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A COUPLED FIXED POINT PROBLEM UNDER A FINITE NUMBER OF EQUALITY CONSTRAINTS IN A BANACH SPACE PARTIALLY ORDERED BY A CONE

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Abstract. Let $(E, \|\cdot\|)$ be a Banach space with a cone P. Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be a finite number of mappings. We obtain sufficient conditions for the existence and uniqueness of solutions to the problem

$$\begin{cases} F(x,y) &= x, \\ F(y,x) &= y, \\ \varphi_i(x,y) &= 0_E, \ i = 1,2,\ldots,r, \end{cases}$$

where 0_E is the zero vector of E.

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1. Introduction and preliminaries

Let X be a nonempty set and $F : X \times X \to X$ be a given operator. A coupled fixed point of F is an element $(x, y) \in X \times X$ satisfying the system of equations

$$\begin{cases} F(x,y) &= x, \\ F(y,x) &= y. \end{cases}$$

Observe that $(x, y) \in X \times X$ is a coupled fixed point of F if and only if $(y, x) \in X \times X$ is a coupled fixed point of F.

The coupled fixed point's concept was introduced by Opoitsev [14, 15] and then, by Guo and Lakshmikantham [4] in connection with coupled quasi-solutions of an initial value problem for ordinary differential equations. Various existence results of coupled fixed points for different classes of operators were obtained by many authors. The motivation of such contributions is the usefulness of the coupled fixed point approach in studying the existence of solutions to nonlinear functional equations. For more details on coupled fixed point theory, we refer the reader to [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 20, 19, 21] and the references therein.

Let $(E, \|\cdot\|)$ be a Banach space with zero vector 0_E. Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \dots, r)$ be a finite number of given mappings. In this paper, we are interested to study the existence and uniqueness of solutions to the following problem: Find $(x, y) \in E \times E$ such that

$$\begin{cases} F(x,y) &= x, \\ F(y,x) &= y, \\ \varphi_i(x,y) &= 0_E,\ i = 1,2,\dots,r. \end{cases} \tag{1.1}$$

We obtain sufficient conditions for the existence and uniqueness of solutions to Pb. (1.1). Some interesting consequences are deduced from our main results.

At first, let us recall some basic definitions and some preliminary results that will be used later. In this paper, the considered Banach space $(E, \|\cdot\|)$ is supposed to be partially ordered by a cone P. Recall that a nonempty closed convex set $P \subset E$ is said to be a cone (see [7]) if it satisfies the following conditions:
(P1) $\lambda \geq 0,\ x \in P \Longrightarrow \lambda x \in P$.
(P2) $-x, x \in P \Longrightarrow x = 0_E$.
We define the partial order $\leq_P$ in E induced by the cone P by

$$(x, y) \in E \times E, \quad x \leq_P y \Longleftrightarrow y - x \in P.$$

Definition 1.1 ([1])**.** Let $\varphi : E \times E \to E$ be a given mapping. We say that φ is level closed from the right if for every $e \in E$, the set

$$\mathrm{lev}\varphi_{\leq_P}(e) := \{(x, y) \in E \times E : \varphi(x, y) \leq_P e\}$$

is closed.

Definition 1.2. Let $\varphi : E \times E \to E$ be a given mapping. We say that φ is level closed from the left if for every $e \in E$, the set

$$\mathrm{lev}\varphi_{\geq_P}(e) := \{(x, y) \in E \times E : e \leq_P \varphi(x, y)\}$$

is closed.

We denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions:
(Ψ_1) ψ is non-decreasing.
(Ψ_2) For all $t > 0$, we have

$$\sum_{k=0}^{\infty} \psi^k(t) < \infty.$$

Here, ψ^k is the k-th iterate of ψ.

The following properties are not difficult to prove.

Lemma 1.3. *Let $\psi \in \Psi$. Then*

(i) *$\psi(t) < t$, for all $t > 0$;*
(ii) *$\psi(0) = 0$;*
(iii) *ψ is continuous at $t = 0$.*

Example 1.4. As examples, the following functions belong to the set Ψ:

$\psi(t) = k\,t,\ k \in (0,1)$.

$$\psi(t) = \begin{cases} t/2 & \text{if} \quad 0 \le t \le 1, \\ 1/2 & \text{if} \quad t > 1. \end{cases}$$

$$\psi(t) = \begin{cases} t/2 & \text{if} \quad 0 \le t < 1, \\ t - 1/3 & \text{if} \quad t \ge 1. \end{cases}$$

Now, we are ready to state and prove our main results. This is the aim of the next section.

2. Main results

Through this paper, $(E, \|\cdot\|)$ is a Banach space partially ordered by a cone P and 0_E denotes the zero vector of E.

Let us start with the case of one equality constraint.

2.1. **A coupled fixed point problem under one equality constraint.** We are interested with the existence and uniqueness of solutions to the problem: Find $(x, y) \in E \times E$ such that

$$\begin{cases} F(x,y) &=& x, \\ F(y,x) &=& y, \\ \varphi(x,y) &=& 0_E, \end{cases} \tag{2.1}$$

where $F, \varphi : E \times E \to E$ are two given mappings.

The following theorem provides sufficient conditions for the existence and uniqueness of solutions to Pb. (2.1).

Theorem 2.1. *Let $F, \varphi : E \times E \to E$ be two given mappings. Suppose that the following conditions are satisfied:*
(i) *φ is level closed from the right.*
(ii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi(x_0, y_0) \le_P 0_E$.*
(iii) *For every $(x, y) \in E \times E$, we have*

$$\varphi(x,y) \le_P 0_E \Longrightarrow \varphi(F(x,y), F(y,x)) \ge_P 0_E.$$

(iv) *For every $(x, y) \in E \times E$, we have*

$$\varphi(x,y) \ge_P 0_E \Longrightarrow \varphi(F(x,y), F(y,x)) \le_P 0_E.$$

(v) *There exists some $\psi \in \Psi$ such that*

$$\|F(u,v) - F(x,y)\| + \|F(y,x) - F(v,u)\| \le \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi(x, y) \le_P 0_E$, $\varphi(u, v) \ge_P 0_E$.
Then Pb. (2.1) *has a unique solution.*

Proof. Let $(x_0, y_0) \in E \times E$ be such that

$$\varphi(x_0, y_0) \le_p 0_E.$$

Such a point exists from (ii). From (iii), we have

$$\varphi(x_0, y_0) \le_P 0_E \Longrightarrow \varphi(F(x_0, y_0), F(y_0, x_0)) \ge_P 0_E.$$

Define the sequences $\{x_n\}$ and $\{y_n\}$ in E by

$$x_{n+1} = F(x_n, y_n),\ y_{n+1} = F(y_n, x_n), \quad n = 0, 1, 2, \ldots$$

Then we have

$$\varphi(x_1, y_1) \geq_P 0_E.$$

From (iv), we have

$$\varphi(x_1, y_1) \geq_P 0_E \Longrightarrow \varphi(F(x_1, y_1), F(y_1, x_1)) \leq_P 0_E,$$

that is,

$$\varphi(x_2, y_2) \leq_P 0_E.$$

Again, using (iii), we get from the above inequality that

$$\varphi(x_3, y_3) \geq_P 0_E.$$

Then, by induction, we obtain

$$\varphi(x_{2n}, y_{2n}) \leq_P 0_E,\ \varphi(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad n = 0, 1, 2, \ldots \tag{2.2}$$

Using (v) and (2.2), by symmetry, we obtain

$$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right), \quad n = 1, 2, 3, \ldots \tag{2.3}$$

From (2.3), since ψ is a non-decreasing function, for every $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right) \\
&\leq \psi^2\left(\|x_{n-1} - x_{n-2}\| + \|y_{n-1} - y_{n-2}\|\right) \\
&\leq \ldots \\
&\leq \psi^n\left(\|x_1 - x_0\| + \|y_1 - y_0\|\right).
\end{aligned} \tag{2.4}$$

Suppose that

$$\|x_1 - x_0\| + \|y_1 - y_0\| = 0.$$

In this case, we have

$$x_0 = x_1 = F(x_0, y_0) \quad \text{and} \quad y_0 = y_1 = F(y_0, x_0).$$

Moreover, from (iii), since $\varphi(x_0, y_0) \leq_P 0_E$, we obtain $\varphi(x_1, y_1) = \varphi(x_0, y_0) \geq 0_E$. Since P is a cone, the two inequalities $\varphi(x_0, y_0) \leq_P 0_E$ and $\varphi(x_0, y_0) \geq_P 0_E$ yield

$$\varphi(x_0, y_0) = 0_E.$$

Thus we proved that in this case, $(x_0, y_0) \in E \times E$ is a solution to Pb. (2.1).
Now, we may suppose that $\|x_1 - x_0\| + \|y_1 - y_0\| \neq 0$. Set

$$\delta = \|x_1 - x_0\| + \|y_1 - y_0\| > 0.$$

From (2.4), we have

$$\|x_{n+1} - x_n\| \leq \psi^n(\delta), \quad n = 0, 1, 2, \ldots \tag{2.5}$$

Using the triangular inequality and (2.5), for all $m = 1, 2, 3, \ldots$, we have

$$\begin{aligned} \|x_n - x_{n+m}\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \ldots + \|x_{n+m-1} - x_{n+m}\| \\ &\leq \psi^n(\delta) + \psi^{n+1}(\delta) + \ldots + \psi^{n+m-1}(\delta) \\ &= \sum_{i=n}^{n+m-1} \psi^i(\delta) \\ &\leq \sum_{i=n}^{\infty} \psi^i(\delta). \end{aligned}$$

On the other hand, since $\sum_{k=0}^{\infty} \psi^k(\delta) < \infty$, we have

$$\sum_{i=n}^{\infty} \psi^i(\delta) \to 0 \text{ as } n \to \infty,$$

which implies that $\{x_n\}$ is a Cauchy sequence in $(E, \|\cdot\|)$. The same argument gives us that $\{y_n\}$ is a Cauchy sequence in $(E, \|\cdot\|)$. As consequence, there exists a pair of points $(x^*, y^*) \in E \times E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - y^*\| = 0. \tag{2.6}$$

From (2.2), we have

$$\varphi(x_{2n}, y_{2n}) \leq_P 0_E, \quad n = 0, 1, 2, \ldots,$$

that is,

$$(x_{2n}, y_{2n}) \in \text{lev}\varphi_{\leq_P}(0_E), \quad n = 0, 1, 2, \ldots,$$

Since φ is level closed from the right, passing to the limit as $n \to \infty$ and using (2.6), we obtain

$$(x^*, y^*) \in \text{lev}\varphi_{\leq_P}(0_E),$$

that is,

$$\varphi(x^*, y^*) \leq_P 0_E. \tag{2.7}$$

Now, using (2.2), (2.7) and (v), we obtain

$$\begin{aligned} &\|F(x_{2n+1}, y_{2n+1}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n+1}, x_{2n+1})\| \\ &\qquad \leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right), \end{aligned}$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+2} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+2}\| \leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, using (2.6), the continuity of ψ at 0, and the fact that $\psi(0) = 0$ (see Lemma 1.3), we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

that is,

$$x^* = F(x^*, y^*) \quad \text{and} \quad y^* = F(y^*, x^*).$$

This proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Finally, using (2.7) and the fact that (x^*, y^*) is a coupled fixed point of F, it follows from (iii) that

$$\varphi(x^*, y^*) \geq_P 0_E. \tag{2.8}$$

Then (2.7) and (2.8) yield

$$\varphi(x^*, y^*) = 0_E.$$

Thus we proved that $(x^*, y^*) \in E \times E$ is a solution to Pb. (2.1). Suppose now that $(u^*, v^*) \in E \times E$ is a solution to Pb. (2.1) with $(x^*, y^*) \neq (u^*, v^*)$. Using (v), we obtain

$$\|u^* - x^*\| + \|y^* - v^*\| \leq \psi(\|u^* - x^*\| + \|y^* - v^*\|).$$

Since $\|u^* - x^*\| + \|y^* - v^*\| > 0$, from (i) of Lemma 1.3, we have

$$\psi(\|u^* - x^*\| + \|y^* - v^*\|) < \|u^* - x^*\| + \|y^* - v^*\|.$$

Then

$$\|u^* - x^*\| + \|y^* - v^*\| < \|u^* - x^*\| + \|y^* - v^*\|,$$

which is a contradiction. As consequence, (x^*, y^*) is the unique solution to Pb. (2.1).

Remark 2.2. Observe that the conclusion of Theorem 2.1 is still valid if we replace condition (i) by the following condition:
(i') φ is level closed from the left.
In fact, from (2.2), we have

$$\varphi(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad n = 0, 1, 2, \ldots,$$

that is,

$$(x_{2n+1}, y_{2n+1}) \in \mathrm{lev}\varphi_{\geq_P}, \quad n = 0, 1, 2, \ldots$$

Passing to the limit as $n \to \infty$ and using (2.6), we obtain

$$\varphi(x^*, y^*) \geq_P 0_E. \tag{2.9}$$

Using (2.2), (2.9) and (v), we obtain

$$\|F(x_{2n}, y_{2n}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n}, x_{2n})\| \leq \psi\left(\|x_{2n} - x^*\| + \|y_{2n} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+1} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+1}\| \leq \psi\left(\|x_{2n} - x^*\| + \|y_{2n} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

which proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Using (2.9) and the fact that (x^*, y^*) is a coupled fixed point of F, it follows from (iv) that

$$\varphi(x^*, y^*) \leq_P 0_E. \tag{2.10}$$

Then (2.9) and (2.10) yield

$$\varphi(x^*, y^*) = 0_E.$$

Thus $(x^*, y^*) \in E \times E$ is a solution to Pb. (2.1).

2.2. **A coupled fixed point problem under two equality constraints.** Here, we are interested with the existence and uniqueness of solutions to the following problem: Find $(x,y) \in E \times E$ such that

$$\begin{cases} F(x,y) &=& x, \\ F(y,x) &=& y, \\ \varphi_1(x,y) &=& 0_E, \\ \varphi_2(x,y) &=& 0_E, \end{cases} \tag{2.11}$$

where $F, \varphi_1, \varphi_2 : E \times E \to E$ are three given mappings.

We have the following result.

Theorem 2.3. *Let $F, \varphi_1, \varphi_2 : E \times E \to E$ be three given mappings. Suppose that the following conditions are satisfied:*
(i) *φ_i $(i = 1, 2)$ is level closed from the right.*
(ii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi_i(x_0, y_0) \leq_P 0_E$ $(i = 1, 2)$.*
(iii) *For every $(x,y) \in E \times E$, we have*

$$\varphi_i(x,y) \leq_P 0_E,\ i = 1,2 \Longrightarrow \varphi_i(F(x,y), F(y,x)) \geq_P 0_E,\ i = 1,2.$$

(iv) *For every $(x,y) \in E \times E$, we have*

$$\varphi_i(x,y) \geq_P 0_E,\ i = 1,2 \Longrightarrow \varphi_i(F(x,y), F(y,x)) \leq_P 0_E,\ i = 1,2.$$

(v) *There exists some $\psi \in \Psi$ such that*

$$\|F(u,v) - F(x,y)\| + \|F(y,x) - F(v,u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x,y), (u,v) \in E \times E$ with $\varphi_i(x,y) \leq_P 0_E$, $\varphi_i(u,v) \geq_P 0_E$, $i = 1,2$.
Then Pb. (2.11) *has a unique solution.*

Proof. Let $(x_0, y_0) \in E \times E$ be such that

$$\varphi_i(x_0, y_0) \leq_p 0_E, \quad i = 1,2.$$

Then from (iii), we have

$$\varphi_i(F(x_0,y_0), F(y_0,x_0)) \geq_P 0_E, \quad i = 1,2.$$

Define the sequences $\{x_n\}$ and $\{y_n\}$ in E by

$$x_{n+1} = F(x_n, y_n),\ y_{n+1} = F(y_n, x_n), \quad n = 0,1,2,\ldots$$

We have

$$\varphi_i(x_1, y_1) \geq_P 0_E, \quad i = 1,2.$$

Then from (iv), we obtain

$$\varphi_i(x_2, y_2) \leq_P 0_E, \quad i = 1,2.$$

Again, using (iii), we get from the above inequality that

$$\varphi_i(x_3, y_3) \geq_P 0_E, \quad i = 1,2.$$

Then, by induction, we obtain

$$\varphi_i(x_{2n}, y_{2n}) \leq_P 0_E,\ \varphi_i(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad i = 1,2,\ n = 0,1,2,\ldots$$

Then, using (v), we obtain

$$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right), \quad n = 1,2,3,\ldots$$

Now, we argue exactly as in the proof of Theorem 2.1 to show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $(E, \|\cdot\|)$. As consequence, there exists a pair of points $(x^*, y^*) \in E \times E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - y^*\| = 0.$$

On the other hand, we have

$$(x_{2n}, y_{2n}) \in \text{lev}\varphi_{i_{\leq_P}}(0_E), \quad i = 1, 2,\, n = 0, 1, 2, \ldots,$$

Since φ_i $(i = 1, 2)$ is level closed from the right, passing to the limit as $n \to \infty$, we obtain

$$(x^*, y^*) \in \text{lev}\varphi_{i_{\leq_P}}(0_E), \quad i = 1, 2,$$

that is,

$$\varphi_i(x^*, y^*) \leq_P 0_E, \quad i = 1, 2.$$

Then we have

$$\begin{aligned} &\|F(x_{2n+1}, y_{2n+1}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n+1}, x_{2n+1})\| \\ &\leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right), \end{aligned}$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+2} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+2}\| \leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

that is,

$$x^* = F(x^*, y^*) \quad \text{and} \quad y^* = F(y^*, x^*).$$

This proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Since $\varphi_i(x^*, y^*) \leq_P 0_E$ for $i = 1, 2$, from (iii) we have

$$\varphi_i(F(x^*, y^*), F(y^*, x^*)) \geq_P 0_E, \quad i = 1, 2,$$

that is,

$$\varphi_i(x^*, y^*) \geq_P 0_E, \quad i = 1, 2.$$

Finally, the two inequalities $\varphi_i(x^*, y^*) \leq_P 0_E$ and $\varphi_i(x^*, y^*) \geq_P 0_E$, $i = 1, 2$ yield $\varphi_i(x^*, y^*) = 0_E$, $i = 1, 2$. Then we proved that $(x^*, y^*) \in E \times E$ is a solution to Pb. (2.11). The uniqueness can be obtained using a similar argument as in the proof of Theorem 2.1.

Now, replace φ_2 in Theorem 2.3 by $-\varphi_2$, we obtain the following result.

Theorem 2.4. *Let $F, \varphi_1, \varphi_2 : E \times E \to E$ be three given mappings. Suppose that the following conditions are satisfied:*
(i) *φ_1 is level closed from the right and φ_2 is level closed from the left.*
(ii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi_1(x_0, y_0) \leq_P 0_E$ and $\varphi_2(x_0, y_0) \geq_p 0_E$.*
(iii) *For every $(x, y) \in E \times E$ with $\varphi_1(x, y) \leq_P 0_E$ and $\varphi_2(x, y) \geq_P 0_E$, we have*

$$\varphi_1(F(x, y), F(y, x)) \geq_P 0_E,\ \varphi_2(F(x, y), F(y, x)) \leq_P 0_E.$$

(iv) *For every* $(x,y)\in E\times E$ *with* $\varphi_1(x,y)\geq_P 0_E$ *and* $\varphi_2(x,y)\leq_P 0_E$*, we have*

$$\varphi_1(F(x,y),F(y,x))\leq_P 0_E,\ \varphi_2(F(x,y),F(y,x))\geq_P 0_E.$$

(v) *There exists some* $\psi\in\Psi$ *such that*

$$\|F(u,v)-F(x,y)\|+\|F(y,x)-F(v,u)\|\leq\psi\left(\|u-x\|+\|v-y\|\right),$$

for all $(x,y),(u,v)\in E\times E$ *with* $\varphi_1(x,y)\leq_P 0_E$, $\varphi_2(x,y)\geq_P 0_E$, $\varphi_1(u,v)\geq_P 0_E$, $\varphi_2(u,v)\leq_P 0_E$. *Then* Pb. (2.11) *has a unique solution.*

Replace φ_1 in Theorem 2.4 by $-\varphi_1$, we obtain the following result.

Theorem 2.5. *Let* $F,\varphi_1,\varphi_2:E\times E\to E$ *be three given mappings. Suppose that the following conditions are satisfied:*
(i) φ_i $(i=1,2)$ *is level closed from the left.*
(ii) *There exists* $(x_0,y_0)\in E\times E$ *such that* $\varphi_i(x_0,y_0)\geq_P 0_E$ $(i=1,2)$*.*
(iii) *For every* $(x,y)\in E\times E$*, we have*

$$\varphi_i(x,y)\leq_P 0_E,\ i=1,2\Longrightarrow\varphi_i(F(x,y),F(y,x))\geq_P 0_E,\ i=1,2.$$

(iv) *For every* $(x,y)\in E\times E$*, we have*

$$\varphi_i(x,y)\geq_P 0_E,\ i=1,2\Longrightarrow\varphi_i(F(x,y),F(y,x))\leq_P 0_E,\ i=1,2.$$

(v) *There exists some* $\psi\in\Psi$ *such that*

$$\|F(u,v)-F(x,y)\|+\|F(y,x)-F(v,u)\|\leq\psi\left(\|u-x\|+\|v-y\|\right),$$

for all $(x,y),(u,v)\in E\times E$ *with* $\varphi_i(x,y)\leq_P 0_E$, $\varphi_i(u,v)\geq_P 0_E$, $i=1,2$.
Then Pb. (2.11) *has a unique solution.*

2.3. **A coupled fixed point problem under r equality constraints.** Now, we argue exactly as in the proof of Theorem 2.3 to obtain the following existence result for Pb. (1.1).

Theorem 2.6. *Let* $F,\varphi_i:E\times E\to E$ $(i=1,2,\ldots,r)$ *be* $r+1$ *given mappings. Suppose that the following conditions are satisfied:*
(i) φ_i $(i=1,2,\ldots,r)$ *is level closed from the right.*
(ii) *There exists* $(x_0,y_0)\in E\times E$ *such that* $\varphi_i(x_0,y_0)\leq_P 0_E$ $(i=1,2,\ldots,r)$*.*
(iii) *For every* $(x,y)\in E\times E$*, we have*

$$\varphi_i(x,y)\leq_P 0_E,\ i=1,2,\ldots,r\Longrightarrow\varphi_i(F(x,y),F(y,x))\geq_P 0_E,\ i=1,2,\ldots,r.$$

(iv) *For every* $(x,y)\in E\times E$*, we have*

$$\varphi_i(x,y)\geq_P 0_E,\ i=1,2,\ldots,r\Longrightarrow\varphi_i(F(x,y),F(y,x))\leq_P 0_E,\ i=1,2,\ldots,r.$$

(v) *There exists some* $\psi\in\Psi$ *such that*

$$\|F(u,v)-F(x,y)\|+\|F(y,x)-F(v,u)\|\leq\psi\left(\|u-x\|+\|v-y\|\right),$$

for all $(x,y),(u,v)\in E\times E$ *with* $\varphi_i(x,y)\leq_P 0_E$, $\varphi_i(u,v)\geq_P 0_E$, $i=1,2,\ldots,r$.
Then Pb. (1.1) *has a unique solution.*

3. Some consequences

In this section, we present some consequences following from Theorem 2.6.

3.1. **A fixed point problem under symmetric equality constraints.**

We need the following definition.

Definition 3.1. Let X be a nonempty set and let $F : X \times X \to X$ be a given mapping. We say that $x \in X$ is a fixed point of F if $F(x,x) = x$.

Let $F, \varphi : E \times E \to E$ be given mappings. We consider the problem: Find $x \in E$ such that

$$\begin{cases} F(x,x) &= x, \\ \varphi(x,x) &= 0_E. \end{cases} \tag{3.1}$$

We have the following result.

Corollary 3.2. *Let $F, \varphi : E \times E \to E$ be two given mappings. Suppose that the following conditions are satisfied:*
(i) *φ is level closed from the right.*
(ii) *φ is symmetric, that is,*

$$\varphi(x,y) = \varphi(y,x), \quad (x,y) \in E \times E.$$

(iii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi(x_0,y_0) \leq_P 0_E$.*
(iv) *For every $(x,y) \in E \times E$, we have*

$$\varphi(x,y) \leq_P 0_E \Longrightarrow \varphi(F(x,y), F(y,x)) \geq_P 0_E.$$

(v) *For every $(x,y) \in E \times E$, we have*

$$\varphi(x,y) \geq_P 0_E \Longrightarrow \varphi(F(x,y), F(y,x)) \leq_P 0_E.$$

(vi) *There exists some $\psi \in \Psi$ such that*

$$\|F(u,v) - F(x,y)\| + \|F(y,x) - F(v,u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x,y), (u,v) \in E \times E$ with $\varphi(x,y) \leq_P 0_E$ and $\varphi(u,v) \geq_P 0_E$.
Then Pb. (3.1) *has a unique solution.*

Proof. From Theorem 2.1, we know that Pb. (2.1) has a unique solution $(x^*, y^*) \in E \times E$. Since φ is symmetric, (y^*, x^*) is also a solution to (2.1). By uniqueness, we get $x^* = y^*$. Then $x^* \in E$ is the unique solution to Pb. (3.1).

Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be $r + 1$ given mappings. We consider the problem: Find $x \in X$ such that

$$\begin{cases} F(x,x) &= x, \\ \varphi_i(x,x) &= 0_E, \ i = 1, 2, \ldots, r. \end{cases} \tag{3.2}$$

Similarly, from Theorem 2.6, we have the following result.

Corollary 3.3. *Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be $r + 1$ given mappings. Suppose that the following conditions are satisfied:*
(i) *φ_i $(i = 1, 2, \ldots, r)$ is level closed from the right.*

(ii) φ_i *($i=1,2,\ldots,r$) is symmetric.*
(iii) *There exists* $(x_0,y_0)\in E\times E$ *such that* $\varphi_i(x_0,y_0)\leq_P 0_E$ *($i=1,2,\ldots,r$).*
(iv) *For every* $(x,y)\in E\times E$*, we have*

$$\varphi_i(x,y)\leq_P 0_E,\ i=1,2,\ldots,r \Longrightarrow \varphi_i(F(x,y),F(y,x))\geq_P 0_E,\ i=1,2,\ldots,r.$$

(v) *For every* $(x,y)\in E\times E$*, we have*

$$\varphi_i(x,y)\geq_P 0_E,\ i=1,2,\ldots,r \Longrightarrow \varphi_i(F(x,y),F(y,x))\leq_P 0_E,\ i=1,2,\ldots,r.$$

(vi) *There exists some* $\psi\in\Psi$ *such that*

$$\|F(u,v)-F(x,y)\|+\|F(y,x)-F(v,u)\|\leq \psi\left(\|u-x\|+\|v-y\|\right),$$

for all $(x,y),(u,v)\in E\times E$ *with* $\varphi_i(x,y)\leq_P 0_E$*,* $\varphi_i(u,v)\geq_P 0_E$*,* $i=1,2,\ldots,r$*.*
Then Pb. (3.2) *has a unique solution.*

3.2. **A common coupled fixed point result.** We need the following definition.

Definition 3.4. Let X be a nonempty set, $F: X\times X\to X$ and $g: X\to X$ be two given mappings. We say that the pair of elements $(x,y)\in X\times X$ is a common coupled fixed point of F and g if

$$x=gx=F(x,y)\quad\text{and}\quad y=gy=F(y,x).$$

We have the following common coupled fixed point result.

Corollary 3.5. *Let* $F: E\times E\to E$ *and* $g: E\to E$ *be two given mappings. Suppose that the following conditions hold:*
(i) g *is a continuous mapping.*
(ii) *There exists* $(x_0,y_0)\in E\times E$ *such that*

$$gx_0\leq_p x_0\quad\text{and}\quad gy_0\leq_p y_0.$$

(iii) *For every* $(x,y)\in E\times E$*, we have*

$$gx\leq_P x,\ gy\leq_p y \Longrightarrow gF(x,y)\geq_P F(x,y),\ gF(y,x)\geq_P F(y,x).$$

(iv) *For every* $(x,y)\in E\times E$*, we have*

$$gx\geq_P x,\ gy\geq_P y \Longrightarrow gF(x,y)\leq_P F(x,y),\ gF(y,x)\leq_P F(y,x).$$

(v) *There exists some* $\psi\in\Psi$ *such that*

$$\|F(u,v)-F(x,y)\|+\|F(y,x)-F(v,u)\|\leq \psi\left(\|u-x\|+\|v-y\|\right),$$

for all $(x,y),(u,v)\in E\times E$ *with* $gx\leq_P x$*,* $gy\leq_P y$ *and* $gu\geq_P u$*,* $gv\geq_P v$*.*
Then F *and* g *have a unique common coupled fixed point.*

Proof. Let us consider the mappings $\varphi_1,\varphi_2: E\times E\to E$ defined by

$$\varphi_1(x,y)=gx-x,\quad (x,y)\in E\times E$$

and

$$\varphi_2(x,y)=gy-y,\quad (x,y)\in E\times E.$$

Observe that $(x,y)\in E\times E$ is a common coupled fixed point of F and g if and only if $(x,y)\in E\times E$ is a solution to Pb. (2.11). Note that since g is continuous, then

φ_i is level closed from the right (also from the left) for all $i = 1, 2$. Now, applying Theorem 2.3, we obtain the desired result.

3.3. **A fixed point result.** We denote by $\widetilde{\Psi}$ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:
$(\widetilde{\Psi}_1)$ $\psi \in \Psi$.
$(\widetilde{\Psi}_2)$ For all $a, b \in [0,\infty)$, we have

$$\psi(a) + \psi(b) \le \psi(a+b).$$

Example 3.6. As example, let us consider the function

$$\psi(t) = \begin{cases} t/2 & \text{if} \quad 0 \le t < 1, \\ t - 1/3 & \text{if} \quad t \ge 1. \end{cases}$$

It is not difficult to observe that $\psi \in \Psi$. Now, let us consider an arbitrary pair $(a,b) \in [0,\infty) \times [0,\infty)$. We discuss three possible cases.
Case 1. If $(a,b) \in [0,1) \times [0,1)$.
In this case, we have $\psi(a)+\psi(b) = (a+b)/2$. O the other hand, we have $a+b \in [0,2)$. So, if $0 \le a+b < 1$, then $\psi(a)+\psi(b) = (a+b)/2 = \psi(a+b)$. However, if $1 \le a+b < 2$, then $\psi(a+b) - \psi(a) - \psi(b) = (a+b)/2 - 1/3 \ge 0$.
Case 2. If $(a,b) \in [0,1) \times [1,\infty)$.
In this case, we have $\psi(a) + \psi(b) = a/2 + b - 1/3 \le a + b - 1/3 = \psi(a+b)$.
Case 3. If $(a,b) \in [1,\infty) \times [1,\infty)$.
In this case, we have $\psi(a) + \psi(b) = a + b - 2/3 \le a + b - 1/3 = \psi(a+b)$.
Therefore, we have $\psi \in \widetilde{\Psi}$.

Note that the set Ψ is more large than the set $\widetilde{\Psi}$. The following example illustrates this fact.

Example 3.7. Let us consider the function

$$\psi(t) = \begin{cases} t/2 & \text{if} \quad 0 \le t \le 1, \\ 1/2 & \text{if} \quad t > 1. \end{cases}$$

Clearly, we have $\psi \in \Psi$. However,

$$\psi(1+1) = 1/2 < 1 = \psi(1) + \psi(1),$$

which proves that $\psi \notin \widetilde{\Psi}$.

We have the following fixed point result.

Corollary 3.8. *Let $T : E \to E$ be a given mapping. Suppose that there exists some $\psi \in \widetilde{\Psi}$ such that*

$$\|Tu - Tx\| \le \psi(\|u - x\|), \quad (u,x) \in E \times E. \tag{3.3}$$

Then T has a unique fixed point.
Proof. Let us define the mapping $F : E \times E \to E$ by

$$F(x,y) = Tx, \quad (x,y) \in E \times E.$$

Let $g : E \to E$ be the identity mapping, that is,

$$gx = x, \quad x \in E.$$

From (3.3), for all $(x,y),(u,v) \in E \times E$, we have

$$\|Tu - Tx\| \le \psi(\|u - x\|)$$

and

$$\|Ty - Tv\| \le \psi(\|v - y\|).$$

Then

$$\|Tu - Tx\| + \|Ty - Tv\| \le \psi(\|u - x\|) + \psi(\|v - y\|).$$

Using the property $(\widetilde{\Psi}_2)$, we obtain

$$\|Tu - Tx\| + \|Ty - Tv\| \le \psi(\|u - x\| + \|v - y\|), \quad (x,y),(u,v) \in E \times E.$$

From the definitions of F and g, we obtain

$$\|F(u,v) - F(x,y)\| + \|F(y,x) - F(v,u)\| \le \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x,y),(u,v) \in E \times E$ with $gx \le_P x$, $gy \le_P y$ and $gu \ge_P u$, $gv \ge_P v$.
By Corollary 3.5, there exists a unique $(x^*,y^*) \in E \times E$ such that

$$x^* = F(x^*,y^*) = Tx^* \quad \text{and} \quad y^* = F(y^*,x^*) = Ty^*.$$

Suppose that $x^* \ne y^*$. By (3.3), we have

$$\|x^* - y^*\| = \|Tx^* - Ty^*\| \le \psi(\|x^* - y^*)) < \|x^* - y^*\|,$$

which is a contradiction. As consequence, $x^* \in E$ is the unique fixed point of T.

Remark 3.9. Taking

$$\psi(t) = kt, \quad t \ge 0,$$

where $k \in (0,1)$ is a constant, we obtain from Corollary 3.8 the Banach Contraction Principle.

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