b_1 d(y, Ty) \\ &+ b_2 d(Ty, T^2y) + a d(Tx, Ty) + b d(x, y), \end{aligned}$$

where a, b, a_1, a_2, b_1, b_2 are nonnegative real numbers satisfying

$$a + b + a_1 + a_2 + b_1 + b_2 < 1.$$

Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property. In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point. Furthermore, T has a unique fixed point provide that X has the property (H).

The following example indicates that if we replace 'metric' by 'generalized metric' in Theorem 3.3 then the uniqueness of fixed point will not hold.

Example 3.3. Assume that $d : X \times X \to [0, +\infty)$ satisfies $d(u, v) = 1$, $d(z_i, z_j) = 1$, $i \neq j$, $d(u, z_i) = d(v, z_i) = \frac{1}{7}$, $i = 1, 2, \ldots$, $d(x, x) = 0$, for all $x \in X$. Here $X = X_1 \cup X_2$, $X_1 \cap X_2 \neq \emptyset$, $X_1 = \{u, v\}$ and $X_2 = \{z_1, \ldots, z_n, \ldots\}$. The mapping $T : X \to X$ is defined as $Tu = u$, $Tv = v$, $Tz_i = z_{i+1}$, $i = 1, 2, \ldots$.

If we take $a = b = a_1 = a_2 = b_1 = b_2 = \frac{1}{7}$, $\alpha(u, z_i) = \alpha(v, z_i) = \alpha(z_i, u) = \alpha(z_i, v) = 1$, otherwise $\alpha(x, y) = 0$. Then by Example 1.1, (X, d) is a GMS, and

$$d(u, z_{n+2}) = \frac{1}{7} < \frac{1 + 1 + \frac{1}{7} + \frac{1}{7}}{7} = \frac{d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + d(z_n, u) + d(z_{n+1}, u)}{7}$$

holds for any $n \in \mathbb{N}^+$. However, T has two fixed points: u and v.

As corollaries of Theorem 3.3, we can obtain the main results of [11] as follows.

Corollary 3.1 (Theorems 3.2 and 3.4 in [11])**.** *Let (X, d) be a metric space and $\alpha : X \times X \to R$ be a mapping and $T : X \to X$ satisfy $\alpha(x, y) \geq 1 \Rightarrow$*

$$d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y).$$

Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property. In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point. Moreover, adding property (H) to the hypotheses of X and α, we obtain uniqueness of the fixed point of T.

Corollary 3.2 (Theorem 3.10 in [11])**.** *Let (X, d) be a metric space and $\alpha : X \times X \to R$ be a mapping and $T : X \to X$ satisfy $\alpha(x, y) \geq 1 \Rightarrow$*

$$d(T^2x, T^2y) \leq a_1d(x, Tx) + a_2d(Tx, T^2x) + b_1d(y, Ty) + b_2d(Ty, T^2y).$$

Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property. In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point. Moreover, T has a unique fixed point provide that X has the property (H).

The following example shows that in some practical cases Theorem 3.2 cannot be replaced by Theorem 3.3.

Example 3.4. Set $X = \{x_1, x_2, x_3, x_4, x_5\}$, and the symmetric mapping $\alpha : X \times X \to R$ is defined as $\alpha(x_1, x_2) = \alpha(x_2, x_3) = 1$, otherwise $\alpha(x, y) = 0$. We define the metric d on X as follows: $d(x_1, x_2) = 1$, $d(x_2, x_3) = 0.1$, $d(x_3, x_4) = 0.9$, $d(x_4, x_5) = 0.81$, $d(x_1, x_3) = 0.9$, $d(x_2, x_4) = 0.9$, $d(x_3, x_5) = 0.9$, $d(x_1, x_4) = 0.9$, $d(x_2, x_5) = 1$, $d(x_1, x_5) = 1$. One can check that (X, d) is a complete metric space.

$T : X \to X$ be such that $Tx_i = x_{i+1}$, $i = 1, 2, 3, 4$, $Tx_5 = x_5$. Choosing $\lambda = 0.9$, we can easily check that

$$d(x_3, x_4) = 0.9 = \lambda \max\{d(x_1, x_2), d(x_2, x_3)\}$$

and

$$d(x_4, x_5) = 0.81 = \lambda \max\{d(x_2, x_3), d(x_3, x_4)\}.$$

So by Theorem 3.2, T has a fixed point. Now we point out that Theorem 3.3 cannot be applied here. Indeed, for $n = 1, 2$, we have

$$\begin{aligned} d(x_{n+2}, x_{n+3}) \le a_1 d(x_n, x_{n+1}) + a_2 d(x_{n+1}, x_{n+2}) + b_1 d(x_{n+1}, x_{n+2}) \\ + b_2 d(x_{n+2}, x_{n+3}) + a d(x_{n+1}, x_{n+2}) + b d(x_n, x_{n+1}). \end{aligned}$$

So by simplification, we get

$$d(x_{n+2}, x_{n+3}) \le l_1 d(x_n, x_{n+1}) + l_2 d(x_{n+1}, x_{n+2})$$

for $n = 1, 2$, where $l_1 = \frac{a_1+b}{1-b_2}$, $l_2 = \frac{a_2+b_1+a}{1-b_2}$ be two nonnegative numbers satisfying $l_1 + l_2 < 1$.

Taking $n = 1, 2$ respectively, we have $0.9 \le l_1 1 + l_2 0.1$ and $0.81 \le l_1 0.1 + l_2 0.9$. Hence $1.71 \le 1.1 l_1 + l_2 < 1 + 0.1 l_1 \Rightarrow l_1 > 7.1$, which is a contradiction with $l_1 < 1$. Hence, there are no a, b, a_1, b_1, a_2, b_2 satisfying the condition of Theorem 3.3.

4. Proofs of Theorems 3.1, 3.2 and 3.3

First we prepare a lemma which will be used in the context.

Lemma 4.1. *If a positive sequence $\{a_n\}$ and a positive number $k < 1$ satisfy*

$$a_{n+1} \le k \max\{a_n, a_{n-1}\} \textit{ for } n = 2, 3, \ldots,$$

then $a_n \le \alpha^n \beta$, $n = 1, 2, \ldots$, where

$$\alpha = \sqrt{k},\ \beta = \frac{\max\{a_2, ka_1\}}{k} + \frac{\max\{a_1, ka_0\}}{\sqrt{k}}.$$

Thus $\lim\limits_{n\to+\infty} a_n = 0$.

Proof. Notice that

$$a_{n+1} \le k \max\{a_n, a_{n-1}\} \le k \max\{k \max\{a_{n-1}, a_{n-2}\}, a_{n-1}\} = \max\{ka_{n-1}, k^2 a_{n-2}\}$$

and

$$ka_n \le k^2 \max\{a_{n-1}, a_{n-2}\} \le \max\{ka_{n-1}, k^2 a_{n-2}\}$$

for $n = 2, 3, \ldots$. Let $b_n = \max\{a_n, ka_{n-1}\}$, then by the above inequalities, we have

$$b_{n+1} \le kb_{n-1},\ n = 2, 3, \ldots.$$

Hence, $b_n \le \alpha^n \beta$, where $\alpha = \sqrt{k}$, $\beta = \frac{b_2}{k} + \frac{b_1}{\sqrt{k}}$. It follows that $a_n \le \alpha^n \beta$ holds for $n = 1, 2, \ldots$. So we have completed the proof. □

Proof of Theorem 3.1. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \ge 1$. Note that such point x_0 exists due to condition (1). Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \ge 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then clearly x_{n_0} is a fixed point of T. Hence, throughout the proof, we suppose that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \ge 1 \Rightarrow \alpha(x_1, x_2) = \alpha(Tx_0, T^2 x_0) \ge 1.$$

Repeating the process above, we derive,

$$\alpha(x_n, x_{n+1}) \ge 1,\ \text{for all } n = 0, 1, \ldots.$$

By using the same technique above, we get

$$\alpha(x_0, x_2) = \alpha(x_0, T^2 x_0) \geq 1 \Rightarrow \alpha(x_1, x_3) = \alpha(Tx_0, T^3 x_0) \geq 1.$$

The expression above yields

$$\alpha(x_n, x_{n+2}) \geq 1, \text{ for all } n = 0, 1, \ldots.$$

This proof can be divided into 4 steps.
Step 1: $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$.

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n), \\ &\qquad d(x_n, x_{n+1}), d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\} \\ &= \lambda \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Then according to Lemma 4.1, we have

$$d(x_n, x_{n+1}) \leq \alpha^n \beta, \ n = 1, 2, \ldots, \tag{4.1}$$

where

$$\alpha = \sqrt{\lambda}, \ \beta = \frac{\max\{d(x_2, x_3), \lambda d(x_1, x_2)\}}{\lambda} + \frac{\max\{d(x_1, x_2), \lambda d(x_0, x_1)\}}{\sqrt{\lambda}}.$$

Thus, $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$. Therefore, by Proposition 3.1, T has the approximate fixed point property.
Step 2: $x_n \neq x_m$ for any $n \neq m$.

If not, there exist n and $n+q$ such that $x_n = x_{n+q}$, then we have $T^q x_n = x_n$, i.e., $T^q x = x$ for $x = x_n, x_{n+1}, \ldots$. It follows from $x_m \neq x_{m+1}$ and $d(x_m, x_{m+1}) = d(x_{m+q}, x_{m+q+1})$ that the sequence $\{d(x_m, x_{m+1})\}$ is periodic. This is a contradiction with $d(x_m, x_{m+1}) \to 0$ as $m \to +\infty$.
Step 3: $\{x_n\}$ is a Cauchy sequence.

If not, we will prove that there exist $\delta > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, x_{n_{k+1}}) > \delta$, $k = 1, 2, \ldots$.

In fact, since $d(x_n, x_m) \not\to 0, m, n \to +\infty$, we have

$$\lim_{N \to +\infty} \sup_{n,m \geq N} d(x_n, x_m) > 0.$$

Now we show $\liminf\limits_{n \to +\infty} \sup\limits_{m > n} d(x_n, x_m) > 0$. If not, there exist $n_1 < \ldots < n_k < \ldots$ such that $d(x_{n_k}, x_m) < \frac{1}{2^k}$, for any $m > n_k$, $k = 1, 2, \ldots$.
Since $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for distinct x, u, v, y, we can easily get that

$$d(x_n, x_m) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_m) < \frac{3}{2^k} \text{ for any } n, m > n_{k+1}.$$

So

$$\lim_{N \to +\infty} \sup_{n,m \geq N} d(x_n, x_m) = \lim_{k \to +\infty} \sup_{n,m > n_{k+1}} d(x_n, x_m) = 0,$$

which is a contradiction.
Hence, we may denote a positive number $\delta_0 = \liminf\limits_{n \to +\infty} \sup\limits_{m > n} d(x_n, x_m)$.
Thus, there exists $N \in \mathbb{N}^+$ such that for any $n \geq N$, there exists $m > n$ satisfying

$d(x_n, x_m) > \frac{\delta_0}{2}$. Now, let $\delta = \frac{\delta_0}{2}$, $n_1 = N$, n_2 be such that $d(x_{n_1}, x_{n_2}) > \delta$, n_3 be such that $d(x_{n_2}, x_{n_3}) > \delta$, ...

It is easy to see that $\sum_{j=n}^{m} d(x_j, x_{j+1}) < \alpha^n \gamma$ via the inequality (4.1), where $\gamma = \frac{\beta}{1-\alpha}$. Without loss of generality, we may assume that n_1 satisfies $\alpha^n \gamma < \frac{\delta}{3}$ for all $n \geq n_1$.

Notice that $d(x_{n_i}, x_{n_{i+1}}) > \delta$, $\sum_{j=n_i}^{n_{i+1}-1} d(x_j, x_{j+1}) < \dfrac{\delta}{2}$ hold for $i \geq 1$. According to Theorem 2.1, we have $d(x_n, x_{n+1}) = \rho(x_n, x_{n+1})$, and

$$\begin{aligned} d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2}) &\geq d(x_n, x_{n+1}) + d(x_{n_i}, x_{n_{i+1}}) - \sum_{j=n_i}^{n_{i+1}-1} d(x_j, x_{j+1}) \\ &\geq d(x_n, x_{n+1}) + \delta - \frac{\delta}{3} = d(x_n, x_{n+1}) + \frac{2\delta}{3} \end{aligned}$$

for $n \geq n_1 + 2$. So

$$d(x_n, x_{n+2}) \geq d(x_n, x_{n+1}) + \frac{2\delta}{3} - d(x_{n+1}, x_{n+2}) \geq \frac{\delta}{3}$$

holds for large n. However,

$$\begin{aligned} d(x_n, x_{n+2}) \leq \lambda \max\{&d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \\ &d(x_{n+1}, x_{n+2}), d(x_{n-2}, x_n), d(x_{n-1}, x_{n+1})\} \\ = \lambda \max\{&d(x_{n-2}, x_n), d(x_{n-1}, x_{n+1})\} \end{aligned}$$

holds for $n \geq n_1 + 2$. Then by Lemma 4.1, we can easily get that

$$d(x_n, x_{n+2}) \to 0,\ n \to +\infty. \tag{4.2}$$

This is a contradiction with $d(x_n, x_{n+2}) > \frac{\delta}{3}$. Hence, we conclude that $\{x_n\}$ is a GMS Cauchy sequence in the complete GMS (X, d).

Thus, there exists $u \in X$ such that

$$\lim_{n\to+\infty} d(x_n, u) = 0.$$

Since T is α-continuous, $Tx_n \to Tu$, then $Tu = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} x_{n+1} = u$ and thus T has a fixed point.

Step 4: Uniqueness of fixed point.

Suppose the contrary, that u, v are two distinct fixed points of T.

We can choose $z \in X$ such that $\alpha(u, z) \geq 1$, $\alpha(v, z) \geq 1$, $\alpha(Tz, z) \geq 1$ and $\alpha(T^2 z, z) \geq 1$. Since T is α-admissible, we derive that $\alpha(u, T^n z) \geq 1$, $\alpha(v, T^n z) \geq 1$, $\alpha(T^{n+1} z, T^n z) \geq 1$ and $\alpha(T^{n+2} z, T^n z) \geq 1$ for all $n \in \mathbb{N}^+$.

If $z \in \{u, v\}$, then $\alpha(u, v) \geq 1$. Thus $d(u, v) = d(T^2 u, T^2 v) \leq \lambda d(u, v)$, and then $d(u, v) = 0$, i.e. $u = v$, which is a contradiction. So we have $z \neq u, v$ and

$$d(T^{n+2} z, u) \leq \lambda \max\{d(T^n z, T^{n+1} z), d(T^{n+1} z, T^{n+2} z), d(T^n z, u), d(T^{n+1} z, u)\}.$$

Let $a_n = d(T^n z, u)$ and $\epsilon_n = \max\{d(T^n z, T^{n+1} z), d(T^{n+1} z, T^{n+2} z)\}$.

Then we have $0 < a_{n+2} \leq \lambda \max\{\epsilon_n, a_{n+1}, a_n\}$. Similar to Step 1, $d(T^n z, T^{n+1}z) \to 0$ as $n \to +\infty$, and thus $\epsilon_n \to 0$ as $n \to +\infty$. Now we show that $a_n \to 0$ as $n \to +\infty$. Let $b_n = \max\{a_n, \lambda a_{n-1}\}$. We have

$$b_{n+2} \leq \lambda b_n + \eta_n, n = 1, 2, \ldots,$$

where $\eta_n = \max\{\epsilon_n, \lambda\epsilon_{n-1}\}$. Since $\eta_n \to 0$, we can easily prove that $b_n \to 0$, i.e., $a_n \to 0$. Hence, $d(T^n z, u) \to 0$, i.e. $T^n z \to u$. Similarly, we also have $T^n z \to v$ as $n \to +\infty$. By the uniqueness of limit, we have $u = v$ and then T has a unique fixed point. This completes the proof. □

Proof of Theorem 3.2. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. Similar to the proof of Theorem 3.1, we have $d(x_n, x_{n+1}) \to 0$ and

$$d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \to 0 \text{ as } n \to +\infty. \tag{4.3}$$

Thus, repeating Step 3 in the proof of Theorem 3.1, T has a fixed point. Now we turn to the uniqueness.

(1) If we choose the condition (1), then combining (4.3), the proof of uniqueness can refer to Step 4 in the proof of Theorem 3.1.

(2) If we choose the condition (2), then the above method becomes invalid. We present here a new proof. Let u and v be two fixed points of T, and z be the point such that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$. To utilize the adding assumption '$\lambda < 1/2$', we shall note that $d(T^n z, T^{n+1}z) \leq d(T^n z, u) + d(u, T^{n+1}z)$ and

$$\begin{aligned} d(T^{n+2}z, u) &\leq \lambda \max\{d(T^n z, T^{n+1}z), d(T^{n+1}z, T^{n+2}z), d(T^n z, u), d(T^{n+1}z, u)\} \\ &\leq \lambda \max\{d(T^n z, u) + d(u, T^{n+1}z), d(T^{n+2}z, u) + d(u, T^{n+1}z)\} \\ &\leq 2\lambda \max\{d(T^n z, u), d(u, T^{n+1}z), d(T^{n+2}z, u)\}. \end{aligned}$$

Since $2\lambda < 1$ and $T^n z \neq u$, we have

$$d(T^{n+2}z, u) \leq 2\lambda \max\{d(T^n z, u), d(v, T^{n+1}z)\}.$$

By Lemma 4.1, we immediately deduce that $d(T^n z, u) \to 0$, i.e., $T^n z \to u$. Similarly, we can prove that $T^n z \to v$ as $n \to +\infty$. By the uniqueness of limit, we have $u = v$ and then T has a unique fixed point. This completes the proof. □

Proof of Theorem 3.3. We only need to prove the uniqueness of fixed point of T. Without loss of generality, we may assume $a_1 + a_2 \leq b_1 + b_2$, then

$$a + b + 2a_1 + 2a_2 \leq a + b + a_1 + a_2 + b_1 + b_2 < 1.$$

Suppose the contrary, that u, v are two distinct fixed points of T. We can choose $z \in X$ such that $\alpha(u, z) \geq 1$, $\alpha(v, z) \geq 1$. Since T is α-admissible, we get

$$\alpha(u, T^n z) \geq 1,\ \alpha(v, T^n z) \geq 1 \text{ for all } n \in \mathbb{N}^+.$$

Taking $x = T^n z$, $y = u$, we have

$$\begin{aligned} d(T^{n+2}z,u) &\le a_1 d(T^n z, T^{n+1}z) + a_2 d(T^{n+1}z, T^{n+2}z) + ad(T^{n+1}z,u) + bd(T^n z,u) \\ &\le a_1(d(T^n z,u) + d(uT^{n+1}z,)) + a_2(d(T^{n+1}z,u) \\ &+ d(u,T^{n+2}z)) + ad(T^{n+1}z,u) + bd(T^n z,u) \\ &= (a_1+a_2+a)d(T^{n+1}z,u) + (a_1+b)d(T^n z,u) + a_2 d(u,T^{n+2}z). \end{aligned}$$

So it implies that

$$\begin{aligned} d(T^{n+2}z,u) &\le \frac{a_1+a_2+a}{1-a_2} d(T^{n+1}z,u) + \frac{a_1+b}{1-a_2} d(T^n z,u) \\ &\le \lambda \max\{d(T^n z,u), d(T^{n+1}z,u)\}, \end{aligned}$$

where

$$\lambda = \frac{a_1+a_2+a+a_1+b}{1-a_2} < \frac{1-a_2}{1-a_2} < 1.$$

Therefore, according to Lemma 4.1, we have $d(T^n z,u) \to 0$ as $n \to +\infty$. Others are similar to the proof of Theorem 3.2. □

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