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ON THE CLASS OF RELATIVELY WEAKLY DEMICOMPACT NONLINEAR OPERATORS

BILEL KRICHEN* AND DONAL O'REGAN**

*Department of Mathematics,
Faculty of Sciences of Sfax, University of Sfax,
Road Soukra Km 3.5 B.P. 1171, 3000, Sfax Tunisia
E-mail: krichen_ bilel@yahoo.fr

**School of Mathematics, National University of Ireland
Galway, Ireland
E-mail: donal.oregan@nuigalway.ie

Abstract. In this paper we discuss some topological properties of the set

$$\mathcal{F}(S_0, T, z) := \{x \in X : S_0 x \in Tx + z\},$$

where T is a nonlinear multi-valued mappings and S_0 is a single-valued mappings acting on a Banach space X. This study is based on a new concept, the so called weakly relative demicompactness for nonlinear operators.

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1. Introduction

In 1966, W. V. Petryshyn [12] used demicompactness of nonlinear operators acting on Hilbert spaces to study an iterative method for the construction of fixed points. This notion was also used in [14] to investigate the structure of fixed point sets for nonlinear operators defined on a closed subset of a Banach space. A nonlinear operator $T : D \subset X \longrightarrow X$ (here X is a Banach space) is demicompact if every bounded sequence $\{x_n\}$ in X such that $x_n - Tx_n$ converges strongly has a convergent subsequence. For example each of the following conditions imply that T is demicompact, (i) T is compact. (ii) The range of $I-T$ is closed, the inverse $(I-T)^{-1}$ exists and is continuous. We refer the reader to [3, 11, 12, 13] for more information. W. V. Petryshyn [13] and Y. Akashi [1] used the class of demicompact, 1-set contraction linear operators to obtain some results on Fredholm perturbations. Recently, W. Chaker, A. Jeribi and B. Krichen [5] continued this study to investigate the essential spectra of densely defined linear operator. In 2014, B. Krichen [9], gave a generalization of this notion by introducing the class of relative demicompact linear operator with respect to a given linear operator. If X is a Banach space and $T : \mathcal{D}(T) \subset X \longrightarrow X$ and

$S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ are two densely defined linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$, then T is said to be S_0-demicompact (or relative demicompact with respect to S_0), if every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $S_0x_n - Tx_n$ converges, has a convergent subsequence. In [9] the author showed that if $\mathcal{D}(T)$ lies in a finite dimensional subspace of X, the condition of relative demicompactness is automatically satisfied. For examples of S_0-demicompact operators, we cite operators T such that $(S_0 - T)^{-1}$ exists and is continuous on its range $\mathcal{R}(S_0 - T)$. In this paper we discuss relative demicompactness for nonlinear operators. We now recall some definitions from nonlinear analysis. Let X be a Banach space and let us denote by $\mathcal{P}(X)$, the class of all nonempty subsets of X. A correspondence $T : X \longrightarrow \mathcal{P}(X)$ is called a multi-valued operator or multi-valued mapping of X into itself. For any subset A of X, we let $T(A) = \cup_{x\in A} Tx$. A multi-valued operator $T : \mathcal{D}(T) \subset X \longrightarrow \mathcal{P}(X)$ is said to be closed if its graph $Gr(T) = \{(x, y) \in \mathcal{D}(T) \times X$ such that $y \in T(x)\}$ is a closed subset of $X \times X$. We denote by $\mathcal{C}$ the collection of all closed multi-valued operators from X to itself. We recall that T is upper-semi continuous if, and only if, for every closed subset V in X, the set

$$T^{-1}(V) = \{x \in X \text{ such that } T(x) \cap V \neq \emptyset\}$$

is a closed set in X. Now T is called weakly upper semicontinuous if T is upper semicontinuous with respect to the weak topology on X. The operator T is said to be weakly compact if the set $T(X)$ is relatively weakly compact in X. Now T is said to have weakly sequentially closed graph if for every sequence $(x_n)_n$ with $x_n \rightharpoonup x$ in X and for every sequence y_n with $y_n \in T(x_n)$, $y_n \rightharpoonup y$ implies $y \in T(x)$. An operator $S : \mathcal{D}(S) \subset X \longrightarrow X$ is said to be weakly closed if for every sequence $(x_n)_n \subset \mathcal{D}(S)$ such that $x_n \rightharpoonup x$ and $Sx_n \rightharpoonup y$ implies $x \in \mathcal{D}(S)$ and $y = Sx$.
The following Lemma provides a sequential characterization of an upper semi-continuous multi-valued mapping.

Lemma 1.1. *A multi-valued map T is upper semi-continuous at a point $x \in X$ if, and only if, for every sequence $\{x_n\}_{n=0}^{\infty}$ in X which converges to x, and for any open set $V \subset X$ such that $T(x) \subset V$, there exists $n_0 \in \mathbb{N}$ with $T(x_n) \subset V$, for all $n \geq n_0$.*

The following Lemma [7] will be used later.

Lemma 1.2. *Assume that $T : X \to \mathcal{P}(X)$ is an upper semi-continuous multi-valued operator. Then the graph $Gr(T)$ is a closed subset of $X \times X$.*

The measure of weak noncompactness was defined by De Blasi [6]. Let X be a Banach space, Ω_X the collection of all nonempty bounded subsets of X, and $\mathcal{K}^w$ the subsets of Ω_X consisting of all weakly compact subsets of X. The De Blasi [6] measure of weak noncompactness is the map $\omega : \Omega_X \longrightarrow [0, +\infty)$ defined in the following way:

$$\omega(\mathcal{M}) = \inf\big\{r > 0 : \text{ there exists } \mathrm{K} \in \mathcal{K}^w \text{ such that } \mathcal{M} \subset \mathrm{K} + B_r\big\},$$

for all $\mathcal{M} \in \Omega_X$. Let us recall some properties of $\omega(.)$ needed below (see, for example, [2, 6]) (see also [4], where an axiomatic approach to the notion of a measure of weak noncompactness is presented).

Lemma 1.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two elements of Ω_X. Then, the following conditions are satisfied:*

(i) $\mathcal{M}_1 \subset \mathcal{M}_2$ *implies* $\omega(\mathcal{M}_1) \leq \omega(\mathcal{M}_2)$.

(ii) $\omega(\mathcal{M}_1) = 0$ *if, and only if,* $\overline{\mathcal{M}_1}^w \in \mathcal{K}^w$, *where* $\overline{\mathcal{M}_1}^w$ *is the weak closure of* $\mathcal{M}_1$.

(iii) $\omega(\overline{\mathcal{M}_1}^w) = \omega(\mathcal{M}_1)$.

(iv) $\omega(\mathcal{M}_1 \cup \mathcal{M}_2) = \max\{\omega(\mathcal{M}_1), \omega(\mathcal{M}_2)\}$.

(v) $\omega(\mathcal{M}_1 + \mathcal{M}_2) \leq \omega(\mathcal{M}_1) + \omega(\mathcal{M}_2)$.

Definition 1.1. [8] Let X be a topological vector space. An operator $T : \mathcal{D}(T) \subset X \longrightarrow X$ is said to be ω-condensing map if $\omega(T(V)) < \omega(V)$ for all bounded subsets V of $\mathcal{D}(T)$ with $\omega(V) > 0$.

2. Relative weak demicompact nonlinear operators

In this section we first introduce a new concept. In our results in this section X is a Banach space and S_0 a single-valued operator from $\mathcal{D}(S_0) \subset X$ into X.

Definition 2.1. Let X be a topological vector space. Let $T : \mathcal{D}(T) \subset X \longrightarrow X$ be a single-valued operator with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$. Now T is called weakly S_0-demicompact if whenever $S_0x_n - Tx_n$ converges weakly and $(x_n)_n$ is contained in a bounded set of X, then the sequence $(x_n)_n$ has a weakly convergent subsequence in $\mathcal{D}(T)$. If $S_0 = I$, T is simply said to be weakly demicompact.

Definition 2.2. A single-valued operator $T : \mathcal{D}(T) \subset X \longrightarrow X$ with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ is said to be weakly S_0-semiclosed if for any weakly closed subset $V \subset$ of X, the set $(S_0 - T)V$ is weakly closed.

Remark 2.1. We note that there is no relationship between the concepts of weak S_0-semiclosedness and weak S_0-demicompactness. Consider the map

$$\begin{cases} T : \mathbb{R} & \longrightarrow \mathbb{R} \\ \quad x & \longrightarrow \dfrac{x|x|}{1+|x|} \end{cases}$$

Obviously, $(Id_{\mathbb{R}} - T)(\mathbb{R}) =]-1, 1[$, so it follows that T is not $Id_{\mathbb{R}}$-semiclosed. However, we know that in finite dimensional spaces, every bounded sequence has a weakly convergent subsequence. Therefore, T is weakly $Id_{\mathbb{R}}$-demicompact.

For $T \in \mathcal{C}$, S_0 a single-valued operator from $\mathcal{D}(S_0) \subset X$ into X such that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ and an element $z \in X$ we denote by $\mathcal{F}(S_0, T, z)$ the set of solutions of

$$S_0x \in Tx + z. \tag{2.1}$$

Here, $Tx + z = \{y + z : y \in Tx\}$, and x is a solution if the relation (2.1) holds. Note we do not assume existence or uniqueness so the set $\mathcal{F}(S_0, T, z)$ might be empty or contain many elements.

Remark 2.2. Let X be a Banach space. Assume that $T \in \mathcal{C}$ and S_0 is a continuous single-valued operator. Then, for every $z \in X$ the subset $\mathcal{F}(S_0, T, z)$ is closed.

If X is endowed with its weak topology, then we have the following.

Theorem 2.1. *Let X be a Banach space. Assume that T is a multi-valued operator with weakly sequentially closed graph and S_0 is a weakly sequentially continuous operator. Then, for every $z \in X$ the subset $\mathcal{F}(S_0, T, z)$ is weakly closed.*

Proof. Let $(x_n)_n$ be a sequence of $\mathcal{F}(S_0, T, z)$ such that $x_n \rightharpoonup x$. Then, there exists a sequence $(y_n)_n$ of $(Tx_n)_n$ such that $S_0 x_n = y_n + z$ for every $n \in \mathbb{N}$. From the weak sequential continuity of S_0 it follows that $S_0 x_n \rightharpoonup S_0 x$ and then

$$(x_n, y_n) \rightharpoonup (x, S_0 x - z).$$

Since T has a weakly sequentially closed graph, it follows that $x \in \mathcal{F}(S_0, T, z)$ and so, $\mathcal{F}(S_0, T, z)$ is weakly closed. □

Notice that if S_0 is only weakly closed, then we have the following:

Theorem 2.2. *Let X be a Banach space. Assume that T is a weakly compact multi-valued operator and S_0 is a weakly closed operator. Then, for every $z \in X$ the subset $\mathcal{F}(S_0, T, z)$ is weakly closed.*

Proof. Let $(x_n)_n \subset \mathcal{D}(T)$ be a sequence of $\mathcal{F}(S_0, T, z)$ such that $x_n \rightharpoonup x$. Then, there exists a sequence $(y_n)_n$ of $(Tx_n)_n$ such that $S_0 x_n = y_n + z$ for every $n \in \mathbb{N}$. Since $\{y_n : n \in \mathbb{N}\} \subset \cup_{x \in \mathcal{D}(T)} Tx$, it follows that $\omega(\{y_n : n \in \mathbb{N}\}) = 0$. Hence, there exists a subsequence $(y_{\varphi(n)})_n$, such that $S_0 x_{\varphi(n)} \rightharpoonup y + z$. The weak closedness of S_0 shows that $x \in \mathcal{D}(T)$ and $S_0 x \in Tx + z$. Then, $x \in \mathcal{F}(S_0, T, z)$ and consequently, $\mathcal{F}(S_0, T, z)$ is weakly closed. □

Theorem 2.3. *Let X be a Banach space and let the multi T be a weakly compact range operator with weakly sequentially closed graph. Assume that S_0 is weakly closed and $I - S_0$ is a weakly demicompact mapping. Then for every $z \in X$, the set $\mathcal{F}(S_0, T, z)$ is relatively weakly compact.*

Proof. Let $(x_n)_n \subset \mathcal{D}(T)$ be a sequence of $\mathcal{F}(S_0, T, z)$. Then, $S_0 x_n \in Tx_n + z \subset TX + z$. Since $TX + z$ is weakly compact, there exists a subsequence $x_{\varphi(n)} \subset \mathcal{D}(T)$ such that $S_0 x_n \rightharpoonup y$, $y \in X$. Using the weak demicompactness of $I - S_0$, we deduce the existence of a subsequence $(x_{\varphi\circ\psi(n)})_n$ of $(x_{\varphi(n)})_n$ such that $x_{\varphi\circ\psi(n)}$ converges weakly to some $x \in X$. From the weak closedness of S_0, we obtain $x \in \mathcal{D}(T)$ and $y = S_0 x$. Now, since $Gr(T)$ is weakly sequentially closed, we deduce that $x \in \mathcal{F}(S_0, T, z)$ and so the result follows from the Eberlein-Šmulian theorem [10, Theorem 2.8.6]. □

The next result shows that in the case of condensing operators, $\mathcal{F}(S_0, T, z)$ may be relatively weakly compact.

Theorem 2.4. *Let X be a Banach space and let T be a single-valued ω-condensing operator. Then, for every bounded subset D of X, the set $D \cap \mathcal{F}(I, T, z)$ is relatively weakly compact.*

Proof. Since $x \in \mathcal{F}(I, T, z)$ means that $x = Tx + z$, it follows that

$$T(\mathcal{F}(I, T, z) \cap D) = \mathcal{F}(I, T, z) \cap D - z,$$

so $\omega(T(\mathcal{F}(I, T, z) \cap D)) = \omega(\mathcal{F}(I, T, z) \cap D)$. If $\omega(\mathcal{F}(I, T, z) \cap D) > 0$ then since T is ω-condensing we have $\omega(T(\mathcal{F}(I, T, z) \cap D)) < \omega(\mathcal{F}(I, T, z) \cap D)$, a contradiction. Thus $\omega(\mathcal{F}(I, T, z) \cap D) = 0$ so $\mathcal{F}(I, T, z) \cap D$ is relatively weakly compact. □

Theorem 2.5. *Let $T : \mathcal{D}(T) \subset X \longrightarrow X$, $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be two single-valued closed operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ such that $S_0 - T$ is weakly closed. If T is weakly S_0-demicompact, then for every closed, convex bounded set D, the multi-valued map*

$$F_D : X \longrightarrow \mathcal{P}(X),$$
$$y \longrightarrow D \cap (S_0 - T)^{-1}y,$$

is weakly compact-valued and weakly upper-semicontinuous.

Proof. Suppose T is weakly S_0-demicompact. First we show F_D is weakly upper-semicontinuous. Let $(y_n)_n$ be a sequence with $y_n \rightharpoonup y$ and Q a weakly open set such that $F_D(y) \subset Q$. From Lemma 1.1 it suffices to show the existence of $n_0 \in \mathbb{N}$ such that $F_D(y_n) \subset Q$, $\forall n \geq n_0$. If not, then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ (here $x_n \in D \cap (S_0 - T)^{-1}y_n$) such that $x_{\varphi(n)} \in D \cap (S_0 - T)^{-1}y_n$ and $x_{\varphi(n)} \notin Q$. Note $S_0 x_{\varphi(n)} - T x_{\varphi(n)}$ converges weakly to y. Since T is weakly S_0-demicompact we infer that there exists a subsequence $(x_{\varphi\circ\psi(n)})_n$ of $(x_{\varphi(n)})_n$ such that $(x_{\varphi\circ\psi(n)})_n \rightharpoonup x$, $x \in X$. Since D is a closed and convex subset of X we deduce that $x \in D$. Moreover, taking into account that $(S_0 - T)x_{\varphi\circ\psi(n)} \rightharpoonup y$ and $S_0 - T$ is a weakly closed mappings, we deduce that $y = (S_0 - T)x$. Consequently, $x \in D \cap (S_0 - T)^{-1}y = F_D(y)$. Therefore $x_{\varphi\circ\psi(n)} \in Q$ for n large enough, a contradiction to the construction of $(x_{\varphi(n)})_n$. Thus F_D is weakly upper-semicontinuous. Next fix $y \in X$. Now since T is weakly S_0-demicompact then if $(x_n)_n$ is a sequence in $D \cap (S_0 - T)^{-1}y$, then it has a weakly converging subsequence, and so the weak closedness of $S_0 - T$ implies that the limit is also in $D \cap (S_0 - T)^{-1}y$. Thus, $D \cap (S_0 - T)^{-1}y$ is weakly compact. □

The following theorem provides a sufficient condition to an nonlinear operator to be weakly relative demicompact with respect to a given nonlinear operator.

Theorem 2.6. *Let $T : \mathcal{D}(T) \subset X \longrightarrow X$, $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be two single-valued closed operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ such that $S_0 - T$ have a weakly sequentially closed graph. Assume that for every weakly closed convex, bounded set D, the multi-valued map*

$$F_D : X \longrightarrow \mathcal{P}(X),$$
$$y \longrightarrow D \cap (S_0 - T)^{-1}y,$$

is compact-valued and weakly upper-semicontinuous. Then, T is weakly S_0-demicompact.

Proof. Let $y_n := (S_0 - T)x_n$ be a weakly converging sequence and assume that $(x_n)_n$ is included in the weakly closed, and bounded set D. The weak upper-semicontinuity of $y \longrightarrow D \cap (S_0 - T)^{-1}y$ implies that $(x_n)_n$ converges weakly to the set $D \cap (S_0 - T)^{-1}y_0$ where $y_0 = \lim y_n$, with respect to the weak topology. The weak compactness of $D \cap (S_0 - T)^{-1}y_0$ implies that there exists a subsequence of $(x_n)_n$ that converges weakly to an element $x \in D \cap (S_0 - T)^{-1}y_0$. Therefore, T is weakly S_0-demicompact. □

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