Fixed Point Theory, 19(2018), No. 2, 595-610
DOI: 10.24193/fpt-ro.2018.2.47
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON THE SIZE OF A MAP

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Abstract. Some properties depending on an upper bound of the diameter of fibers of a continuous map f from the n-dimensional unit cube I^n to the Euclidean space are investigated. In particular, we consider the problem when the image $f(I^n)$ has the nonempty interior. Obtained results are consequences of the Poincaré theorem and some theorems on extensions of maps. Generalizations of the De Marco theorem and the Borsuk theorem are presented.

Key Words and Phrases: Domain invariance theorem, Bolzano-Poincaré theorem, Brouwer fixed point theorem, size of a map.
2010 Mathematics Subject Classification: 54H25, 55M20, 54F45, 54B25.

1. Introduction

For a map $f : X \to Y$, $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, the *size of the map* f is

$$s(f) = \sup\{\mathrm{diam} f^{-1}(f(x)) : x \in X\},$$

where $\mathrm{diam} f^{-1}(f(x))$ denotes the diameter of the set $f^{-1}(f(x))$ in some norm. Let $\theta = (0, \dots, 0) \in \mathbb{R}^n$ denote the origin of $\mathbb{R}^n$.

In this paper the following facts will be established:

- for a continuous function $f : I^n \to Y$ defined on a cube $I^n = [-a, a]^n$ ($a > 0$, $n \in \mathbb{N}$), endowed with the maximum metric (see notation):
  (1.1) if $s(f) < a$, then $f(\theta) \in \mathrm{int} f(I^n)$ (Theorem 5.6),

The authors would like to thank the anonymous referees for their comments which enabled to improve the paper.

(1.2) if $s(f) < 2a$, then $\mathrm{int} f(I^n) \neq \emptyset$ (Theorem 4.8),

- for a continuous function $f : \mathrm{fr} I^n \to \mathbb{R}^m$ $(m \in \mathbb{N})$:

(1.3) if $f(\mathrm{fr} I^n)$ is homeomorphic to a convex set, then $s(f) = 2a$ (Theorem 4.11).

From (1.1) we obtain the domain invariance theorem [9] stating that if $f : U \to \mathbb{R}^n$ is a continuous and one to one map from an open set $U \subset \mathbb{R}^n$, then $f(U)$ is an open subset of $\mathbb{R}^n$, and the Borsuk theorem [3]: if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map of a finite size $(s(f) < \infty)$, then $f(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^n$.

Fact (1.2) easily implies

(1.4) if $f(I^n)$ is a boundary set in $\mathbb{R}^n$, then $s(f) = 2a$,

which is similar to the Borsuk-Ulam theorem stating that $s(f) = 2$ for any continuous map $f : S^n \to \mathbb{R}^n$, where S^n is the n-dimensional unit sphere. Let us notice that the map $f : [-a, a] \to \mathbb{R}$, $f(x) = 0$ for $x \leqslant 0$ and $f(x) = x$ for $x \geqslant 0$ confirms that (1.1) does not hold for $s(f) = a$. Furthermore, since for any constant map from I^n to $\mathbb{R}^n$ we have $s(f) = 2a$, the inequality in (1.2) cannot be improved.

Moreover, from fact (1.2) we can derive the non-retraction theorem for convex sets and subsequently the Brouwer fixed point theorem. In fact a more general version of the non-retraction theorem is true:

(1.5) for a set $D \subset \mathbb{R}^n$ homeomorphic to I^n, there is no continuous map $f : D \to \mathrm{fr} D$ which is a homeomorphism on $\mathrm{fr} D$.

Without loss of generality we can assume that $D = I^n$. To see (1.5) observe that for the maximum norm $||\cdot||_m$ and $x, y \in I^n$, the equality $||x - y||_m = 2a$ holds if and only if the points x and y belong to a pair of opposite faces of I^n. If f is continuous, then by (1.2) $s(f) = 2a$. There exists $x \in I^n$ such that $\mathrm{diam} f^{-1}(f(x)) = 2a$ and by the compactness of $f^{-1}(f(x))$ there exist $x_1, x_2 \in I^n : ||x_1 - x_2||_m = 2a > 0$, hence $x_1, x_2 \in \mathrm{fr} I^n$. Therefore $f(x) = f(x_1) = f(x_2)$, which is impossible.

Results of this paper are based on some combinatorial lemma presented in Section 3.

2. Notation

Let $\mathbb{Z}$ denote the set of integers, $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and $[n] = \{1, \ldots, n\}$, $[n]_0 = [n] \cup \{0\}$, $n \in \mathbb{N}$. For $x = (x_1, \ldots, x_n)$ $(x_i \in \mathbb{R}, i \in [n])$, $||x||_E = (\sum_{i \in [n]} x_i^2)^{1/2}$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. The norm $||\cdot||_E$ induces a metric $\rho_E(x, y) = ||x - y||_E$ on $\mathbb{R}^n$ $(x, y \in \mathbb{R}^n)$. We also use the maximum norm on $\mathbb{R}^n$: $||x||_m = \max_{i \in [n]} |x_i|$, where $|x_i|$ denotes the absolute value of $x_i \in \mathbb{R}$, $x \in \mathbb{R}^n$; a metric induced by the maximum norm is called a maximum metric and it is denoted by ρ_m.

Let X denote a metric space with a metric ρ. For $x \in X$ and a real number $\varepsilon > 0$ $B(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}$ denotes the open ball centered at x with radius ε (briefly, ε-ball at x). The closed ε-ball at x is denoted by $\overline{B}(x, \varepsilon)$. For a set $A \subset X$ we denote by $d(x, A) = \inf\{\rho(x, a) : a \in A\}$ the distance of a point x from the set A, $\mathrm{diam} A = \sup\{\rho(x, y) : x, y \in A\}$ is the diameter of the set A, and $B(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ stands for the open ε-hull of the set A. For sets $A, C \subset X$ in a metric space (X, ρ) and $\varepsilon > 0$ $B_A(C, \varepsilon) = \{x \in A : d(x, C) < \varepsilon\}$ and $\overline{B}_A(C, \varepsilon)$ is the closure of $B_A(C,\varepsilon)$ in A. Moreover, int$A$, $\overline{A}$, fr$A$ denote interior, closure, and boundary of the set A, respectively. $A \subset \mathbb{R}^n$ is said to be a *boundary set* if $A \subset \mathrm{fr}A = \overline{A} \cap \overline{X\backslash A} = \overline{A}\backslash \mathrm{int}A$; so, a set is boundary if its interior is the empty set. For $A \subset \mathbb{R}^n$ the convex hull of A is denoted by conv$A$. Notice that in $\mathbb{R}^n$ endowed with the maximum metric, the cube $[-1,1]^n$ is exactly the unit ball $\overline{B}(\theta,1)$. If $f : X \to Y$ (Y a metric space) is a function, then for any non-empty set $A \subset X$ the restriction of f to A is denoted by $f|_A$, $f|_A : A \to Y$, $f|_A(x) = f(x)$, $x \in A$.

Unless otherwise stated we endow the space $\mathbb{R}^n$ with an arbitrary norm $||\cdot||$ (and the induced metric). We also tacitly use the convention that if a space is normed with a norm $||\cdot||$, then the metric induced by this norm is denoted by ρ.

3. Combinatorial lemma and topological lemma

In this section we prove the Poincaré theorem using a combinatorial lemma for cubes which can be considered as a counterpart of the well-known Sperner lemma for cubes.

Let us fix numbers $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $a > 0$. Let $C_k = \{ja/k : j \in [k]_0 \text{ or } -j \in [k]_0\}$. Define a *combinatorial n-cube* C_k^n as the Cartesian product of n copies of the set C_k:

$$C_k^n = \underbrace{C_k \times \ldots \times C_k}_{n \text{ times}}.$$

If there is no ambiguity we write briefly C instead of C_k^n. It is clear that

$$C = \{x \in I^n : x_j \in \{-ja/k, ja/k\} \text{ where } j \in [k]_0\},$$

where $I^n = [-a,a]^n$. Define the *i-th combinatorial faces* of C by

$$C_i^- = C_i^-(k) = \{z \in C : z_i = -a\} \text{ and } C_i^+ = C_i^+(k) = \{z \in C : z_i = a\},\ i \in [n]$$

and the *c-boundary* (*combinatorial boundary*) of C by

$$\mathrm{bnd}C = \bigcup_{i\in[n]} (C_i^- \cup C_i^+).$$

Let $e^i = (0,\ldots,0,\frac{a}{k},0,\ldots,0)$, $e^i_i = \frac{a}{k}$, be the i-th basic vector. By $P(n)$ we denote the set of all permutations of $[n]$.

A set $S = \{z^0,\ldots,z^n\} \subset C$ is said to be a combinatorial *n-simplex* (with vertices z^i, $i \in [n]_0$) if there exists a permutation $\alpha = (\alpha_1,\ldots,\alpha_n) \in P(n)$ such that

$$z^1 = z^0 + e^{\alpha_1},\ z^2 = z_1 + e^{\alpha_2},\ldots, z^n = z^{n-1} + e^{\alpha_n}.$$

We have $C = \bigcup_{S\subset C, S \text{ is } n\text{-simplex}} S$. If $S = \{z^0,\ldots,z^n\}$ is a combinatorial n-simplex, then the set $S_i = \{z^0,\ldots,z^{i-1},z^{i+1},\ldots,z^n\}$, $i \in [n]_0$ (we denote $z^{-1} = z^n$, $z^{n+1} = z^0$), is said to be the i-th $(n-1)$-*face* of the combinatorial n-simplex S. We say that an $(n-1)$-face S_j, $j \in [n]_0$, of a combinatorial n-simplex $S \subset C$ belongs to the c-boundary bnd$C$ of C if there are $i \in [n]$ and $\varepsilon \in \{-,+\}$ such that

$$S_j \subset C_i^\varepsilon.$$

Observe that any $(n-1)$-face of a combinatorial n-simplex contained in the combinatorial n-cube C is an $(n-1)$-face of exactly one or two combinatorial n-simplices contained in C, depending on whether or not it belongs to the c-boundary of C (see

[12], [13]). The following lemma is a slight modification of a fact contained in the proof of the Poincaré theorem presented in [8] and [11]. Let us also note that a version of the proof below leads directly to the Brouwer fixed point theorem [14].

Lemma 3.1 (combinatorial lemma)**.** *Let $n \in \mathbb{N}$ be fixed. Suppose that $\{F_i^- : i \in [n]\}$ is a family of subsets of the combinatorial cube C such that*

$$C_i^- \subset F_i^- \subset C \backslash C_i^+,\ i \in [n]. \tag{3.1}$$

Then there exists a combinatorial n-simplex $S \subset C$ such that for each $i \in [n]$

$$F_i^- \cap S \neq \emptyset \neq S \cap (C \backslash F_i^-). \tag{3.2}$$

Proof. Define a map $\varphi : C \to [n]_0$ by

$$\varphi(x) = \min\{i - 1 : x \in F_i^-,\ i \in [n+1]\},$$

where $F_{n+1}^- = C$. The map φ has the following properties:

(a) if $x \in C_i^-$, then $\varphi(x) < i$, and if $x \in C_i^+$, then $\varphi(x) \neq i - 1$,
(b) for each simplex $S \subset C$ such that $\varphi(S \cap C_i^\varepsilon) = [n-1]_0$ it holds that $i = n$ and $\varepsilon = -$,
(c) if $\varphi(x) = i - 1$ and $\varphi(y) = i$, then $x \in F_i^-$ and $y \notin F_i^-$.

Property (a) and (c) easily follow from the definition of φ and inclusions (3.1). Property (b) is a consequence of property (a).
We say that an n-simplex S is *proper* if $\varphi(S) = [n]_0$. Similarly, we call an $(n-1)$-face s of an n-simplex contained in C *proper* if $\varphi(s) = [n-1]_0$. The property (c) implies that the claim is proved if the number ρ of proper simplices is odd. We prove the lemma by induction on the dimension n of C. Notice that the lemma holds for $n = 1$. According to (b) for any proper face s that belongs to the c-boundary of C is contained in C_n^- and by our inductive hypothesis the number α of such faces is odd. Let $\alpha(S)$ denote the number of proper faces of a simplex $S \subset C$. Now, if S is a proper simplex, clearly $\alpha(S) = 1$; while S is not a proper simplex, we have $\alpha(S) = 2$ or $\alpha(S) = 0$ according to either $\varphi(S) = [n-1]_0$ or $\varphi(S) \neq [n-1]_0$, respectively. Hence

$$\rho = \sum \alpha(S) \mod 2,$$

where the sum is taken over the set of n-simplices S contained in C. On the other hand, a proper face appears exactly once or twice in the sum $\sum \alpha(S)$ depending if it is on the boundary of C or not. Accordingly,

$$\sum \alpha(S) = \alpha \mod 2,$$

and thus

$$\alpha = \rho \mod 2.$$

But since α is odd, ρ is odd too. □

Now we present some consequences of the combinatorial lemma. For the cube I^n, we denote

$$I_i^- = \{x \in I^n : x_i = -a\} \text{ and } I_i^+ = \{x \in I^n : x_i = a\},\ i \in [n].$$

Lemma 3.2 (topological lemma; Eilenberg-Otto, Theorem 1.8.1 in [6])**.** *Let* $\{H_i^-, H_i^+ : i \in [n]\}$*,* $n \in \mathbb{N}$*, be a family of closed subsets of the cube* I^n *such that* $I^n = H_i^- \cup H_i^+$*,* $I_i^- \subset H_i^-$*,* $I_i^+ \subset H_i^+$*,* $i \in [n]$*. Then*

$$\bigcap_{i\in[n]} (H_i^- \cap H_i^+) \neq \emptyset.$$

Proof. Let us fix $\varepsilon \in (0, a)$ and define

$$H(\varepsilon)_i^- = H_i^- \backslash (B(I_i^+, \varepsilon) \cap I^n), \quad H(\varepsilon)_i^+ = \overline{I^n \backslash H(\varepsilon)_i^-}, \quad i \in [n].$$

Notice that $H(\varepsilon)_i^+ \subset \overline{B(H_i^+, \varepsilon)}$, $i \in [n]$. For each $k \in \mathbb{N}\backslash\{1\}$ define

$$F_i^{k-} = C_k^n \cap H(\varepsilon)_i^-, \ i \in [n].$$

Assumptions on the sets H_i^-, $i \in [n]$, imply that the sets F_i^{k-}, $i \in [n]$, satisfy the hypotheses of the combinatorial lemma for the combinatorial cube C_k^n. We conclude that for the fixed $\varepsilon > 0$ for each $k \geqslant 2$ there exists an n-simplex $S_\varepsilon^k \subset C_k^n$ satisfying the condition (3.2) (with S, F_i^-, C, replaced with S_ε^k, F_i^{k-}, C_k^n, respectively). By the compactness of $H(\varepsilon)_i^-$, $H(\varepsilon)_i^+$, $i \in [n]$, and the fact that $\lim_{k\to+\infty} \operatorname{diam} S_\varepsilon^k = 0$, there exists a point $x^\varepsilon \in \bigcap_{i\in[n]}(H(\varepsilon)_i^- \cap H(\varepsilon)_i^+)$. So, for each sufficiently large $q \in \mathbb{N}$ there exists $x^q \in \bigcap_{i\in[n]}(H(1/q)_i^- \cap H(1/q)_i^+) \subset I^n$ and we see that any cluster point of the sequence x^q, $q \in \mathbb{N}$, belongs to $\bigcap_{i\in[n]}(H_i^- \cap H_i^+)$. To finish the proof, observe that the compactness of I^n implies that such a cluster point exists. □

Corollary 3.3. *The Eilenberg-Otto lemma implies the combinatorial lemma.*

Proof. Indeed, let $\{F_i^- : i \in [n]\}$ be a family of subsets of the combinatorial cube $C = C_k^n$ such that $C_i^- \subset F_i^- \subset C\backslash C_i^+$ for each $i \in [n]$. Define for $i \in [n]$:

$$H_i^- = \bigcup \{\operatorname{conv} S : S \cap F_i^- \neq \emptyset, S \subset C \text{ is a combinatorial } n\text{-simplex}\}.$$

and

$$H_i^+ = \bigcup \{\operatorname{conv} S : S \cap (C\backslash F_i^-) \neq \emptyset, S \subset C \text{ is a combinatorial } n\text{-simplex}\}.$$

According to the Eilenberg-Otto lemma there is a point $c \in \bigcap_{i\in[n]}(H_i^- \cap H_i^+)$. A combinatorial simplex $S \subset C(k)$ with $c \in \operatorname{conv} S$ satisfies the assertion of the combinatorial lemma. □

We also obtain a few other corollaries.

Corollary 3.4 (Poincaré theorem, [8])**.** *Let* $f : I^n \to \mathbb{R}^n$*,* $f = (f_1, \ldots, f_n)$*, be a continuous map such that for each* $i \in [n]$*,* $f_i(I_i^-) \subset (-\infty, 0]$ *and* $f_i(I_i^+) \subset [0, \infty)$*. Then there exists a point* $c \in I^n$ *such that* $f(c) = \theta$*.*

Proof. For each $i \in [n]$ let us put $H_i^- = f_i^{-1}((-\infty, 0])$, $H_i^+ = f_i^{-1}([0, \infty))$. These sets satisfy the assumptions of the topological lemma and therefore

$$C = \bigcap_{i\in[n]} (H_i^- \cap H_i^+) \neq \emptyset.$$

It is clear that $f(c) = \theta$ for each $c \in C$. □

Observe that the Poincaré theorem can be strengthened to a weak version of the invariance of domain theorem:

Corollary 3.5 (Bolzano-Poincaré theorem)**.** *If* $f = (f_1, \ldots, f_n) : I^n \to \mathbb{R}^n$ *is a continuous map such that for each* $i \in [n]$*:* $f_i(I_i^-) \subset (-\infty, 0)$ *and* $f_i(I_i^+) \subset (0, +\infty)$*, then* $\theta \in \operatorname{int} f(I^n)$*.*

Proof. From compactness of I^n and the assumptions on f, there exists $\delta > 0$ such that $f_i(I_i^-) \subset (-\infty, -\delta)$ and $f_i(I_i^+) \subset (\delta, +\infty)$ for each $i \in [n]$. Now observe that for each $b \in J^n = [-\delta, \delta]^n$ the map $f_b(x) = f(x) - b$, $x \in I^n$, also satisfies the assumptions of the Poincaré theorem. Therefore, there is $c \in I^n$ such that $f_b(c) = \theta$, i.e. $f(c) = b$. Thus we have proved that $J^n \subset f(I^n)$. □

Corollary 3.6 (coincidence theorem)**.** *If maps* g, $h : I^n \to I^n$ *are continuous and for each* $i \in [n]$*:* $h(I_i^-) \subset I_i^-$ *and* $h(I_i^+) \subset I_i^+$*, then they have the coincidence property, i.e. there exists a point* $c \in I^n$ *such that* $g(c) = h(c)$*.*

Proof. Let us put $f(x) = h(x) - g(x)$. The map f satisfies the assumptions of the Poincaré theorem and therefore there is a point $c \in I^n$ such that $f(c) = \theta$. But this means that $g(c) = h(c)$. □

If h is the identity map,Hence the function g satisfies assumptions of the Poincaré theorem and there exists then we get a fixed-point theorem discovered by Bohl [2]:

Corollary 3.7 (Bohl-Brouwer fixed point theorem)**.** *Any continuous map* $g : I^n \to I^n$ *has a fixed point.*

Applying the coincidence theorem to the constant map $g : I^n \to I^n$ we get

Corollary 3.8. *Any continuous map* $h : I^n \to I^n$ *satisfying* $h(I_i^-) \subset I_i^-$ *and* $h(I_i^+) \subset I_i^+$*,* $i \in [n]$*, is surjective.*

Corollary 3.9 (Borsuk non-retraction theorem)**.** *Let* $f : X \to \mathbb{R}^n$ *be a continuous map from a compact set* $X \subset \mathbb{R}^n$*. If* $f(x) = x$ *for each* $x \in \operatorname{fr} X$*, then* $X \subset f(X)$*.*

Proof. Without loss of generality we assume that $X \subset I^n$ and $f(X) \subset I^n$, where I^n is an n-dimensional cube. Let us define the map $h : I^n \to I^n$ by $h(x) = x$ for each $x \in I^n \setminus X$ and $h(x) = f(x)$, $x \in X$. It is obvious that the map h is continuous, $h(I_i^-) \subset I_i^-$ and $h(I_i^+) \subset I_i^+$, $i \in [n]$. From the Corollary 3.8 we infer that $I^n \subset h(I^n)$, and due to the fact that $x \in I^n \setminus X$ implies $h(x) = x \in I^n \setminus X$ we conclude with the inclusion $X \subset f(X)$. □

Let us observe that if we replace the assumption on behavior of f on $\operatorname{fr} X$ by the assumption that f maps the boundary $\operatorname{fr} X$ homeomorphically onto $\operatorname{fr} X$, then the assertion of Theorem 3.9 may be false. This is shown in the following

Example 3.10. From (1.5), for a compact and convex set $X \subset \mathbb{R}^n$ and a continuous function $f : X \to \mathbb{R}^n$ such that $f|_{\operatorname{fr} X}$ is a homeomorphism onto $\operatorname{fr} X$ we have $X \subset f(X)$. This is no longer true for a set $X = \overline{\operatorname{int} X}$, $X \subset R^n$.

Let $A = \bigcup_{k,l \in \mathbb{Z}} (([2k, 2k+1] \times [2l, 2l+1]) \cup ([2k-1, 2k] \times [2l-1, 2l])) \subset \mathbb{R}^2$ (we can think that the set A consists of all closed black squares of a 2-dimensional *unbounded* *chessboard*) and let B denote the open unit ball in $\mathbb{R}^2$. Define continuous mappings $g:\mathbb{R}^2\to B$ and $h:A\to\mathbb{R}^2$ by $g(x)=\frac{1}{1+||x||}x$ and $h(x)=(x_1+1,x_2)$, for $x=(x_1,\,x_2)$ respectively. Observe that the function g is a homeomorphism, the function $h|_{\mathrm{fr}A}$ is a homeomorphism onto $\mathrm{fr}A$ and $\mathrm{int}A\cap h(A)=\emptyset$. Notice that $\mathrm{fr}g(A)=\mathrm{fr}(\mathrm{int}g(A))$, $\mathrm{fr}B\subset\mathrm{fr}g(A)$, $\overline{g(A)}=\mathrm{fr}B\cup g(A)$ and $\mathrm{fr}B\cap g(A)=\emptyset$. Now, let $X=\overline{g(A)}$ and define a function $f:X\to\mathbb{R}^2$ by

$$f(x)=\begin{cases} x, & \text{if } x\in\mathrm{fr}B,\\ g(h(g^{-1}(x))), & \text{if } x\in g(A).\end{cases}$$

It is not difficult to check that f is a continuous function, the function $f|_{\mathrm{fr}X}$ is a homeomorphism onto $\mathrm{fr}X$. Furthermore, $f(X)\cap\mathrm{int}X=\emptyset$ and $\mathrm{int}X\neq\emptyset$. The compact set X is not contained in the image of f in spite of the fact that f maps the boundary $\mathrm{fr}X$ homeomorphically onto $\mathrm{fr}X$.

Let us emphasize that $X=\overline{\mathrm{int}X}$ in our example. Our construction can be easily adapted for the n-dimensional case, $n\in\mathbb{N}$, as well.

Another application of the Poincaré theorem is a generalization of the exploding point theorem [9].

Corollary 3.11. *Let $\delta>0$ and $(-\delta,\delta)^n\subset I^n$. If $f:I^n\to\mathbb{R}^n\backslash(-\delta,\delta)^n$ satisfies $f_i(I_i^-)\subset(-\infty,-\delta]$, $f_i(I_i^+)\subset[\delta,+\infty)$, $i\in[n]$, then there exists $j\in[n]$ and a point $c\in I^n$ such that for any $\varepsilon>0$ there are points $x,y\in B(c,\varepsilon)$ with $f_j(x)\leqslant-\delta$ and $f_j(y)\geqslant\delta$. Moreover, $f_j(c)\leqslant-\delta$ or $f_j(c)\geqslant\delta$.*

Proof. For the point-set distance function d generated by the Euclidean metric, let $g_i(x)=d(x,f_i^{-1}((-\infty,-\delta]))-d(x,f_i^{-1}([\delta,+\infty)))$, $x\in I^n$, $i\in[n]$. Obviously, $g=(g_1,\dots,g_n)$ is a continuous function on I^n. For $i\in[n]$ and $x\in I_i^-$ we have

$$g_i(x)=d(x,f_i^{-1}((-\infty,-\delta]))-d(x,f_i^{-1}([\delta,+\infty)))=-d(x,f_i^{-1}([\delta,+\infty)))\leqslant 0,$$

and for $x\in I_i^+$, $i\in[n]$

$$g_i(x)=d(x,f_i^{-1}((-\infty,-\delta]))-d(x,f_i^{-1}([\delta,+\infty)))=d(x,f_i^{-1}((-\infty,-\delta]))\geqslant 0.$$

Hence the function g satisfies assumptions of the Poincaré theorem and there exists $c\in I^n$ such that $g(c)=\theta$. It means that $d(c,f_i^{-1}((-\infty,-\delta]))=d(c,f_i^{-1}([\delta,+\infty)))$, $i\in[n]$. Since $f(c)\notin(-\delta,\delta)^n$, there exists $j\in[n]$ such that $f_j(c)\notin(-\delta,\delta)$. Therefore either $d(c,f_j^{-1}((-\infty,-\delta]))=0$ or $d(c,f_j^{-1}([\delta,+\infty)))=0$ and thus $d(c,f_j^{-1}((-\infty,-\delta]))=0=d(c,f_j^{-1}([\delta,+\infty)))$. The index j and the point c satisfy our assertion, since c belongs to the closures of both $f_j^{-1}((-\infty,-\delta])$ and $f_j^{-1}([\delta,+\infty))$. □

4. Squeezing and non-squeezing of balls

Let us start with auxiliary lemmas. Some of them are well-known. First, recall that two continuous maps $f_0,f_1:X\to Y$ between topological spaces are *homotopic* if there exists a continuous map $h:X\times[0,1]\to Y$ such that $h(x,t)=f_t(x)$ for $t\in\{0,\,1\}$ and $x\in X$; the map h is called a homotopy between f_0 and f_1 (we write $f_0\simeq f_1$). A homotopy extension lemma due to Borsuk is an important tool in our paper.

Lemma 4.1 (see Borsuk homotopy extension lemma 1.9.7 in [6]). *For $c \in \mathbb{R}^n$, let $f, g : A \to \mathbb{R}^n \backslash \{c\}$ be homotopic continuous maps from a closed subset A of $X \subset \mathbb{R}^n$. If f has a continuous extension $\widetilde{f} : X \to \mathbb{R}^n \backslash \{c\}$, then g also admits a continuous extension $\widetilde{g} : X \to \mathbb{R}^n \backslash \{c\}$.*

Proof. If $A = X$, there is nothing to prove. Since $\mathbb{R}^n \backslash \{\theta\}$ is homeomorphic to $\mathbb{R}^n \backslash \{c\}$, without loss of generality we may assume that $c = \theta$. For $I = [0, 1]$, let $h : A \times I \to \mathbb{R}^n \backslash \{\theta\}$ be a homotopy between f and g. Let $D = X \times \{0\} \cup A \times I$ and define a continuous map $h' : D \to \mathbb{R}^n \backslash \{\theta\}$ as

$$h'(x, t) := \begin{cases} \widetilde{f}(x), & \text{if } (x, t) = (x, 0), \\ h(x, t), & \text{otherwise.} \end{cases}$$

Observe that D is a closed subset of the space $X \times I$. Let us fix a point $b \in (X \times I) \backslash D$ and consider a continuous extension $\widetilde{h}' : X \times I \to \mathbb{R}^n$ of the map: $(x, t) \mapsto h'(x, t)$, $(x, t) \in D$, and $b \mapsto \theta \in \mathbb{R}^n$. $\widetilde{h}'$ exists by the Tietze extension theorem. It follows that $U = \widetilde{h}'^{-1}(\{y \in \mathbb{R}^n : y \neq \theta\})$ is an open subset of $X \times I$ not containing b and $D \subset U$, $\theta \notin \widetilde{h}'(U)$. From the compactness of I it follows that there exists an open set $V \subset X$ such that $A \times I \subset V \times I \subset U$. By the Urysohn lemma there exists a continuous function $u : X \to [0, 1]$ such that $u(X \backslash V) = \{0\}$ and $u(A) = \{1\}$. The map $H : X \times I \to \mathbb{R}^n \backslash \{\theta\}$ defined by $H(x, t) = \widetilde{h}'(x, tu(x))$, $(x, t) \in X \times I$, is a homotopy between the maps $\widetilde{f} : X \to \mathbb{R}^n \backslash \{\theta\}$, $\widetilde{f}(x) = H(x, 0)$ and $\widetilde{g} : X \to \mathbb{R}^n \backslash \{\theta\}$, $\widetilde{g}(x) = H(x, 1)$, $x \in X$, $\widetilde{f} \simeq \widetilde{g}$, and $\widetilde{g}$ is a continuous extension of the map g. □

Theorem 4.2 (Borsuk theorem on the homotopy of identity). *Let $c \in \mathrm{int}X$ and $f : X \to \mathbb{R}^n$ be a continuous map from a compact subset $X \subset \mathbb{R}^n$ such that $c \in \mathrm{int}X \backslash f(\mathrm{fr}X)$. If the maps $f|_{\mathrm{fr}X} : \mathrm{fr}X \to \mathbb{R}^n \backslash \{c\}$ and the identity $id : \mathrm{fr}X \to \mathbb{R}^n \backslash \{c\}$ are homotopic, then $c \in \mathrm{int}f(X)$. In particular, if $f(\mathrm{fr}X) \subset \mathrm{fr}X$ and the maps $id, f|_{\mathrm{fr}X} : \mathrm{fr}X \to \mathrm{fr}X$ are homotopic, then $\mathrm{int}X \subset f(X)$.*

Proof. Let $h : \mathrm{fr}X \times [0, 1] \to \mathbb{R}^n \backslash \{c\}$ be a homotopy between $f|_{\mathrm{fr}X} : \mathrm{fr}X \to \mathbb{R}^n \backslash \{c\}$ and the identity $id|_{\mathrm{fr}X}$. Put $U = \mathrm{int}X \backslash h(\mathrm{fr}X \times [0, 1])$. The set U is open and $c \in U$. If there is a point $x_0 \in U \backslash f(X)$, then the map $f|_{\mathrm{fr}X} : \mathrm{fr}X \to \mathbb{R}^n \backslash \{x_0\}$ is homotopic to the identity map $id|_{\mathrm{fr}X}$. According to the Borsuk homotopy extension lemma the identity map $id|_{\mathrm{fr}X} : \mathrm{fr}X \to \mathbb{R}^n \backslash \{x_0\}$ has a continuous extension $id^* : X \to \mathbb{R}^n \backslash \{x_0\}$. Hence, $x_0 \notin id^*(X)$, which is impossible due to Corollary 3.9. Thus $c \in U \subset f(X)$. □

A map $f : X \to Y$, where X, Y are topological spaces, is said to be *fine* if for each compact boundary set $A \subset X$ the image $f(A)$ is a boundary set.

Lemma 4.3 (approximation by fine maps). *For a continuous map $f : X \to \mathbb{R}^n$ from a compact subset $X \subset \mathbb{R}^n$ and $\varepsilon > 0$, there is a continuous fine map $h : X \to \mathbb{R}^n$ such that $||h(x) - f(x)|| < \varepsilon$ for each $x \in X$.*

Proof. Let $\varepsilon > 0$ and $\delta > 0$ satisfy the condition: for x, $x' \in X$, $||x - x'|| < \delta$ implies $||f(x) - f(x')|| < \varepsilon$. Let $I^n = [-a, a]^n$ be a cube such that $X \subset I^n$. Extend the map f to a continuous map $g : I^n \to \mathbb{R}^n$. Let $C = C^n_k$ denote the combinatorial n-cube (see Section 3). We can assume that combinatorial n-simplices in C^n_k have diameters less than δ. Define a piecewise linear map $h : I^n \to \mathbb{R}^n$ in the following way: if $x \in S$ and $S = [z_0, ..., z_n] \subset C$, then we put

$$h(x) = \sum_{i=0}^{n} t_i g(z_i), \quad \text{where} \quad x = \sum_{i=0}^{n} t_i z_i,$$

where the coefficients t_i, $i \in [n]_0$, are the barycentric coordinates of x in the simplex S. Observe that there is no ambiguity in this definition if the point x belongs to more than one combinatorial simplices contained in C, so the map h is well-defined; h is continuous, as well. Moreover, $||f(x)-h(x)|| < \varepsilon$ for $x \in I^n$. Notice that for a simplex $S \subset C$ it is possible to extend $h|_{\mathrm{conv}S}$ affinely to the whole $\mathbb{R}^n$. It is obvious that affine mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ transform boundary compact subsets of $\mathbb{R}^n$ onto boundary compact sets in $\mathbb{R}^n$. Since there are a finite number of combinatorial simplices in $\mathbb{R}^n$, the function h is a fine approximation we are looking for. $\square$

Lemma 4.4 (on extensions of maps)**.** *Suppose that $A \subset \mathbb{R}^n$, $L \subset \mathbb{R}^n$ are compact sets, L is a boundary set, $c \in \mathbb{R}^n$ and let $X = A \cup L$. Then any continuous map $f : A \to \mathbb{R}^n \backslash \{c\}$ has a continuous extension $\widetilde{f} : X \to \mathbb{R}^n \backslash \{c\}$, $\widetilde{f}|_A = f$.*

Proof. Without loss of generality we may assume that $c = \theta$. Fix an arbitrary small $\varepsilon > 0$ such that $f(A) \cap B(\theta, 3\varepsilon) = \emptyset$. According to Lemma 4.3 there is a continuous fine map $h : X \to \mathbb{R}^n$ such that $||f(x)-h(x)|| < \varepsilon$ for each $x \in A$. Since $h(L)$ is a boundary set, we can fix a point $b \in B(\theta, \varepsilon) \backslash L$. Next, let us put $k(x) = h(x) - b$, $x \in X$. For $x \in A$ we have $||f(x) - k(x)|| < 2\varepsilon$ and $\varepsilon < ||k(x)||$. The map $g(x, t) = (1 - t)k(x) + tf(x)$, $t \in [0, 1]$, $x \in A$, is a homotopy between $k|_A$ and f, $k|_A \simeq f : A \to \mathbb{R}^n \backslash \{\theta\}$, and according to Lemma 4.1 the map f has a continuous extension $\widetilde{f} : X \to \mathbb{R}^n \backslash \{\theta\}$. $\square$

The following approximate selection lemma related to an upper bound of the diameter of fibers was first considered by Alexandroff in 1928 (see [1]).

Lemma 4.5 ([4], [10])**.** *Let $f : X \to Y$ be a continuous map from a compact space $X \subset \mathbb{R}^n$ onto a compact space $Y \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) and let $a > 0$ be a real number such that for each $y \in Y$: $\mathrm{diam} f^{-1}(y) < a$. Then there exists a continuous map $g : Y \to \mathbb{R}^n$ such that for each $x \in X$ $||x - g(f(x))|| < a$.*

Now, we formulate a result motivated by the Hurewicz theorem (see [6], Theorem 1.9.2), which we apply to prove a theorem on non-squeezing of cubes:

Theorem 4.6. *Let $X \subset \mathbb{R}^n$ be a compact set and $A \subset X$ be a closed subset of X such that $\mathrm{int}(X \backslash A) = \emptyset$. Then for every continuous map $f : A \to \mathrm{fr}B^n$ there exists a continuous extension $\widetilde{f} : X \to \mathrm{fr}B^n$ of f over X, where B^n is the closed unit ball $\overline{B}(\theta, 1) \subset \mathbb{R}^n$ (the metric is induced by a norm $|| \cdot ||$).*

Proof. Since $A \subset X$ and both sets A and X are compact and $\mathrm{int}(X \backslash A) = \emptyset$, $\mathrm{int}(\overline{X \backslash A}) = \emptyset$ and the set $L = \overline{X \backslash A} \subset X$ is a boundary subset of $\mathbb{R}^n$. Indeed, if $x \in X \backslash A$, then $x \in \mathrm{fr}X$, and $\mathrm{fr}X = \overline{\mathrm{fr}X}$ together with $\mathrm{int}(\mathrm{fr}X) = \emptyset$ imply now that $L \subset \mathrm{fr}X$. Hence L is a compact set with the empty interior, so it is a boundary set. By

Lemma 4.4 there exists a continuous extension $\overline{f} : X \to \mathbb{R}^n \backslash \{\theta\}$ of f. The mapping $\widetilde{f} : X \to \mathrm{fr}B^n$ defined by

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in A, \\ \frac{\overline{f}(x)}{||\overline{f}(x)||}, & \text{if } x \in X \backslash A, \end{cases}$$

satisfies our assertion. □

Theorem 4.7 (on non-squeezing of balls)**.** *Let* $B^n = \overline{B}(\theta, 1)$ *be the closed unit ball in* $\mathbb{R}^n$ *(endowed with a norm* $|| \cdot ||$*). If* $f : B^n \to Y$ *is a continuous map onto* $Y \subset \mathbb{R}^n$ *such that* $\mathrm{diam}(f|_{\mathrm{fr}B^n})^{-1}(f|_{\mathrm{fr}B^n}(x)) < 1$ *for* $x \in \mathrm{fr}B^n$*, then* $\mathrm{int}(f(B^n) \backslash f(\mathrm{fr}B^n)) \neq \emptyset$*.*

Proof. Suppose that $\mathrm{int}(f(B^n) \backslash f(\mathrm{fr}B^n)) = \emptyset$. According to Lemma 4.5 there is a continuous map $g : f(\mathrm{fr}B^n) \to \mathbb{R}^n$ such that for $x \in \mathrm{fr}B^n$: $||x - g(f(x))|| < 1$, hence $\theta \notin g(f(\mathrm{fr}B^n))$ and $tx + (1-t)g(f(x)) \neq \theta$, $t \in [0, 1]$, $x \in \mathrm{fr}B^n$. In view of the continuity of f and the assumption $\mathrm{int}(f(B^n) \backslash f(\mathrm{fr}B^n)) = \emptyset$, the set $f(B^n) \backslash f(\mathrm{fr}B^n)$ is a compact boundary set and Lemma 4.4 implies that the map g has a continuous extension $\widetilde{g} : f(B^n) \to \mathbb{R}^n \backslash \{\theta\}$; observe that $\widetilde{g}(f(x)) = g(f(x))$ for $x \in \mathrm{fr}B^n$. Now define a continuous function $\overline{g} : B^n \to \mathbb{R}^n \backslash \{\theta\}$ by $\overline{g}(x) = \widetilde{g}(f(x))$, $x \in B^n$. Let also $h : \mathrm{fr}B^n \times [0, 1] \to \mathrm{fr}B^n$ be a map defined by

$$h(x, t) = \frac{(1-t)x + t\overline{g}(x)}{||(1-t)x + t\overline{g}(x)||},\ x \in \mathrm{fr}B^n,\ t \in [0, 1].$$

The map h is a homotopy between the identity map (on $\mathrm{fr}B^n$) and the map $x \mapsto \frac{\overline{g}(x)}{||\overline{g}(x)||}$, $x \in \mathrm{fr}B^n$. By Theorem 4.2 we have $\theta = \frac{\overline{g}(x)}{||\overline{g}(x)||}$ for some $x \in B^n$, which is not possible since $\overline{g}(x) \neq \theta$ for $x \in B^n$. □

The assumptions in Theorem 4.7 may be weakened if $\mathbb{R}^n$ is endowed with the maximum metric (norm). This is shown in

Theorem 4.8 (Theorem on non-squeezing of cubes)**.** *Let* $f : I^n \to \mathbb{R}^n$*, where* $I^n = [-a, a]^n$ *is endowed with the maximum metric, be a continuous map. If* $\mathrm{diam}(f|_{\mathrm{fr}I^n})^{-1}(f|_{\mathrm{fr}I^n}(x)) < 2a$ *for* $x \in \mathrm{fr}I^n$*, then* $\mathrm{int}(f(I^n) \backslash f(\mathrm{fr}I^n)) \neq \emptyset$*.*

Proof. Assume that $\mathrm{int}(f(I^n) \backslash f(\mathrm{fr}I^n)) = \emptyset$. Let us put $A_i = f(I_i^-)$, $B_i = f(I_i^+)$, for $i \in [n]$, and $X = f(I^n)$, $A = f(\mathrm{fr}I^n)$. The assumption

$$\mathrm{diam}(f\ |_{frI^n})^{-1}(f\ |_{frI^n}\ (x)) < 2a, x \in frI^n$$

implies that $A_i \cap B_i = \emptyset$ for each $i \in [n]$. Since $A = f(\mathrm{fr}I^n)$ is a normal space, there exists a continuous map $g : A \to I^n$, $g = (g_1, \ldots, g_n)$ such that $g_i(A_i) = \{-a\}$ and $g_i(B_i) = \{a\}$ for each i. This yields $g(A_i) \subset I_i^-$, $g(B_i) \subset I_i^+$, and in consequence $g(A) \subset \mathrm{fr}I^n$. Now, applying Theorem 4.6 to the map g we get a continuous extension $\widetilde{g} : X \to \mathrm{fr}I^n$. The composition $\widetilde{g} \circ f$ satisfies the assumptions of the Poincaré theorem (Corollary 3.4), so $\theta \in \widetilde{g}(f(I^n))$, but this contradicts the inclusion $\widetilde{g}(f(I^n)) \subset \mathrm{fr}I^n$. □

Observe that for $m, n \in \mathbb{N}$, $m < n$, the space $\mathbb{R}^m$ can be naturally embedded into $\mathbb{R}^n$ as a boundary subset (subspace) of $\mathbb{R}^n$. Let us also notice that if $I^n = [-a, a]^n$, $a >$

0, is endowed with the maximum metric and $f : I^n \to \mathbb{R}^m$ $(m \in \mathbb{N})$, then $s(f) = 2a$ is equivalent to $\mathrm{diam} f^{-1}(f(x)) = 2a$ for some $x \in \mathrm{fr} I^n$.

These imply

Corollary 4.9. *If $f : I^n \to \mathbb{R}^m$, $I^n = [-a, a]^n$, is a continuous map and $m < n$, then $s(f) = 2a$.*

Theorem 4.10. *Suppose that $I^n = [-a, a]^n$ is endowed with the maximum metric. Let $f : I^n \to Y$, $I^n = [-a, a]^n$, be a continuous map into $Y \subset \mathbb{R}^m$ $(m \in \mathbb{N})$ such that $f(\mathrm{fr} I^n) = f(I^n)$. Then $s(f) = 2a$.*

Proof. As in the previous proof, let us put $A_i = f(I_i^-)$, $B_i = f(I_i^+)$ for $i \in [n]$, and $X = f(I^n)$, $A = f(\mathrm{fr} I^n)$. Assume that $s(f) < 2a$. It means that $A_i \cap B_i = \emptyset$ for each $i \in [n]$. Since $A = f(I^n)$ is a normal space, there exists a continuous map $g : A \to I^n$, $g = (g_1, \dots, g_n)$ such that $g_i(A_i) = \{-a\}$ and $g_i(B_i) = \{a\}$ for each i. One can verify that $g(A_i) \subset I_i^-$, $g(B_i) \subset I_i^+$, and $g(f(I^n)) \subset \mathrm{fr} I^n$. But this contradicts the Poincaré theorem (Corollary 3.4) which says that $\theta \in g(f(I^n))$. $\square$

Theorem 4.11. *Suppose that $I^n = [-a, a]^n$ is endowed with the maximum metric. Let $f : \mathrm{fr} I^n \to Y$ be a continuous map onto a set $Y \subset \mathbb{R}^m$ homeomorphic to a convex subset of $\mathbb{R}^m$ $(m \in \mathbb{N})$. Then $s(f) = 2a$.*

Proof. According to the Dugundji extension Theorem [5, p.163] Y is an absolute extensor and therefore f has a continuous extension $F : I^n \to Y$. Thus the equality $F(\mathrm{fr} I^n) = F(I^n)$ holds and in consequence $s(f) = s(F) = 2a$. $\square$

Let $|| \cdot ||$ be a norm on $\mathbb{R}^n$ and $B^n = \{x \in \mathbb{R}^n : || \cdot || \leqslant 1\}$ be the closed unit ball. Define *n-size $s_n(|| \cdot ||)$ of the norm* $|| \cdot ||$ as the greatest real number such that for any continuous function $f : B^n \to \mathbb{R}^n$ from the closed unit ball $B^n \subset \mathbb{R}^n$ into $\mathbb{R}^n$ we have: if $\mathrm{diam} f^{-1}(f(x)) < s_n(|| \cdot ||)$ for each $x \in B^n$, then $\mathrm{int}(f(B^n) \backslash f(\mathrm{fr} B^n)) \neq \emptyset$.
It has been proved that for a norm $|| \cdot ||$ we have $1 \leqslant s_n(|| \cdot ||) \leqslant 2$ and $s_n(|| \cdot ||_m) = 2$ for the maximum norm $|| \cdot ||_m$. It is interesting to know

Problem 4.12. What is the n-size $s_n(|| \cdot ||_E)$ for the Euclidean norm?

Some results related to the above problem were obtained by Fedeli and Le Donne [7].

5. An extension of the domain invariance theorem

In this part we give an elementary proof of a generalization of the Brouwer domain invariance theorem, Theorem 5.6.

Lemma 5.1. *Let $Y \subset \mathbb{R}^n$ be a compact set, $G \subset \mathbb{R}^n$ an open set and $A \subset Y \cap G$ a compact connected set such that $A \cap \mathrm{fr} Y \neq \emptyset$. Then there exist an open set $V \subset \mathbb{R}^n$ with $A \subset V \subset \overline{V} \subset G$, a compact boundary set $L \subset G$ and a continuous map $r : Y \to \mathbb{R}^n$ such that $r(Y) \subset L \cup (Y \backslash V)$ with $r(y) = y$ for each $y \in Y \backslash V$.*

Proof. From the fact that $A \subset Y$ is a compact and disjoint from the closed set $\mathbb{R}^n \backslash G$, there exists a finite family of open balls $Q = \{B_i \subset \mathbb{R}^n : i \in [m]\}$, $m \in \mathbb{N}$, with $A \subset \bigcup_{i \in [m]} B_i \subset \bigcup_{i \in [m]} \overline{B_i} \subset G$, and $B_i \cap A \neq \emptyset$, $i \in [m]$, and $A \not\subset \bigcup_{i \in [m] \backslash \{j\}} B_i$, $j \in$

$[m]$. Define $V = \bigcup_{i\in[m]} B_i$ and $L = \bigcup_{i\in[m]} \mathrm{fr}B_i$. Let us now construct a function $r : Y \to \mathbb{R}^n$ satisfying the statement. Let $Y_1 = Y$, $M_1 = [m]$, $k = 1$. To construct the function $r : Y \to \mathbb{R}^n$ we apply the following *sweeping out* procedure:

Step 1. Let i_k be the smallest number in M_k for which $B_{i_k} \cap \mathrm{fr}Y_k \neq \emptyset$. Choose a point $a_k \in B_{i_k}\backslash Y_k$ and define the continuous function $r_k : Y \to \mathbb{R}^n$ as follows: $r_k(y) = y$ for $y \in Y \cap (Y_k\backslash B_{i_k})$, and to each $y \in B_{i_k} \cap Y_k$ assign the projection $r_k(y)$ of the point y onto $\mathrm{fr}B_{i_k}$ along the ray $a_k + t(y - a_k)$, $t \geqslant 0$. Go to STEP 2.

Step 2. If $k < m$, put $Y_{k+1} = Y_k\backslash B_{i_k}$, $M_{k+1} = M_k\backslash\{i_k\}$, $k = k + 1$ and go to STEP 1. If $k = m$, define $r(y) = (r_m \circ r_{m-1} \circ \ldots \circ r_1)(y)$, $y \in Y$, and STOP: the required function has been obtained.

The correctness of STEP 1 comes from the assumptions on the family Q. □

Lemma 5.2. *Let $U \subset \mathbb{R}^n$ be an open, connected and bounded set. Then for each number $\delta > 0$ there is an open and connected set W, $\overline{W} \subset U$, such that $d(x, \mathrm{fr}U) < \delta$ for each $x \in U\backslash W$.*

Proof. Fix $\delta > 0$. Since $\overline{U}$ is a compact set, there exists a finite family $\{B_i : i \in [m]\}$, $m \in \mathbb{N}$, of open balls whose diameters are less than δ and such that $\overline{U} \subset \bigcup_{i\in[m]} B_i$ and $U \cap B_i \neq \emptyset$, $i \in [m]$. For each $i \in [m]$ choose a point $x_i \in B_i \cap U$. We recall that a connected set and locally connected set is the arcwise connected set. Next, since the set U is open and connected we can fix an arcwise connected compact set $P \subset U$ consisting of $m - 1$ arcs connecting the set $\{x_i : i \in [m]\}$. The compactness of P and the openness of U imply that there is a family of open balls $\{Q_i \subset \mathbb{R}^n : i \in [l]\}$, $l \in \mathbb{N}$, with $P \subset \bigcup_{j\in[l]} Q_j$ and $\overline{Q}_j \subset U$, $j \in [l]$. Define $W = \left(\bigcup_{j\in[l]} Q_j\right) \cup \left(\bigcup_{i\in[m]:\, \overline{B}_i\subset U} B_i\right)$. Observe that $\overline{W} \subset U$ and the set W is (arcwise) connected because P is, balls are connected and $B_i \cap P \neq \emptyset$, $i \in [m]$, $Q_j \cap P \neq \emptyset$, $j \in [l]$. Now, fix $x \in U\backslash W$ and choose $i \in [m]$ such that $x \in B_i$. Since $x \notin W$ by definition of W we infer that $\overline{B}_i \not\subset U$. Hence $\overline{B}_i \cap (X\backslash U) \neq \emptyset$. This implies that $d(x, \mathrm{fr}U) \leqslant \mathrm{diam}B_i < \delta$. □

Lemma 5.3. *Let $f : X \to \mathbb{R}^n$ be a continuous map from a compact subset $X \subset \mathbb{R}^n$, a point $c \in \mathrm{int}X$ such that $c \notin \mathrm{conv}f^{-1}(f(x))$ for $x \in \mathrm{fr}X$. Then there exist $\varepsilon > 0$ such that $c \notin B(\mathrm{conv}f^{-1}(y), \varepsilon)$, $y \in f(\mathrm{fr}X)$, a compact set $K \subset \mathbb{R}^n$ with $f(\mathrm{fr}X) \subset \mathrm{int}K$ and a continuous map $g : K \to \mathbb{R}^n\backslash\{c\}$ with the following property: for $x \in f^{-1}(K)$ there is $y \in f(\mathrm{fr}X)$ such that $\{x, g(f(x))\} \subset B(\mathrm{conv}f^{-1}(y), \varepsilon)$.*

Proof. By the continuity of f the set $f(\mathrm{fr}X)$ is compact and for each $x \in \mathrm{fr}X$ there exists $\varepsilon_x > 0$ such that $d(c, \mathrm{conv}f^{-1}(f(x))) > \varepsilon_x$. Suppose that $\inf_{x\in\mathrm{fr}X} d(c, \mathrm{conv}f^{-1}(f(x))) = 0$. Then there exists a sequence $x^q \in \mathrm{fr}X$, $q \in \mathbb{N}$, with $\lim_{q\to+\infty} d(c, \mathrm{conv}f^{-1}(f(x^q))) = 0$. By the compactness of $\mathrm{fr}X$ we may assume that $\lim_{q\to+\infty} x^q = x \in \mathrm{fr}X$. The continuity of f implies that for a sufficiently large q we have $f^{-1}(f(x^q)) \subset B(f^{-1}(f(x)), \varepsilon_x/2) \subset B(\mathrm{conv}f^{-1}(f(x)), \varepsilon_x/2)$, and hence $\mathrm{conv}f^{-1}(f(x^q)) \subset B(\mathrm{conv}f^{-1}(f(x)), \varepsilon_x/2)$. Thus $d(c, \mathrm{conv}f^{-1}(f(x^q))) \geqslant \varepsilon_x/2 > 0$ for a sufficiently large q; but this is impossible.

Now, let us fix $\varepsilon > 0$ such that $d(c, \mathrm{conv} f^{-1}(f(x))) > \varepsilon$, $x \in \mathrm{fr}X$; hence $d(c, \mathrm{conv} f^{-1}(y)) > \varepsilon$, $y \in f(\mathrm{fr}X)$, and $c \notin B(\mathrm{conv} f^{-1}(y), \varepsilon)$, $y \in f(\mathrm{fr}X)$. For $y \in f(\mathrm{fr}X)$ let $U_y \subset \mathbb{R}^n$ be an an open ball with the center y such that $f^{-1}(y) \subset f^{-1}(U_y) \subset B(\mathrm{conv} f^{-1}(y), \varepsilon) \subset X \backslash \{c\}$. Since the set $f(\mathrm{fr}X)$ is compact, there exist $y_i \in f(\mathrm{fr}X)$, $i \in [m]$, ($m \in \mathbb{N}$) such that the balls $U_i = U_{y_i}$, $i \in [m]$, cover $f(\mathrm{fr}X)$. Let $\gamma > 0$ be a Lebesgue number of the cover. By the compactness of $f(\mathrm{fr}X)$ there are points $v_j \in f(\mathrm{fr}X)$, $j \in [m_1]$, ($m_1 \in \mathbb{N}$) such that the balls V_j, $j \in [m_1]$, where V_j is the open ball centered at v_j with radius $\gamma/4$, are an open cover of $f(\mathrm{fr}X)$. Observe that the choice of the radius of the balls V_j, $j \in [m_1]$, ensures that for a subset $A \subset [m_1]$ the diameter of the set $\bigcup_{j \in A} V_j$ is not greater than γ whenever $\bigcap_{j \in A} V_j \neq \emptyset$, and thus there exists $i \in [m]$ for which $\bigcup_{j \in A} V_j \subset U_i$. Observe that there exists a compact set $K \subset \mathbb{R}^n$ such that $f(\mathrm{fr}X) \subset \mathrm{int}K \subset K \subset \bigcup_{j \in [m_1]} V_j$. For $j \in [m_1]$ pick a point $x_j \in \mathrm{fr}X$ such that $v_j = f(x_j)$ and define the function $g : K \to \mathbb{R}^n \backslash \{c\}$ by

$$g(y) = \sum_{j \in [m_1]} \frac{d_j(y)}{d(y)} x_j,$$

where $d_j(y) = d(y, \mathbb{R}^n \backslash V_j) = \inf\{\rho(y, z) : z \in \mathbb{R}^n \backslash V_j\}$, $d(y) = \sum_{j \in [m_1]} d_j(y)$.

Let us fix $x \in f^{-1}(K)$ and define $A_x = \{j \in [m_1] : f(x) \in V_j\}$. Hence, we can also fix $\bar{i} \in [m]$ such that $f(x) \in \bigcap_{j \in A_x} V_j \subset \bigcup_{j \in A_x} V_j \subset U_{\bar{i}}$. We have $x \in \mathrm{conv} f^{-1}(U_{\bar{i}}) \subset B(\mathrm{conv} f^{-1}(y_{\bar{i}}), \varepsilon) \subset X \backslash \{c\}$ and since $g(f(x)) = \sum_{j \in A_x} \frac{d_j(f(x))}{d(f(x))} x_j$ and $f(x) \in \bigcap_{j \in A_x} V_j$, also $v_j = f(x_j) \in U_{\bar{i}}$ for $j \in A_x$. So, for $j \in A_x$ we have $x_j \in f^{-1}(U_{\bar{i}}) \subset B(\mathrm{conv} f^{-1}(y_{\bar{i}}), \varepsilon) \subset X \backslash \{c\}$. By the convexity of $B(\mathrm{conv} f^{-1}(y_{\bar{i}}), \varepsilon)$, $g(f(x)) \in B(\mathrm{conv} f^{-1}(y_{\bar{i}}), \varepsilon) \subset X \backslash \{c\}$.

Now, let $y \in K$ and $A_y = \{j \in [m_1] : y \in V_j\}$. Analogously as in the previous paragraph we see that there exists $i \in [m]$ such that $y \in \bigcup_{j \in A_y} V_j \subset U_i$ and $x_j \in B(\mathrm{conv} f^{-1}(y_i), \varepsilon) \subset X \backslash \{c\}$, $j \in A_y$. Since $g(y) = \sum_{j \in A_y} \frac{d_j(y)}{d(y)} x_j$ and the set $B(\mathrm{conv} f^{-1}(y_i), \varepsilon)$ is convex, $g(y) \neq c$. Thus, $c \notin g(K)$ and the function g is well-defined. □

The proof of the next lemma is based on some ideas from the paper [9].

Lemma 5.4. *Let U be an open connected bounded subset of $\mathbb{R}^n$, $c \in U$ and let $f : \overline{U} \to \mathbb{R}^n$ be a continuous map such that $f^{-1}(f(\mathrm{fr}U)) = \mathrm{fr}U$ and $c \notin \mathrm{conv} f^{-1}(f(x))$ for $x \in \mathrm{fr}U$. Then $f(c) \in \mathrm{int} f(\overline{U})$.*

Proof. For $X = \overline{U}$ by Lemma 5.3 there exist $\varepsilon > 0$ such that $c \notin B(\mathrm{conv} f^{-1}(y), \varepsilon)$ for $y \in f(\mathrm{fr}U)$, a compact set $K \subset \mathbb{R}^n$ with $f(\mathrm{fr}U) \subset \mathrm{int}K$ and a continuous map $g : K \to \mathbb{R}^n \backslash \{c\}$ fulfilling the condition: for $x \in f^{-1}(K)$ there exists $y \in f(\mathrm{fr}U)$ such that $\{x, g(f(x))\} \subset B(\mathrm{conv} f^{-1}(y), \varepsilon)$. Since the set $f^{-1}(f(\mathrm{fr}U))$ is a compact subset of $\overline{U}$, there exists $\delta > 0$ with $B(f^{-1}(f(\mathrm{fr}U)), \delta) \subset f^{-1}(\mathrm{int}K)$. According to Lemma 5.2 there exists a connected open set W, $c \in W \subset \overline{W} \subset U$, such that $d(x, \mathrm{fr}U) < \delta$ for $x \in \overline{U} \backslash W$. Thus, by our assumption $\overline{U} \backslash W \subset B(\mathrm{fr}U, \delta) = B(f^{-1}(f(\mathrm{fr}U)), \delta) \subset f^{-1}(K)$ and this implies $f(\overline{U}) \backslash f(\overline{W}) \subset f(\overline{U} \backslash W) \subset K$. Let $Y = f(\overline{U})$ and suppose that $f(c) \in \mathrm{fr}Y$. Applying Lemma 5.1 to the sets $A = f(\overline{W})$ and $G = \mathbb{R}^n \backslash f(\mathrm{fr}\overline{U})$ we obtain an open set V such that $f(\overline{W}) \subset V \subset \overline{V} \subset \mathbb{R}^n \backslash f(\mathrm{fr}\overline{U})$, a compact boundary set $L \subset \mathbb{R}^n$ and a continuous map $r : Y \to \mathbb{R}^n$ such that $r(Y) \subset L \cup (Y \backslash V)$ and $r(y) = y$ for $y \in Y \backslash V$. By our construction $Y \backslash V \subset K$. Since $g(K) \subset \mathbb{R}^n \backslash \{c\}$ and L is a compact boundary subset of $\mathbb{R}^n$, according to the lemma on extension of maps (Lemma 4.4), g has a continuous extension $g_1 : L \cup K \to \mathbb{R}^n \backslash \{c\}$. Let $h : \overline{U} \to \mathbb{R}^n \backslash \{c\}$ be the composition $h = g_1 \circ r \circ f$. Since $h|_{\mathrm{fr}\overline{U}} = (g \circ f)|_{\mathrm{fr}\overline{U}}$, from the property of the map g (Lemma 5.3) it follows that for $x \in \mathrm{fr}\overline{U}$ there is $y \in f(\mathrm{fr}\overline{U})$ such that the points x and $h(x) = g(f(x))$ belong to the convex set $B(\mathrm{conv} f^{-1}(y), \varepsilon)$ and c does not belong to $B(\mathrm{conv} f^{-1}(y), \varepsilon)$. Thus for $x \in \mathrm{fr}\overline{U}$, the point c is not in the segment with the endpoints x and $h(x)$. The map $p : \mathrm{fr}\overline{U} \times [0,1] \to \mathbb{R}^n \backslash \{c\}$ defined by $p(x,t) = (1-t)x + th(x)$ omits the point c and is a homotopy between the identity map $\mathrm{id} : \mathrm{fr}\overline{U} \to \mathbb{R}^n \backslash \{c\}$ and the map $h|_{\mathrm{fr}\overline{U}}$. Since $h : \overline{U} \to \mathbb{R}^n \backslash \{c\}$ is an extension of the map $h|_{\mathrm{fr}\overline{U}}$, according to the Borsuk homotopy theorem (Theorem 4.2), the identity map $\mathrm{id}|_{\mathrm{fr}\overline{U}} : \mathrm{fr}\overline{U} \to \mathrm{fr}\overline{U} \subset \mathbb{R}^n \backslash \{c\}$ has a continuous extension $\mathrm{id}^* : \overline{U} \to \mathbb{R}^n \backslash \{c\}$. But this contradicts the Borsuk non-retraction theorem (Corollary 3.9). □

Theorem 5.5. *Let $X \subset \mathbb{R}^n$ be a compact set. Assume that a point $c \in \mathrm{int} X$. If $f : X \to \mathbb{R}^n$ is a continuous map such that $c \notin \mathrm{conv} f^{-1}(f(x))$ for $x \in \mathrm{fr} X$, then $f(c) \in \mathrm{int} f(X)$.*

Proof. Consider the restriction f_1 of f to the set $\overline{U}$, where $U \subset X$ is the connected component of $X \backslash f^{-1}(f(\mathrm{fr} X))$ with $c \in U$. Observe that if $x \in \mathrm{fr} U$, then either $x \in \mathrm{fr} X$ or $x \in \mathrm{int} X$. In the latter case it must hold $x \in f^{-1}(f(\mathrm{fr} X))$ because otherwise, by the continuity of f, $x \in U$ and $x \notin \mathrm{fr} U$, since U open. Hence $x \in f^{-1}(f(\mathrm{fr} X))$ and $c \notin \mathrm{conv} f^{-1}(f(x))$ for $x \in \mathrm{fr} U$. We infer that $c \notin \mathrm{conv} f_1^{-1}(f_1(x))$, $x \in \mathrm{fr} U \subset \overline{U}$. Suppose now that $x \in f_1^{-1}(f_1(\mathrm{fr} U))$. This implies that $f(x) = f_1(x) = f_1(z) = f(z)$ for some $z \in \mathrm{fr} U$ and thus $x \in f^{-1}(f(\mathrm{fr} X))$. We conclude that $f_1^{-1}(f_1(\mathrm{fr} U)) \subset \overline{U} \backslash U = \mathrm{fr} U$ and hence $f_1^{-1}(f_1(\mathrm{fr} U)) = \mathrm{fr} U$. Now it suffices to apply Lemma 5.4 for the function f_1. □

The following theorem is a consequence of our Theorem 5.5:

Theorem 5.6 (De Marco [4])**.** *Let $X \subset \mathbb{R}^n$ be a compact set. Assume that a point $c \in \mathrm{int} X$. If $f : X \to \mathbb{R}^n$ is a continuous function such that $\mathrm{diam} f^{-1}(f(x)) < d(c, \mathrm{fr} X)$ for $x \in \mathrm{fr} X$, then $f(c) \in \mathrm{int} f(X)$.*

Proof. Observe that the assumption $\mathrm{diam} f^{-1}(f(x)) < d(c, \mathrm{fr} X)$, $x \in \mathrm{fr} X$, implies $c \notin \mathrm{conv} f^{-1}(f(x))$, $x \in \mathrm{fr} X$, and then apply Theorem 5.5. □

Let us emphasize that our proof of Theorem 5.6 is a purely elementary topological proof without the reference to more complicated machinery of algebraic topology or degree theory. In fact, in the proof of Lemma 5.4 we have proved that the restriction $f|_{\mathrm{fr} X} : \mathrm{fr} X \to \mathbb{R}^n \backslash \{c\}$ is homotopic to the identity map $\mathrm{id} : \mathrm{fr} X \to \mathrm{fr} X$, and therefore, the degree of the map f is equal to 1, $\deg(f, X, c) = 1$. Moreover, from the degree theory it follows that if $\deg(f, X, c) \neq 0$, then $c \in \mathrm{int} f(X)$ (for the details see [4] and references therein).

For metric spaces X and Y Borsuk [3] defined an ε-map $f : X \to Y$ as: for every point $y \in Y$ the diameter of the set $f^{-1}(y)$ is less than $\varepsilon > 0$. He has shown, that for every continuous ε-map $f : \mathbb{R}^n \to \mathbb{R}^n$, the set $f(\mathbb{R}^n)$ is open in $\mathbb{R}^n$. In our notation, for an ε-map f we have $s(f) \leqslant \varepsilon$.

Let us recall his theorem:

Theorem 5.7 (Borsuk [3]). *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map and there is $r \geqslant 0$ such that for each $x \in \mathbb{R}^n$ $\mathrm{diam} f^{-1}(f(x)) \leqslant r$, then $f(\mathbb{R}^n)$ is an open set.*

From Theorem 5.6 we immediately obtain an extension of Theorem 5.7:

Theorem 5.8. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map such that*

$$\limsup_{||x||\to\infty} \frac{\mathrm{diam} f^{-1}(f(x))}{||x||} < 1,$$

then $f(\mathbb{R}^n)$ is an open set.

Example 5.9. The above result cannot be improved because for the following map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ for $x \leqslant 1$ and $f(x) = \frac{1}{|x|}$ for $x \geqslant 1$, the image $f(\mathbb{R}) = [0, \infty)$ is not an open subset of $\mathbb{R}$ and

$$\limsup_{||x||\to\infty} \frac{\mathrm{diam} f^{-1}(f(x))}{||x||} = 1.$$

An interesting discussion on the domain invariance theorem can be found on Terence Tao blog [15].

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Received: March 3, 2016; Accepted: May 25, 2016.