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FIXED POINTS OF HAMMERSTEIN-TYPE EQUATIONS ON GENERAL CONES

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Abstract. We obtain new results on the existence and multiplicity of fixed points of Hammerstein equations in very general cones. In order to achieve this, we combine a new formulation of cones in terms of continuous functionals with fixed point index theory. Many examples and an application to boundary value problems are also included.
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1. Introduction

In the last years, a vast amount of literature devoted to fixed point index theory in cones has been written. Ever since the publication of the well-known Krasnosel'skiĭ's Fixed Point Theorem [31], some authors have attempted to obtain new results in order to generalize and apply it to a large class of problems [1, 32, 37]. Probably, one of the most useful applications of Krasnosel'skiĭ-type theorems is the localization of solutions of differential equations satisfying certain boundary conditions [5, 6, 11]. A classical approach in this direction consists in rewriting the original differential problem in terms of an operator defined in a normed space. The next step is to use some fixed-point technique to ensure that the operator has a fixed point that will correspond to a solution of the boundary value problem.

In the light of this background, we develop a unified framework that allows us to look for solutions in a large class of boundary value problems. As it is well-known, most of these problems can be rewritten in terms of a *Hammerstein*-type equation, so our goal will be to obtain new results on the existence and localization of fixed points

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for this equation. On this ground, we develop a new general formulation to obtain abstract Krasnosel'skiĭ-type results in general cones.

The paper is organized as follows: in Section 2 we deal with abstract cones in normed spaces and show how these sets can be characterized in terms of continuous functionals; also, we include many examples of the application of this new perspective to some of the cones which are most often used in the literature. In Section 3 we obtain the main results of this work, which are about the existence and localization of solutions of Hammerstein-type equations in cones. Finally, in Section 4 we illustrate the theory providing an example to which we apply our results.

2. Characterization of cones in terms of functionals

We begin by recalling some concepts about cones in normed spaces.

Definition 2.1. Let $(N, \|\cdot\|)$ be a real normed space. A *cone* in N is a closed set such that

(1) $u+v \in K$ for all $u, v \in K$;
(2) $\lambda\, u \in K$ for all $u \in K$, $\lambda \in [0,+\infty)$;
(3) $K \cap (-K) = \{0\}$.

In the sequel, $(N, \|\cdot\|)$ will denote a real normed space and

$$N^* := \{\alpha : N \to \mathbb{R} \ : \ \alpha \text{ continuous}\}$$

will be the set of continuous functionals defined on N. Moreover, we will consider $\mathcal{A} \subset N^*$ to be the set of those $\alpha \in N^*$ which satisfy the following three conditions:

$$\alpha(u+v) \ge \alpha(u) + \alpha(v) \ \text{ for all } u, v \in N; \tag{2.1}$$

$$\alpha(\lambda u) \ge \lambda \alpha(u) \ \text{ for all } u \in N,\ \lambda \in [0,+\infty); \tag{2.2}$$

$$[\alpha(u) \ge 0, \alpha(-u) \ge 0] \Rightarrow u = 0. \tag{2.3}$$

Notice that, in general, we cannot ensure that $\mathcal{A}$ is a vector subspace of N^*. However, it follows from (2.1)-(2.3) that if $\alpha, \beta \in \mathcal{A}$ and $\lambda \in [0,+\infty)$ then $\min\{\alpha, \beta\} \in \mathcal{A}$ and $\lambda\alpha \in \mathcal{A}$.

Condition (2.3) could be quite difficult to check in practice. Nevertheless, notice that a sufficient condition to guarantee that (2.3) is satisfied is the following:

$$\alpha(u) + \alpha(-u) \le 0 \ \text{ for all } u \in N \text{ and } \alpha(u) = \alpha(-u) = 0 \ \text{ implies } u = 0. \tag{2.4}$$

The following Lemma will be useful in subsequent applications. In the sequel, given $\alpha \in N^*$ we will denote $\widetilde{\alpha}(u) = \alpha(-u)$.

Lemma 2.2. *Let $\{\alpha_j\}_{j\in J} \subset N^*$ such that $\sum_{j\in J}\alpha_j \in N^*$, $u \in N$. Assume that:*

$$\alpha_j + \widetilde{\alpha_j} \le 0 \ \text{ for all } j \in J, \tag{2.5}$$

and

$$\bigcap_{j\in J} (\alpha_j + \widetilde{\alpha_j})^{-1}(\{0\}) = \{0\}. \tag{2.6}$$

Then $\sum_{j\in J} \alpha_j$ satisfies condition (2.4).

Proof. By condition (2.5), we have that $\sum_{j\in J}\alpha_j + \sum_{j\in J}\widetilde{\alpha_j} \le 0$. Now, let $u \in N$ be an element satisfying $\sum_{j\in J}\alpha_j(u) = \sum_{j\in J}\alpha_j(-u) = 0$. This implies that $\sum_{j\in J}[\alpha_j(u) + \widetilde{\alpha_j}(u)] = 0$ and thus, because of (2.5), $\alpha_j(u) + \widetilde{\alpha_j}(u) = 0$ for every $j \in J$. Hence we have $u \in (\alpha_j + \widetilde{\alpha_j})^{-1}(\{0\})$ for every $j \in J$. Therefore,

$$u \in \bigcap_{j\in J}(\alpha_j + \widetilde{\alpha_j})^{-1}(\{0\}),$$

and, by virtue of (2.6), condition (2.4) is satisfied. □

Now we introduce the main result of this section, which characterizes all cones in N in terms of suitable functionals. For this purpose, we denote by $\mathcal{K}$ be the set of all cones in N and, given $\alpha \in \mathcal{A}$, we define $K_\alpha := \{u \in N \ : \ \alpha(u) \ge 0\}$.

Remark 2.3. With the notation introduced above, it is clear that

$$K_\alpha \cap K_\beta = K_{\min\{\alpha,\beta\}}$$

for $\alpha, \beta \in \mathcal{A}$. In the same way, $\bigcap_{\alpha\in\mathcal{A}'} K_\alpha = K_{\inf_{\alpha\in\mathcal{A}'}\alpha}$ for every $\mathcal{A}' \subset \mathcal{A}$. In [28] we can see a cone constructed in this way.

Theorem 2.4. $\mathcal{K} = \{K_\alpha \ : \ \alpha \in \mathcal{A}\}$.

Proof. To see that $\mathcal{K} \supset \{K_\alpha,\ \alpha \in \mathcal{A}\}$ we only have to notice that for every $\alpha \in \mathcal{A}$, K_α is a cone by properties (2.1) – (2.3).

Now we show that $\mathcal{K} \subset \{K_\alpha,\ \alpha \in \mathcal{A}\}$. Let $K \in \mathcal{K}$ and define

$$\alpha(u) := -\inf_{w\in K}\|u - w\|.$$

Thus defined, α is a continuous functional ($\alpha(u)$ is actually minus the distance from u to the cone K). Clearly, $K_\alpha = K$. Furthermore, for $u, v \in N$ and $\lambda \in [0, +\infty)$, we have that

$$\begin{aligned}\alpha(u+v) &= -\inf_{w\in K}\|u+v-w\| = -\inf_{w\in K}\|u+v-2w\| \\ &\ge -\inf_{w\in K}(\|u-w\| + \|v-w\|) = \alpha(u) + \alpha(v), \\ \alpha(\lambda\, u) &= -\inf_{w\in K}\|\lambda\, u - w\| = -\inf_{w\in K}\|\lambda\, u - \lambda\, w\| = -\inf_{w\in K}\lambda\|u-w\| = \lambda\alpha(u).\end{aligned}$$

Finally, if $u \in K\backslash\{0\}$, $-u \notin K$ and so $\alpha(u) < 0$. Therefore $\alpha \in \mathcal{A}$. □

Remark 2.5. In the previous result we have proved something even stronger: we can take α to satisfy $\alpha(\lambda u) = \lambda\alpha(u)$ and $\alpha(u) = 0$ for every $u \in K$, $\lambda \in [0, +\infty)$.

This last result shows that any cone on a normed space is given by a functional satisfying properties (2.1) – (2.3). Now, we may wonder under which circumstances two different functionals define the same cone. In order to elucidate this, given a cone K in N, define the functional $\varphi_K(u) := d(u, \partial K) - 2d(u, K)$ where ∂K is the boundary of K and $d(u,X) := \inf_{w\in X} \|u-w\|$ is the distance from u to the set $X \subset N$. The way it is defined, φ_K is clearly continuous. Actually, we have that

$$\varphi_K(u) = \begin{cases} -d(u,K) < 0, & u \in N\backslash K, \\ d(u,\partial K) > 0, & u \in \operatorname{Int}(K), \\ 0, & u \in \partial K. \end{cases}$$

With this, it is easy to prove the following Lemma.

Lemma 2.6. *Let $\alpha,\beta \in \mathcal{A}$. Then $K_\alpha = K_\beta$ if and only if $\beta = \xi\,\varphi_{K_\alpha}$ for some $\xi : N \to [0,+\infty)$ such that $\xi(u) > 0$ for $u \in N\backslash K_\alpha$.*

Proof. Assume $K_\alpha = K_\beta$ and define $\xi = \beta/\varphi_{K_\alpha}$ in $N\backslash\partial K_\alpha$ and $\xi = 0$ in ∂K_α. ξ is well defined since $\varphi_{K_\alpha} \neq 0$ in $N\backslash\partial K_\alpha$. Clearly, $\beta = \xi\varphi_{K_\alpha}$ in $N\backslash\partial K_\alpha$. Also, since β is continuous, $\beta \geq 0$ in K_α and $\beta < 0$ in $N\backslash K_\alpha$, we have that $\beta = 0$ in ∂K_α, so $\beta = \xi\varphi_{K_\alpha}$ in N.

Assume now $\beta = \xi\varphi_{K_\alpha}$. Then $\xi\varphi_{K_\alpha} \geq 0$ in K_α and so $K_\alpha \subset K_\beta$. On the other hand, if $u \in N\backslash K_\alpha$ then $\xi(u) > 0$ and $\varphi_{K_\alpha} < 0$, so $\xi\varphi_{K_\alpha} < 0$ and $u \in N\backslash K_\beta$. Hence, $K_\alpha = K_\beta$. □

Now we show some examples of functionals satisfying (2.1) - (2.3). As we will see, these functionals will be related to some cones which frequently appear in the literature.

In the following, consider the interval $I = [0,1]$ and the Banach space of continuous functions with the maximum norm $(\mathcal{C}(I), \|\cdot\|)$.

Example 2.7. Let $\|\cdot\|_*$ be a continuous norm (possibly different from the maximum norm $\|\cdot\|$) in $\mathcal{C}(I)$, K be a cone in $\mathcal{C}(I)$ and $\sigma \in \mathcal{C}(I)$ a positive function. The functional

$$\alpha(u) := -\inf_{v\in K} \|\sigma\, u - v\|_*$$

satisfies properties (2.1) and (2.2).

If $\alpha(u) \geq 0$ and $\alpha(-u) \geq 0$, then $\alpha(u) = \alpha(-u) = 0$. Then, being α continuous and K closed, there exist $v, w \in K$ such that $\|\sigma\, u - v\|_* = \|-\sigma\, u - w\|_* = 0$, so $\sigma\, u = v = -w$. Since K is a cone, $K \cap (-K) = \{0\}$, and hence $\sigma\, u = v = -w = 0$. σ is positive, which implies $u = 0$.

From the above discussion we deduce that $K_\alpha = \{u/\sigma \ :\ u \in K\}$. For the particular choices $\sigma = 1$, $K = \{0\}$ and $\|\cdot\|_* = c\|\cdot\|$, $c \in (0,+\infty)$, we have $\alpha(u) = -c\|u\|$.

Also, we can choose $\|\cdot\|_* = \|\cdot\|_p$, the p norm of $\mathrm{L}^p(I)$, to have $\alpha(u) = -\|\cdot\|_p$.

Example 2.8. For $u \in \mathcal{C}(I)$, let $\max u$ ($\min u$) be the maximum (minimum) of u on its domain. It is clear that the functionals $\min u$, $-\max u$, $-\|u\|$, satisfy conditions (2.1), (2.2) and (2.4). In fact, for a function $\sigma \geq 0$ we can generalize this to the functionals $\min(\sigma\, u)$, $-\max(\sigma\, u)$, $-\|\sigma\, u\|$, which satisfy properties (2.1) and (2.2). If we define $\chi_{[a,b]}$ to be the characteristic function of the interval $[a,b] \subset I$ and take $c \in (0,+\infty)$, we can combine the above functionals using Lemma 2.2 to derive the functional

$$\alpha(u) = \min(\chi_{[a,b]}\, u) - c\|u\|,$$

which satisfies conditions (2.1), (2.2) and (2.4). Observe that, for $u \in K_\alpha$,

$$\alpha(u) = \min_{t\in[a,b]} u(t) - c\|u\|,$$

and in this form it is used in [10, 12, 15–17, 19, 21–24, 26–28, 35, 36, 38–41, 43–51].

We can derive, in the same way, the more general functional

$$\alpha(u) = \min(\chi_{[a,b]}\,\sigma\, u) - \|u\|,$$

where $\sigma \in \mathcal{C}(I)$, $\sigma > 0$, which appears in the cones of the form

$$K = \{u \in \mathcal{C}(I)\ :\ u(t) \geq \sigma(t)\|u\|,\ t \in I\},$$

used in [3, 14, 34, 45].

Example 2.9. Let $t_0 \in I$, $a, b \in \mathcal{C}(I)$, $a, b \geq 0$, $a + b > 0$. The functional

$$\alpha(u) = \min(a\, u) - \max(b\, u) - |u(t_0)|,$$

satisfies properties (2.1) and (2.2). Also,

$$\alpha(u) + \widetilde{\alpha}(u) = \min(a\, u) - \max(a\, u) + \min(b\, u) - \max(b\, u) - 2|u(t_0)| \leq 0,$$

and $\alpha(u) + \widetilde{\alpha}(u) = 0$ only for $u = 0$, so condition (2.4) is also satisfied.

Example 2.10. Consider

$$\nparallel u \nparallel := \max\{\min u, -\max u\}.$$

For every $u, v \in \mathcal{C}(I)$ and $\lambda \in \mathbb{R}$, the function $\nparallel \cdot \nparallel$ satisfies the following conditions:

- $\nparallel u + v \nparallel \geq \nparallel u \nparallel + \nparallel v \nparallel$,
- $\nparallel \lambda u \nparallel = |\lambda| \nparallel u \nparallel$,
- $\|u\| - \nparallel u \nparallel = \max u - \min u \geq 0$,
- $\nparallel u \nparallel - \min u = \|u\| - \max u \geq 0$,
- $\max\{\max u, \nparallel u \nparallel\} = |\max u|$.

Consider then the functional

$$\alpha(u) = \nparallel a\, u \nparallel - \|b\, u\|,$$

where $a, b \in \mathcal{C}(I)$ are such that $|a| \leq |b|$ and $|a| + |b| > 0$. α satisfies conditions (2.1) and (2.2). Then,

$$\alpha(u) = \|a\, u\| - \|b\, u\| + \min(a\, u) - \max(a\, u) \leq 0,$$

In fact, if $\alpha(u) = 0$, then $\|a\, u\| - \|b\, u\| = 0$ and $\min(a\, u) - \max(a\, u) = 0$. Therefore $a\, u = b\, u = 0$. Since $|a| + |b| > 0$ we obtain $u = 0$ and so condition (2.4) is satisfied.

Example 2.11. Let $S \subset \mathcal{C}(I)$ be a bounded set such that for every $t \in I$ there exists $\sigma \in S$ satisfying $\sigma(t) \neq 0$ in an open neighborhood of t. Also, assume $\bigcup_{\sigma\in S} \sigma(I)$ has at least two elements. Define

$$\alpha(u) = \inf_{\sigma\in S} \min(\sigma\, u).$$

Thus defined, α satisfies conditions (2.1) and (2.2) and also

$$\alpha(u) + \widetilde{\alpha}(u) = \inf_{\sigma\in S} \min(\sigma\, u) - \sup_{\sigma\in S} \max(\sigma\, u) \leq \inf_{\sigma\in S} (\min(\sigma\, u) - \max(\sigma\, u)) \leq 0.$$

Now, assume $\alpha(u)+\widetilde{\alpha}(u)=0$. This implies $\min(\sigma\, u)=\max(\sigma\, u)$ for every $\sigma\in S$, so σu is constant for every $\sigma\in S$. Furthermore, for every t there exists $\sigma\in S$ such that $\sigma(t)\neq 0$ in an open neighborhood of t, so u is constant in an open neighborhood of t. Consider the set $A=u^{-1}(\{u(0)\})$. Being the inverse image by a continuous function of a closed set, A is closed in I. On the other hand, A is open in I, since for every $t\in A$ there is a neighborhood U of t such that u is constant in U. Then $u(t)=u(t_0)$ for all $t\in U\subset A$. As A is both closed and open in I, $A=I$, so u is constant in all of I. Hence,

$$\alpha(u)+\widetilde{\alpha}(u)=u(0)\left[\inf_{\sigma\in S}\min\sigma-\sup_{\sigma\in S}\max\sigma\right].$$

Now, there exist $\sigma_1,\sigma_2\in S$ and $t_1,t_2\in I$ such that $\sigma_1(t_1)>\sigma_2(t_2)$. Hence, we obtain $\inf\limits_{\sigma\in S}\min\sigma-\sup\limits_{\sigma\in S}\max\sigma<0$ and therefore $u=0$.

Example 2.12. Consider now a function $h:I\times\mathbb{R}\to\mathbb{R}$ such that for $u\in\mathcal{C}(I)$ the composition $t\in I\longmapsto h(t,u(t))$ is integrable and which moreover satisfies that $h(t,x+y)\geq h(t,x)+h(t,y)$ and $h(t,\lambda x)\geq\lambda h(t,x)$ for $x\in\mathbb{R}$, $t\in I$ and $\lambda\geq 0$ (we could consider, for instance, the function $e^t\chi_{[0,+\infty)}(t)x$ where χ is the characteristic function). Consider also a positive measure given by a function of bounded variation A and the functional given by the Stieltjes integral

$$\alpha(u)=\int_0^1 h(t,u(t))\,\mathrm{d}\,A(t).$$

Cones defined by functionals involving integrals can be found in a number of works, for instance [36, 42, 52], and functionals given by a measure of bounded variation in [18, 24, 25, 29, 46, 48, 49, 51].

Example 2.13. The set of continuous concave functions is given by

$$C=\{u\in\mathcal{C}(I)\ :\ \alpha(u)\geq 0\},$$

where

$$\alpha(u):=\inf_{t,s\in I}\left[u\left(\frac{t+s}{2}\right)-\frac{u(t)+u(s)}{2}\right]^{1}.$$

The functional α satisfies conditions (2.1) and (2.2). If $u\in\mathcal{C}(I)$ and $\alpha(u),\alpha(-u)=0$ then we have what is called Jensen's functional equation:

$$u\left(\frac{t+s}{2}\right)=\frac{u(t)+u(s)}{2}\ \text{for all}\ t,s\in I.$$

To solve it, just define $I_n:=[0,2^{-n}]$, $n=0,1,\dots,$ and observe that, for $t\in I_n$ and $s=2^{-n}-t$ we have that

$$u\left(2^{-n-1}\right)=\frac{u(t)+u(2^{-n}-t)}{2};\ t,s\in I.$$

[1] Actually, this functional gives the set of mid-point concave continuous functions. A theorem of Sierpiński (see [13]) shows that in the case of measurable functions, mid-point concave and concave functions coincide.

That is, u is symmetric with respect to 2^{-n-1} in the interval I_n, which means that $u(I_{n+1}) = u(I_n)$ for every $n = 0, 1, \dots$ or, equivalently, $u(I) = \bigcap_{k=0}^{\infty} u(I_n)$.

If $y \in \bigcap_{k=0}^{\infty} u(I_n)$ for every $n = 0, 1, \dots$, there exists $x_n \in I_n$ such that $u(x_n) = y$. Since $0 \le x_n \le 2^{-n}$, we have that $x_n \to 0$ and, since u is continuous, $u(x_n) \to u(0)$. Therefore, $y = u(0)$ and so u is a constant. Reciprocally, every constant satisfies Jensen's equation and, in conclusion, C is not a cone.

Now, if we consider $\eta \in I$ and define the closed vector subspace

$$N_\eta := \{u \in \mathcal{C}(I) \ : \ u(\eta) = 0\},$$

we have that $\mathcal{C}(I) = N_\eta \oplus \mathbb{R}$ and in this case $C \bigcap N_\eta$ is a cone.

Cones in which concave functions are involved appear, for instance, in [4, 30].

3. Fixed point results for Hammerstein equations

In this section we obtain some results regarding the existence of solutions of integral equations of Hammerstein-type in abstract cones. To do this, we will work in cones characterized by functionals satisfying (2.1)-(2.3).

Consider again the interval $I := [0, 1]$ and the Banach space of continuous functions with the maximum norm $(\mathcal{C}(I), \|\cdot\|)$. Given a functional $\alpha : \mathcal{C}(I) \to \mathbb{R}$, $\alpha \in \mathcal{A}$, we look for fixed points in K_α of an operator $T : \mathcal{C}(I) \to \mathcal{C}(I)$ given by

$$Tu(t) := \int_0^1 k(t, s) g(s) f(s, u(s)) \, \mathrm{d}\, s. \tag{3.1}$$

An equation of the form (3.1) is usually known as a *Hammerstein*-type equation, and there are many papers in the literature which deal with this type of equations, see for instance [7–9]. Typically, as we said in Section 1, these equations appear when looking for solutions of certain type of boundary value problems. In this context, the kernel k uses to be the Green's function of a related problem and g and f are, respectively, the linear and the nonlinear part of the differential equation in that problem. In this context, our work provides a new point of view from which all these problems can be considered in a unified framework.

The way we look for solutions of equation (3.1) is the well-known technique in fixed point index theory. For the sake of completeness, we recall now a classical result for continuous compact maps (cf. [2] or [20]).

Let K be a cone in $\mathcal{C}(I)$. If Ω is a bounded open subset of K (in the relative topology) we now denote by $\overline{\Omega}$ and $\partial\Omega$ respectively its closure and its boundary. Moreover, we will denote $\Omega_K = \Omega \cap K$, which is an open subset of K.

Lemma 3.1. *Let Ω be an open bounded set with $0 \in \Omega_K$ and $\overline{\Omega_K} \neq K$. Assume that $F : \overline{\Omega_K} \to K$ is a continuous compact map such that $x \neq Fx$ for all $x \in \partial\Omega_K$. Then the fixed point index $i_K(F, \Omega_K)$ has the following properties.*

1. *If there exists $e \in K \setminus \{0\}$ such that $x \neq Fx + \lambda e$ for all $x \in \partial\Omega_K$ and all $\lambda > 0$, then $i_K(F, \Omega_K) = 0$.*
2. *If $\mu x \neq Fx$ for all $x \in \partial\Omega_K$ and for every $\mu \ge 1$, then $i_K(F, \Omega_K) = 1$.*

(3) *If* $i_K(F,\Omega_K) \neq 0$*, then* F *has a fixed point in* Ω_K*.*
(4) *Let* Ω^1 *be open in* X *with* $\overline{\Omega^1} \subset \Omega_K$*. If* $i_K(F,\Omega_K) = 1$ *and* $i_K(F,\Omega^1_K) = 0$*, then* F *has a fixed point in* $\Omega_K \backslash \overline{\Omega^1_K}$*. The same result holds if* $i_K(F,\Omega_K) = 0$ *and* $i_K(F,\Omega^1_K) = 1$*.*

Now we state the main results of this paper. In order to do so, we consider the following list of assumptions for equation (3.1) and the cone K_α with $\alpha \in \mathcal{A}$:

(C_1) The kernel k is measurable and the function $k(\cdot,s)$ is uniformly continuous with respect to s, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $|k(t_1,s) - k(t_2,s)| < \varepsilon$ for all $s \in I$;
(C_2) $\psi_\alpha(s) := \alpha(k(\cdot,s)) \geq 0$ for a. a. (almost all) $s \in I$;
(C_3) the functions g, $k(t,\cdot)g$ and $\psi_\alpha\, g$ are integrable and $g(t) \geq 0$ for a. a. $t \in I$;
(C_4) the nonlinearity $f : I \times \mathbb{R} \to [0,+\infty)$ satisfies L^∞-Carathéodory conditions, that is, $f(\cdot,u)$ is measurable for each fixed u and $f(t,\cdot)$ is continuous for a. a. $t \in I$, and, for each $r > 0$, there exists $\phi_r \in \mathrm{L}^\infty(I)$ such that $f(t,u) \leq \phi_r(t)$ for all $u \in [-r,r]$ and a. a. $t \in I$;
(C_5) $\alpha(Tu) \geq \displaystyle\int_0^1 \psi_\alpha(s)g(s)f(s,u(s))\,\mathrm{d}\,s$ for all $u \in K_\alpha$;
(C_6) there exist two continuous functionals $\beta,\gamma : \mathcal{C}(I) \to \mathbb{R}$ satisfying that, for $u,v \in K_\alpha$ and $\lambda \in [0,+\infty)$,

$$\beta(u+v) \leq \beta(u) + \beta(v),\ \beta(\lambda u) = \lambda\beta(u),\ \beta(Tu) \leq \int_0^1 \psi_\beta(s)g(s)f(s,u(s))\,\mathrm{d}\,s,$$

$$\gamma(u+v) \geq \gamma(u) + \gamma(v),\ \gamma(\lambda u) \geq \lambda\gamma(u),\ \gamma(Tu) \geq \int_0^1 \psi_\gamma(s)g(s)f(s,u(s))\,\mathrm{d}\,s,$$

$$\psi_\beta,\psi_\gamma \in L^1(I) \text{ and } \int_0^1 \psi_\beta(s)g(s)\,\mathrm{d}\,s, \int_0^1 \psi_\gamma(s)g(s)\,\mathrm{d}\,s > 0;$$

(C_7) there exists $e \in K_\alpha \backslash \{0\}$ such that $\gamma(e) \geq 0$;
(C_8) for every $\rho > 0$ there exist $b(\rho), c(\rho) > 0$ such that $\beta(u) \leq b(\rho)$ for every $u \in K_\alpha$ satisfying $\gamma(u) \leq \rho$ and $\gamma(u) \leq c(\rho)$ for every $u \in K_\alpha$ satisfying $\beta(u) \leq \rho$.

Remark 3.2. Notice that if the kernel $k(t,s)$ is a. e. differentiable with respect to t and $\partial k/\partial t$ is uniformly bounded with respect to s then condition (C_1) is satisfied. This formulation is useful in applications when k corresponds to a Green's function.

Theorem 3.3. *Assume hypotheses* (C_1)-(C_5)*. Then* T *is continuous, compact and maps* K_α *to* K_α*.*

Proof. Continuity and compactness are derived from standard arguments involving Lebesgue's Dominated Convergence Theorem, but we include it for completeness.

Continuity: Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence which converges to u in $\mathcal{C}(I)$. In particular, $\{u_n\}_{n\in\mathbb{N}}$ is bounded, that is, there exists $r > 0$ such that $||u_n|| \leq r$ for all $n \in \mathbb{N}$. Moreover, we have by virtue of (C_4) that $f(s,u_n(s)) \to f(s,u(s))$ for a. e. $s \in I$.

Then, conditions $(C_3)-(C_4)$ imply now that

$$|Tu_n(t)| \leq \phi_r(t)\int_0^1 |k(t,s)\,g(s)|\,\mathrm{d}\,s \ \text{ for all } t\in I,$$

and we obtain, by application of Lebesgue's Dominated Convergence Theorem that $Tu_n \to Tu$, in $\mathcal{C}(I)$. Hence, operator T is continuous.

Compactness: Let $B \subset K_\alpha$ a bounded set, that is, $||u|| \leq R$ for all $u \in B$ and some $R > 0$. Then similar arguments as above show that

$$|Tu(t)| \leq \phi_R(t)\int_0^1 |k(t,s)\,g(s)|\,\mathrm{d}\,s \ \text{ for all } t\in I \text{ and all } u\in B.$$

Therefore, the continuity of ϕ_R and $k(\cdot,s)$ imply that the set $T(B)$ is totally bounded. On the other hand, given $t,s\in I$, we have

$$|Tu(t)-Tu(s)| \leq \int_0^1 |k(t,r)-k(s,r)|\,g(r)\,\phi_R(r)\,\mathrm{d}\,r,$$

Hence, by virtue of (C_1), (C_3) and (C_4), $T(B)$ is equicontinuous. In conclusion, we derive, by application of Ascoli-Arzela's Theorem, that $T(B)$ is relatively compact in $\mathcal{C}(I)$ and derive that T is a compact operator.

Finally, we obtain from conditions (C_2) and (C_5) that

$$\alpha(Tu) \geq \int_0^1 \psi_\alpha(s)g(s)f(s,u(s))\,\mathrm{d}\,s \geq 0 \text{ for all } u\in K_\alpha.$$

Thus, $Tu \in K_\alpha$. □

In the sequel, we give a condition that ensures that, for a suitable $\rho>0$, the index is 1 or 0 in certain open subsets of K_α. In order to see this, we define the sets

$$\begin{aligned} K_\alpha^{\beta,\rho} :=& \beta^{-1}([0,\rho))\cap K_\alpha = \{u\in\mathcal{C}(I) \ : \ \alpha(u)\geq 0,\ 0\leq \beta(u)<\rho\},\\ K_\alpha^{\gamma,\rho} :=& \gamma^{-1}([0,\rho))\cap K_\alpha = \{u\in\mathcal{C}(I) \ : \ \alpha(u)\geq 0,\ 0\leq \gamma(u)<\rho\}. \end{aligned}$$

We can define now two functions $b,c:\mathbb{R}^+\to\mathbb{R}^+$ in the conditions of (C_8) in the following way:

$$b(\rho) := \sup\{\beta(u) \ : \ u\in K_\alpha,\ \gamma(u)<\rho\}, \quad c(\rho) := \sup\{\gamma(u) \ : \ u\in K_\alpha,\ \beta(u)<\rho\}.$$

With these definitions, $K_\alpha^{\beta,\rho}\subset K_\alpha^{\gamma,c(\rho)}$ and $K_\alpha^{\gamma,\rho}\subset K_\alpha^{\beta,b(\rho)}$.

Lemma 3.4. *Assume that*

(I_ρ^1) *there exists $\rho>0$ such that*

$$f^\rho\cdot\int_0^1 \psi_\beta(s)g(s)\,\mathrm{d}\,s<1,$$

where

$$f^\rho = \sup\left\{\frac{f(t,u(t))}{\rho} : \ t\in I,\ u\in K_\alpha,\ \beta(u)=\rho\right\}.$$

Then the fixed point index $i_K(T,K_\alpha^{\beta,\rho})$ is equal to 1.

Proof. We show that $\mu u \neq Fu$ for every $u \in \partial K_\alpha^{\beta,\rho} = \beta^{-1}(\rho) \cap K_\alpha$ and for every $\mu \geq 1$. In fact, if this does not happen there exist $\mu \geq 1$ and $u \in \partial K_\alpha^{\beta,\rho}$ such that $\mu u = Fu$, that is

$$\mu u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}\,s.$$

Taking β on both sides,

$$\mu\beta(u) = \mu\rho \leq \int_0^1 \psi_\beta(s)g(s)f(s,u(s))\,\mathrm{d}\,s \leq \rho f^\rho \cdot \int_0^1 \psi_\beta(s)g(s)\,\mathrm{d}\,s < \rho.$$

This contradicts the fact that $\mu \geq 1$ and proves the result. □

Lemma 3.5. *Assume that*

(I_ρ^0) *there exist* $\rho > 0$ *such that such that*

$$f_\rho \cdot \int_0^1 \psi_\gamma(s)g(s)\,\mathrm{d}\,s > 1,$$

where

$$f_\rho = \inf\left\{\frac{f(t,u(t))}{\rho} : \ t \in I, \ u \in K_\alpha, \ \gamma(u) = \rho\right\}.$$

Then $i_K(T, K_\alpha^{\gamma,\rho}) = 0$.

Proof. Take e as in (C_7). Now we show that $u \neq Fu + \lambda e$ for every $u \in \partial K_\alpha^{\gamma,\rho} = \gamma^{-1}(\rho) \cap K_\alpha$ and $\lambda \geq 0$. Assume otherwise that there exist $u \in \partial K_\alpha^{\gamma,\rho}$ and $\lambda \geq 0$ such that $u = Fu + \lambda e$. Then we have

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}\,s + \lambda\, e.$$

Therefore, applying γ on both sides,

$$\rho = \gamma(u) \geq \int_0^1 \psi_\gamma(s)g(s)f(s,u(s))\,\mathrm{d}\,s + \lambda\gamma(e) \geq \rho f_\rho \int_0^1 \psi_\gamma(s)g(s)\,\mathrm{d}\,s > \rho,$$

which is a contradiction. □

Now we can combine the above Lemmas to prove the following Theorem. The proof of such is straightforward from the properties of the fixed point index stated in Lemma 3.1.

Theorem 3.6. *The integral equation* (3.1) *has at least one non-zero solution in* K *if either of the following conditions hold.*

(S_1) *There exist* $\rho_1, \rho_2 \in (0, +\infty)$ *with* $\rho_2 > b(\rho_1)$ *such that* ($\mathrm{I}_{\rho_1}^0$) *and* ($\mathrm{I}_{\rho_2}^1$) *hold.*

(S_2) *There exist* $\rho_1, \rho_2 \in (0, +\infty)$ *with* $\rho_2 > c(\rho_1)$ *such that* ($\mathrm{I}_{\rho_1}^1$) *and* ($\mathrm{I}_{\rho_2}^0$) *hold.*

The integral equation (3.1) *has at least two non-zero solutions in* K *if one of the following conditions hold.*

(S_3) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ *with* $\rho_2 > b(\rho_1)$ *and* $\rho_3 > c(\rho_2)$ *such that* ($\mathrm{I}_{\rho_1}^0$), ($\mathrm{I}_{\rho_2}^1$) *and* ($\mathrm{I}_{\rho_3}^0$) *hold.*

(S_4) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ *with* $\rho_2 > c(\rho_1)$ *and* $\rho_3 > b(\rho_2)$ *such that* ($\mathrm{I}_{\rho_1}^1$), ($\mathrm{I}_{\rho_2}^0$) *and* ($\mathrm{I}_{\rho_3}^1$) *hold.*

Remark 3.7. The list of conditions can be extended to obtain more multiplicity results (cf. Lan [33]).

4. An example

We finish this paper with an example to illustrate the applications of Theorem 3.6. *Consider the problem*

$$-u''(t) = f(t, u(t)) := \frac{4}{|u(t)| + 4}, \ t \in I, \quad u(0) = u(1) = 0. \tag{4.1}$$

Is there any concave solution of problem (4.1) *satisfying that*

$$\int_0^1 u(s) \,\mathrm{d}\, s \geq \frac{1}{20} \quad \textit{and} \quad \|u\|_2 \leq \frac{1}{2}?$$

To answer this question we will work on a cone K_α of the type given in Example 2.13, where

$$\alpha(u) := \min \left\{ \inf_{t,s \in I} \left[u\left(\frac{t+s}{2}\right) - \frac{u(t) + u(s)}{2} \right], \, u(0), \, -u(0), \, u(1), \, -u(1) \right\}.$$

K_α is precisely the cone of continuous concave functions that vanish at 0 and 1. Moreover, this cone is contained in the cone of nonnegative continuous functions. Observe that we can rewrite problem (4.1) in terms of a fixed-point problem for the operator

$$u(t) = \int_0^1 k(t, s) f(s, u(s)) \,\mathrm{d}\, s,$$

where

$$k(t, s) := \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Notice that k is continuous, non-negative, $k(0, s) = k(1, s) = 0$ for every $s \in I$ and the function $k(\cdot, s)$ is concave for every $s \in I$ since it is piecewise defined as two line segments, one increasing in the first part of the interval and the other decreasing in the second part. Moreover, $k(\cdot, s)$ is a.e. differentiable with uniformly bounded derivative. Hence, conditions (C_1) and (C_2) are satisfied.

In this case $g \equiv 1$ and $\psi_\alpha \equiv 0$, so (C_3) is also satisfied. Furthermore, (C_4) is satisfied by the definition of f and (C_5) holds since $\psi_\alpha \equiv 0$ and α is non-negative in K_α.

On the other hand, if we take $\beta(u) = \|u\|_2$ and $\gamma(u) = \displaystyle\int_0^1 u(s) \,\mathrm{d}\, s$ we have that

$$\psi_\beta(s) = \frac{1}{\sqrt{3}} s(1-s), \ \psi_\gamma(s) = \frac{1}{2} s(1-s) \ \text{ for all } s \in I.$$

Hence,

$$\int_0^1 \psi_\beta(s) g(s) \,\mathrm{d}\, s = \frac{1}{6\sqrt{3}}, \ \int_0^1 \psi_\gamma(s) g(s) \,\mathrm{d}\, s = \frac{1}{12} \ \text{ for all } s \in I.$$

Thus, β and γ satisfy (C_6). Now, $\psi_\gamma(s) \geq 0$ for every $s \in I$, so condition (C_7) is also satisfied. Observe that, since the functions in K_α are nonnegative, $\gamma(u) = \|u\|_1$ for $u \in K_\alpha$.

Now, in order to check that (C_8) also holds, we construct the functions b and c using some inequalities comprising the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$.

First, it is a known fact that $\|u\|_1 \leq \|u\|_2 \leq \|u\|_\infty$ for functions $u \in \mathcal{C}(I)$, so we can choose $c(\rho) = \rho$. Now, for $u \in K_\alpha$, take $t_u := \inf\{t \in I \ : \ u(t) = \|u\|_\infty\}$ and define

$$\widetilde{u}(t) := \begin{cases} \dfrac{\|u\|_\infty}{t_u} t, & t \in [0, t_u], \\ \dfrac{\|u\|_\infty}{1-t_u}(1-t), & t \in [0, t_u]. \end{cases}$$

We have that $u, \widetilde{u} \in K_\alpha$ and $\widetilde{u} \leq u$. Therefore,

$$\|u\|_1 \geq \|\widetilde{u}\|_1 = \frac{1}{2} t_u \|u\|_\infty + \frac{1}{2}(1-t_u)\|u\|_\infty = \frac{1}{2}\|u\|_\infty \geq \frac{1}{2}\|u\|_2.$$

Hence, it is enough to choose $b(\rho) = 2\rho$ to guarantee that (C_8) is satisfied.

Finally, if we take $\rho_1 = \dfrac{1}{20}$ and $\rho_2 = \dfrac{1}{2}$ then we have that $\rho_2 > b(\rho_1)$. Observe that

$$f_{\rho_1} \geq \frac{f(t,u(t))}{\rho_1} = \frac{80}{|u(t)|+4} \geq \frac{80}{\|u\|_\infty + 4} \geq \frac{80}{2\|u\|_1 + 4} = \frac{80}{2/20+4} = \frac{800}{41},$$

for $t \in I$ and $\gamma(u) = \|u\|_1 = \rho_1$. Hence,

$$f_{\rho_1} \int_0^1 \psi_\gamma(s) g(s) \,\mathrm{d}\, s \geq \frac{800}{41}\frac{1}{12} = \frac{200}{123} > 1.$$

Therefore, condition $(I^0_{\rho_1})$ holds.

On the other hand,

$$f^{\rho_2} \leq \frac{f(t,u(t))}{\rho_2} = \frac{8}{|u(t)|+4} \geq \frac{8}{4} = 2,$$

for $t \in I$ and $\gamma(u) = \|u\|_2 = \rho_2$ Thus,

$$f^{\rho_2} \int_0^1 \psi_\beta(s) g(s) \,\mathrm{d}\, s \geq 2\frac{1}{6\sqrt{3}} = \frac{1}{3\sqrt{3}} < 1.$$

Therefore, condition $(I^1_{\rho_2})$ is satisfied. This means that condition (S_1) in Theorem 3.6 holds and, hence, there exists a solution u of problem (4.1) in $K_\alpha^{\beta,\rho_2} \backslash K_\alpha^{\gamma,\rho_1}$. That is, such a solution is concave, nonnegative and satisfies

$$\int_0^1 u(s) \,\mathrm{d}\, s \geq \frac{1}{20} \quad \text{and} \quad \|u\|_2 \leq \frac{1}{2}.$$

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