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REMARKS ABOUT A P.K. LIN FIXED-POINT FREE MAP

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Abstract. In 1987 P.K. Lin found a fixed-point free selfmapping f of a closed convex subset of the unit ball of ℓ_2. Here we point out some remarkable features of this mapping. In particular we will show that if $|\cdot|$ is any equivalent renorming of ℓ_2, then f is not nonexpansive with respect to $|\cdot|$.
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1. Introduction

Since the mid-60s of last century, the study of nonexpansive mappings, (i.e. mappings with Lipschitz constant equal to one), have been object of considerable attention.

In 1965 F. Browder (see [1]) showed that every nonexpansive selfmapping of a closed convex bounded subset of a Hilbert space has a fixed point. On the other hand, from an example given in 1943 by S. Kakutani in [5] it was known that there are fixed point free selfmappings of the unit ball of the standard Hilbert space ℓ_2 with Lipschitz constant greater than 1, but as close to 1 as desired.

A famous longstanding open problem in this branch of the Fixed Point Theory is whether or not a selfmapping, say T, of a nonempty closed convex bounded subset C of the classical Hilbert space ℓ_2 has a fixed point provided that T is nonexpansive with respect to some equivalent norm $|\cdot|$ on ℓ_2.

A strategy to solve this problem in the negative is to start with a previously known fixed point free mapping, say $T : C \to C$, and to search for an equivalent renorming $|\cdot|$ in ℓ_2 such that T is $|\cdot|$-nonexpansive.

This line of thinking succeeded in other spaces. For instance, in the classical space of sequences ℓ_1, if T is the mapping given by

$$T((x_n)) := (1 - \|x\|_1, x_1, x_2, \ldots),$$

were $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$, it is straightforward to check that T admits the Lipschitz constant 2 on the closed unit ball B_1 of ℓ_1. But T. C. Lim showed in 1980 (see [6]) that the restriction of T to the positive part of B_1 , that is, to

$$B_1^+ := \{x \in B_1 : x_n \geq 0, n = 1, \ldots\}$$

is nonexpansive with respect to the equivalent norm $\|x\| := \max\{\|x^+\|_1, \|x^-\|_1\}$, where x^+, and x^- are respectively the positive and the negative part of $x \in \ell_1$. Thus, dramatic changes of the Lipschitz constant of a mapping are possible either by a suitable renorming of the space, or by a suitable restriction of the domain of the mapping under consideration.

In the case of ℓ_2 for the above mentioned Kakutani mapping as well as for another two well known fixed point free mapings of the closed unit ball B due to Baillon and Goebel, Kirk and Thele, it was shown that is impossible to find an equivalent renorming of ℓ_2, for which these mappings can become nonexpansive (see, for instance [3, 8]).

In 1987, P.K. Lin (see [7]) constructed a fixed-point-free, Lipschitzian self-mapping f of a weakly compact convex subset K of the closed unit ball B of ℓ_2 which is uniformly asymptotically regular, that is, satisfying that $\|f^{n+1}(x) - f^n(x)\| \to 0$ uniformly for all $x \in K$. For this mapping f (whose domain is indeed different than B) it seems to be unknown if ℓ_2 admits a renorming, say $|\cdot|$, such that f is $|\cdot|$-nonexpansive on K.

In this paper we will give a negative answer to this, that is, we will show that f cannot become nonexpansive on K with respect to any equivalent renorming of ℓ_2. To do this, we will try to get a better knowledge of the most relevant features of this mapping. In particular we will calculate its Lipschitz contant and we will obtain some information about the asymptiotic behaviour of the iterates of f.

2. Preliminaries. The example

In the ordinary Hilbert space ℓ_2 endowed with the Euclidean norm

$$\|x\| = \|(x_n)\| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}},$$

let (e_n) be the standard orthonormal basis. The unit ball, the unit sphere and the zero vector of $(\ell_2, \|\cdot\|)$ will be denoted as B, S_{ℓ_2}, and 0_{ℓ_2} respectivelly. The right shift S on ℓ_2 is the linear isometry defined as $S((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots)$. Let

$$K := \{x \in \ell_2 : \|x\| \leq 1, x_1 \geq x_2 \geq x_3 \geq \ldots \geq 0\}.$$

For $x = (x_n) \in \ell_2$, we define $\psi(x) := \max\{|x_1|, 1 - \|x\|\}$, and $g(x) = \psi(x)e_1 + S(x)$. Consider the mapping $f : K \to K \cap S_{\ell_2}$ given by

$$f(x) = \frac{g(x)}{\|g(x)\|}.$$

In [7] Lin introduced this mapping and showed that f is fixed point free on K and that

- If $x, y \in K$, $\|f(x) - f(y)\| \leq 20\|x - y\|$, ([7, Lemma 1]).
- f is uniformly asymptotically regular on K, ([7, Lemma 2]).

3. The Lipschitz constant of f

To begin with, we refine [7, Lemma 1].

Lemma 3.1. *For all $x, y \in \ell_2$,*

$$|\psi(x) - \psi(y)| \leq \|x - y\|$$

Proof.

$$\begin{aligned} |\psi(x) - \psi(y)|^2 &= |\max\{x_1, 1 - \|x\|\} - \max\{y_1, 1 - \|y\|\}|^2 \\ &= |\,\|(x_1, 1 - \|x\|)\|_\infty - \|(y_1, 1 - \|y\|)\|_\infty\,|^2 \\ &\leq \|\big(x_1 - y_1, (1 - \|x\|) - (1 - \|y\|)\big)\|_\infty^2 \\ &= \|(x_1 - y_1, \|y\| - \|x\|)\|_\infty^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

□

Theorem 3.1. *The (exact) Lipschitz constant of f on K is 2.*

Proof. From the above lemma, if $x, y \in K$,

$$\begin{aligned} \|g(x) - g(y)\|^2 &= |\psi(x) - \psi(y)|^2 + \|S(x - y)\|^2 \\ &\leq 2\|x - y\|^2. \end{aligned}$$

Moreover, for every $x \in K$, $\|g(x)\|^2 \geq (1 - \|x\|)^2 + \|x\|^2 \geq \frac{1}{2}$. Thefore, from the well known Dunkl-Williams inequality (see [2])

$$\begin{aligned} \|f(x) - f(y)\| &= \left\| \tfrac{g(x)}{\|g(x)\|} - \tfrac{g(y)}{\|g(y)\|} \right\| \\ &\leq \tfrac{2}{\|g(x)\| + \|g(y)\|} \|g(x) - g(y)\| \\ &\leq \tfrac{2\sqrt{2}}{\|g(x)\| + \|g(y)\|} \|x - y\| \\ &\leq \tfrac{2\sqrt{2}}{2\sqrt{\frac{1}{2}}} \|x - y\| \\ &= 2\|x - y\|. \end{aligned}$$

Then, if $\mathrm{lip}(f)$ denotes the (exact) Lipschitz constant of f on K, one has that $\mathrm{lip}(f) \leq 2$. On the other hand, take $\alpha \in (0, \frac{1}{2})$ and let $x = \frac{1}{2}e_1$, $y_\alpha = (\frac{1}{2} - \alpha)e_1$.

Of course, $x, y_\alpha \in K$, $\|x\| = \frac{1}{2}$, $\|y_\alpha\| = \frac{1}{2} - \alpha$.

$$f(x) = \frac{\big(\frac{1}{2}, \frac{1}{2}, 0, \dots\big)}{\|\big(\frac{1}{2}, \frac{1}{2}, 0, \dots\big)\|} = \big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots\big)$$

$$f(y_\alpha) = \frac{\big((\frac{1}{2} + \alpha), (\frac{1}{2} - \alpha), 0, \dots\big)}{\|\big((\frac{1}{2} + \alpha), (\frac{1}{2} - \alpha), 0, \dots\big)\|} = \frac{(\frac{1}{2} + \alpha)e_1 + (\frac{1}{2} - \alpha)e_2}{\sqrt{(\frac{1}{2} - \alpha)^2 + (\frac{1}{2} + \alpha)^2}}$$

Then,

$$\begin{aligned} \mathrm{lip}(f) &\geq \tfrac{\|f(x) - f(y_\alpha)\|}{\|x - y_\alpha\|} \\ &= \frac{\left\| \left(\frac{1/2 + \alpha}{\sqrt{(\frac{1}{2} - \alpha)^2 + (\frac{1}{2} + \alpha)^2}} - \frac{1}{\sqrt{2}} \right) e_1 + \left(\frac{1/2 - \alpha}{\sqrt{(\frac{1}{2} - \alpha)^2 + (\frac{1}{2} + \alpha)^2}} - \frac{1}{\sqrt{2}} \right) e_2 \right\|}{\alpha} \end{aligned}$$

We need the following result of elementary Calculus.

$$\lim_{\alpha\to 0^+} \frac{\frac{1/2\pm\alpha}{\sqrt{(\frac{1}{2}-\alpha)^2+(\frac{1}{2}+\alpha)^2}} - \frac{1}{\sqrt{2}}}{\alpha} = \pm\sqrt{2}.$$

Then, letting $\alpha \to 0^+$,

$$\begin{aligned} \operatorname{lip}(f) &\geq \lim_{\alpha\to 0} \frac{\left\| \left(\frac{1/2+\alpha}{\sqrt{(\frac{1}{2}-\alpha)^2+(\frac{1}{2}+\alpha)^2}} - \frac{1}{\sqrt{2}} \right) e_1 + \left(\frac{1/2-\alpha}{\sqrt{(\frac{1}{2}-\alpha)^2+(\frac{1}{2}+\alpha)^2}} - \frac{1}{\sqrt{2}} \right) e_2 \right\|}{\alpha} \\ &= \|\sqrt{2}e_1 - \sqrt{2}e_2\| = 2. \end{aligned}$$

□

Remark 3.1. Notice that, for $x, y \in K$ with $\|x\| = \|y\| = 1$ one has $\|g(x)\| \geq 1$ and $\|g(y)\| \geq 1$. Then, repeating the first part of the above reasoning we obtain

$$\|f(x) - f(y)\| \leq \sqrt{2}\|x - y\|.$$

Thus, the Lipschitz constant of f on the set $K_S := \{x \in K : \|x\| = 1\}$ does not exceed $\sqrt{2}$. In fact, the same holds on the set $\{x \in K : \|g(x)\| \geq 1\}$ which strictly contains K_S. Indeed, $x = \sqrt{\frac{1}{2}}e_1 \in K \backslash K_S$, while $\psi(x) = \sqrt{\frac{1}{2}}$ and hence

$$\|g(x)\|^2 = \psi(x)^2 + \|x\|^2 = 1.$$

4. An extension of f

Let us observe that the mapping f is indeed well defined on the whole unit ball B. In fact, $f : B \to S_{\ell_2}$. Let $|\cdot|$ be an equivalent norm on X, and suppose that the mapping f is k-Lipschitzcian with respect to the norm $|\cdot|$ on B.

We recurrently define the sequence (y_n) in B as follows:

$$\begin{cases} y_0 := \frac{1/4}{\|f(0_{\ell_2})\|} f(0_{\ell_2}) \\ y_n := \frac{1/4}{\|f(y_{n-1}) - f(0_{\ell_2})\|} \left(f(y_{n-1}) - f(0_{\ell_2})\right) (n = 1, \ldots) \end{cases}$$

Notice that $f(0_{\ell_2}) = e_1$, and that if x in B satisfies $f(x) = e_1$, then $S(x) = 0_{\ell_2}$ and hence $x = 0_{\ell_2}$. Then, the above algorithm is well defined and it produces a sequence (y_n) such that $\|y_n\| = \frac{1}{4}$ $n = 0, 1, 2, \ldots$. Since f is assumed to be k-Lipschitzian, for a positive integer m, one has

$$|y_m| = \frac{1/4}{\|f(y_{m-1}) - f(0_{\ell_2})\|} \, |f(y_{m-1}) - f(0_{\ell_2})| \leq \frac{k/4}{\|f(y_{m-1}) - f(0_{\ell_2})\|} \, |y_{m-1}|$$

In this way, we obtain that

$$|y_m| \leq \frac{(k/4)^m}{\prod_{i=0}^{m-1} \|f(y_i) - f(0_{\ell_2})\|} \, |y_0| \, .$$

Notice also that for $\|y\| = \frac{1}{4}$, one has $|y_1| \leq \frac{1}{4}$ and $1 - \|y\| = \frac{3}{4}$. Hence, $\psi(y) = \frac{3}{4}$, and

$$\begin{aligned}\|f(y) - f(0_{\ell_2})\| &= \sqrt{2 - 2\langle f(y), e_1\rangle} \\ &= \sqrt{2 - 2\frac{\psi(y)}{\sqrt{\psi(y)^2 + \|y\|^2}}} \\ &= \sqrt{2 - 2\frac{3/4}{\sqrt{10/16}}} \\ &= \sqrt{2 - \frac{6}{\sqrt{10}}}.\end{aligned}$$

Then,

$$|y_m| \leq \frac{(k/4)^m}{\prod_{i=0}^{m-1} \|f(y_i) - f(0_{\ell_2})\|} = \frac{(k/4)^m}{\left(\sqrt{2 - \frac{6}{\sqrt{10}}}\right)^m} |y_0|. \tag{4.1}$$

Suppose that

$$\frac{k/4}{\sqrt{2 - \frac{6}{\sqrt{10}}}} < 1.$$

Then, according to inequality 4.1, it follows that $|y_m| \to 0$, and this is a contradiction because $\|y_m\| \equiv \frac{1}{4}$ and both norms $\|\cdot\|$ and $|\cdot|$ are equivalent. Thus,

$$\frac{k/4}{\sqrt{2 - \frac{6}{\sqrt{10}}}} \geq 1,$$

that is

$$k \geq 4\sqrt{2 - \frac{6}{\sqrt{10}}} \approx 1.281457944$$

We have proven the following result.

Theorem 4.1. *The Lipschitz constant k on B of the (extended) P.K. Lin mapping $f : B \to S$ with respect to any equivalent norm on ℓ_2 satisfies that $k \geq 4\sqrt{2 - \frac{6}{\sqrt{10}}}$.*

In particular,

Corollary 4.1. *The space ℓ_2 cannot be renormed in such a way that the (extended) P.K. Lin mapping $f : B \to B$ becomes nonexpansive.*

Below we will refine this result, in a very different way.

5. On the iterates of f

Let us suppose that there exists an equivalent renorming of $(\ell_2, \|.\|)$, say $|\cdot|$, such that, f is nonexpansive w.r.t. $|\cdot|$ on K. Then, with no loss of generality me may suppose that for all $x \in \ell_2$, $\|x\| \leq |x| \leq b\|x\|$ for some $b > 0$. Then, for every $x, y \in K$, and every positive integer n,

$$\|f^n(x) - f^n(y)\| \leq |f^n(x) - f^n(y)| \leq |x - y| \leq b\|x - y\|.$$

This means that under this assumption, all the iterates of f would admit b as a common Lipschitz constant with respect to the norm $\|\cdot\|$. In other words, f would be b-uniformly Lipschitzian w.r.t. the Euclidean standard norm on K. Uniformly Lipschitzian mappings where first studied by K. Goebel and W.A. Kirk in 1973.

We begin by obtaining a precise expression of the mapping f^n.

Proposition 5.1. *For all $x \in K$, and every positive integer n,*

$$f^n(x) = \frac{\psi(x)(e_1 + e_2 + \ldots + e_n) + S^n(x)}{\sqrt{\|x\|^2 + n\psi(x)^2}}. \tag{5.1}$$

Proof. For $n = 1$ (5.1) is just the definition of f.

Assuming that equality (5.1) holds for some n, then, since $\|f^n(x)\| = 1$, then

$$\begin{aligned} g(f^n(x)) &= f^n(x)(1)e_1 + S(f^n(x)) \\ &= \tfrac{\psi(x)}{\sqrt{\|x\|^2+n\psi(x)^2}} e_1 + \tfrac{\psi(x)(e_2+e_3+\ldots+e_{n+1})+S^{n+1}(x)}{\sqrt{\|x\|^2+n\psi(x)^2}} \end{aligned}$$

$$f(f^n(x)) = \frac{\tfrac{\psi(x)}{\sqrt{\|x\|^2+n\psi(x)^2}} e_1 + \tfrac{\psi(x)(e_2+e_3+\ldots+e_{n+1})+S^{n+1}(x)}{\sqrt{\|x\|^2+n\psi(x)^2}}}{\sqrt{|\tfrac{\psi(x)}{\sqrt{\|x\|^2+n\psi(x)^2}}|^2 + n|\tfrac{\psi(x)}{\sqrt{\|x\|^2+n\psi(x)^2}}|^2 + |\tfrac{\|x\|}{\sqrt{\|x\|^2+n\psi(x)^2}}|^2}}$$

$$= \frac{\psi(x)(e_1 + e_2 + e_3 + \ldots + e_{n+1}) + S^{n+1}(x)}{\sqrt{(n+1)\psi(x)^2 + \|x\|^2}} \qquad \square$$

Regarding the Lipschitz constant of the iterates of f, we know that for every $x, y \in K$, for $n \geq 1$, $f^n(x), f^n(y) \in K_S$, and then, according to Remark 3.1 and Proposition 3.1,

$$\|f^n(x) - f^n(y)\| \leq (\sqrt{2})^{n-1} 2\|x - y\|.$$

A more accurate estimate is:

Proposition 5.2. *For every positive integer n,*

$$\frac{4\sqrt{n}}{n+1} \leq \operatorname{lip}(f^n) \leq \frac{n+1}{\sqrt{n}}. \tag{5.2}$$

Proof. Put $g_n(x) = \psi(x)(e_1 + e_2 + \ldots + e_n) + S^n(x)$. From Lemma 3.1, if $x, y \in K$,

$$\begin{aligned} \|g_n(x) - g_n(y)\|^2 &= |\psi(x) - \psi(y)|^2 \|(e_1 + e_2 + \ldots + e_n)\|^2 + \|S^n(x - y)\|^2 \\ &= n|\psi(x) - \psi(y)|^2 + \|x - y\|^2 \\ &\leq (n+1)\|x - y\|^2. \end{aligned}$$

Thus,

$$\|g_n(x) - g_n(y)\| \leq \sqrt{n+1}\|x - y\|.$$

Since $f^n(x) = \frac{g_n(x)}{\|g_n(x)\|}$, again from the Dunkl-Williams inequality one has

$$\begin{aligned} \|f^n(x) - f^n(y)\| &= \left\| \frac{g_n(x)}{\|g_n(x)\|} - \frac{g_n(y)}{\|g_n(y)\|} \right\| \\ &\leq \frac{2\ \|g_n(x)-g_n(y)\|}{\|g_n(x)\|+\|g_n(y)\|} \\ &\leq \frac{2}{\|g_n(x)\|+\|g_n(y)\|}\sqrt{n+1}\|x-y\|. \end{aligned}$$

Notice that

$$\begin{aligned} \|g_n(x)\|^2 &= |\psi(x)|^2\|e_1+e_2+\ldots+e_n\|^2 + \|S^n(x)\|^2 \\ &= n|\psi(x)|^2 + \|x\|^2 \\ &\geq n(1-\|x\|)^2 + \|x\|^2 \\ &= (n+1)\|x\|^2 - 2n\|x\| + n \\ &\geq (n+1)\left(\frac{n}{n+1}\right)^2 - 2n\left(\frac{n}{n+1}\right) + n \\ &= \frac{n}{n+1}. \end{aligned}$$

Then,

$$\|f^n(x) - f^n(y)\| \leq \frac{2}{2\sqrt{\frac{n}{n+1}}}\sqrt{n+1}\|x-y\| = \frac{n+1}{\sqrt{n}}\|x-y\|.$$

On the other hand, taking $v = \frac{1}{2}e_1, w_t = (1/2-t)e_1$ ($t > 0$ small enough). For $n \geq 1$

$$f^n(v) = \frac{1}{\sqrt{n+1}}(e_1+\ldots+e_{n+1})$$

$$f^n(w_t) = \frac{(1/2+t)(e_1+\ldots+e_n) + (1/2-t)e_{n+1}}{\sqrt{(1/2-t)^2+n(1/2+t)^2}}.$$

Then,

$$\begin{aligned} \operatorname{lip}(f^n) &\geq \frac{\left\| \frac{1}{\sqrt{n+1}}(e_1+\ldots+e_{n+1}) - \frac{(1/2+t)(e_1+\ldots+e_n)+(1/2-t)e_{n+1}}{\sqrt{(1/2-t)^2+n(1/2+t)^2}} \right\|}{t} \\ &= \left\| \frac{\frac{1}{\sqrt{n+1}} - \frac{(1/2+t)}{\sqrt{(1/2-t)^2+n(1/2+t)^2}}}{t}(e_1+\ldots+e_n) + \frac{\frac{1}{\sqrt{n+1}} - \frac{(1/2-t)}{\sqrt{(1/2-t)^2+n(1/2+t)^2}}}{t}e_{n+1} \right\| \\ &= \sqrt{n\left(\frac{\frac{1}{\sqrt{n+1}} - \frac{(1/2+t)}{\sqrt{(1/2-t)^2+n(1/2+t)^2}}}{t}\right)^2 + \left(\frac{\frac{1}{\sqrt{n+1}} - \frac{(1/2-t)}{\sqrt{(1/2-t)^2+n(1/2+t)^2}}}{t}\right)^2} \end{aligned}$$

and letting $t \to 0^+$ we obtain

$$\operatorname{lip}(f^n) \geq \sqrt{n\left(-\frac{4}{(n+1)^{3/2}}\right)^2 + \left(\frac{4n}{(n+1)^{3/2}}\right)^2} = \frac{4\sqrt{n}}{n+1}. \qquad \square$$

Unfortunately, the above result still gives a poor information about the Lipchitz constant of f^n. In fact, there are particular choices of x, y for which

$$\frac{\|f^n(x) - f^n(y)\|}{\|x - y\|} \leq 2$$

for every positive integer n.

Proposition 5.3. *Let $K_0 := \{x \in K \ : \ x_1 \geq \frac{1}{2}\}$. Then, for all $x, y \in K_0$, and every positive integer n.*

$$\| \ f^n(x) - f^n(y) \ \| \leq \ 2 \ \| \ x - y \ \| \ .$$

Proof. For $x, y \in K$, and $n \geq 1$, since

$$f^n(x) \ = \frac{g_n(x)}{\|g_n(x)\|} = \frac{\psi(x)(e_1 + \ldots + e_n) + S^n x}{\sqrt{n\psi(x)^2 + \|x\|^2}},$$

again from Dunkl-Williams inequality one has

$$\begin{aligned} \|f^n(x) - f^n(y)\| \quad &\leq \frac{2\|(\psi(x) - \psi(y)(e_1 + \ldots + e_n) + S^n(x-y)\|}{\sqrt{n\psi(x)^2 + \|x\|^2} + \sqrt{n\psi(y)^2 + \|y\|^2}} \\ &= \frac{2\sqrt{\ n(\psi(x) - \psi(y))^2 + \|x-y\|^2}}{\sqrt{n\psi(x)^2 + \|x\|^2} + \sqrt{n\psi(y)^2 + \|y\|^2}} \\ &\leq \frac{2\sqrt{\ n+1}\|x-y\|}{\sqrt{n\psi(x)^2 + \|x\|^2} + \sqrt{n\psi(y)^2 + \|y\|^2}}. \end{aligned} \tag{5.3}$$

Bearing in mind that for $v \in K_0$,

$$n\psi(v)^2 + \|v\|^2 \ = \ nv_1^2 + \|v\|^2 \geq \ (n+1)v_1^2 \geq \frac{n+1}{4},$$

it follows

$$\|f^n(x) - f^n(y)\| \ \leq \ \frac{2}{2\frac{\sqrt{n+1}}{2}} \sqrt{n+1} \|x - y\| \ = \ 2\|x - y\|. \qquad \square$$

Proposition 5.4. *Let $K_1 := \{x \in K \ : \|x\| \leq \frac{1}{2}\}$. Then, for every positive integer n and all $x, y \in K_1$,*

$$\|f^n(x) - f^n(y)\| \ \leq \ 2\|x - y\|.$$

Proof. We know from inequality 5.3 that, for $x, y \in K$, and $n \geq 1$,

$$\|f^n(x) - f^n(y)\| \leq \frac{2\sqrt{\ n+1}\|x - y\|}{\sqrt{n\psi(x)^2 + \|x\|^2} + \sqrt{n\psi(y)^2 + \|y\|^2}}$$

If $v \in K_1$, then $\|v\| \leq \frac{1}{2}$, $\|v\| \leq \frac{1}{2}$, and therefore

$$1 - \|v\| \ \geq \ \tfrac{1}{2} \ \geq \ \|v\| \ \geq \ v_1 \ \rightarrow \ \psi(v) = 1 - \|v\|.$$

Since $x, y \in K_1$,

$$\begin{aligned} n\psi(x)^2 + \|x\|^2 \ &= \ n(1 - \|x\|)^2 + \|x\|^2 \ = \ (n+1)\|x\|^2 - 2n\|x\| + n \\ n\psi(y)^2 + \|y\|^2 \ &= \ n(1 - \|y\|)^2 + \|y\|^2 \ = \ (n+1)\|y\|^2 - 2n\|y\| + n. \end{aligned}$$

The real function $\varphi(t) := (n+1)t^2 - 2nt + n$, has its absolute minimum at $t = \frac{n}{n+1}$ and it is strictly decreasing in $[0, \frac{n}{n+1}] \supseteq [0, \frac{1}{2}]$. Since we are assuming that $\|x\|,\ \|y\| \in [0, \frac{1}{2}]$, then

$$\begin{aligned} n\psi(x)^2 + \|x\|^2 &= \varphi(\|x\|) \geq \varphi(\tfrac{1}{2}) = \tfrac{n+1}{4} \\ n\psi(y)^2 + \|y\|^2 &= \varphi(\|y\|) \geq \varphi(\tfrac{1}{2}) = \tfrac{n+1}{4}, \end{aligned}$$

which implies

$$\|f^n(x) - f^n(y)\| \leq \frac{2\sqrt{n+1}\|x-y\|}{\sqrt{\frac{n+1}{4}} + \sqrt{\frac{n+1}{4}}} = 2\|x-y\|. \qquad \square$$

Proposition 5.5. *If $x, y \in K$ and $\psi(x)^2 + \psi(y)^2 \geq 1$, then for every positive integer n,*

$$\|f^n(x) - f^n(y)\| \leq 2\|x-y\|.$$

Suppose that for $x, y \in K$ the inequality $\psi(x)^2 + \psi(y)^2 \geq 1$ holds. We claim that,

$$\sqrt{n\psi(x)^2 + \|x\|^2} + \sqrt{n\psi(y)^2 + \|y\|^2} \geq \sqrt{n+1}.$$

Indeed, if $\varphi(t) = (n+1)t^2 - 2nt + n$,

$$\begin{aligned} &\left(\sqrt{n\psi(x)^2 + \|x\|^2} + \sqrt{n\psi(y)^2 + \|y\|^2}\right)^2 \\ &= n[\psi(x)^2 + \psi(y)^2] + \|x\|^2 + \|y\|^2 + 2\sqrt{(n\psi(x)^2 + \|x\|^2)(n\psi(y)^2 + \|y\|^2)} \\ &\geq n + \|x\|^2 + \|y\|^2 + 2\sqrt{(n\psi(x)^2 + \|x\|^2)(n\psi(y)^2 + \|y\|^2)} \\ &\geq n + \|x\|^2 + \|y\|^2 + 2\sqrt{(n(1-\|x\|)^2 + \|x\|^2)(n(1-\|y\|)^2 + \|y\|^2)} \\ &= n + \|x\|^2 + \|y\|^2 + 2\sqrt{\phi(\|x\|)\phi(\|y\|)} \end{aligned}$$

Since $\varphi(\frac{n}{n+1}) = \frac{n}{n+1}$ is the minimum value of $\varphi(t)$ in $[0, 1]$,

$$\begin{aligned} \left(\sqrt{n\psi(x)^2 + \|x\|^2} + \sqrt{n\psi(y)^2 + \|y\|^2}\right)^2 &\geq n + \|x\|^2 + \|y\|^2 + 2\sqrt{\varphi(\|x\|)\varphi(\|y\|)} \\ &\geq n + \|x\|^2 + \|y\|^2 + 2\sqrt{\frac{n}{n+1}\frac{n}{n+1}} \geq n + \frac{2n}{n+1} \geq n+1. \end{aligned}$$

Then, again from inequality 5.3

$$\|f^n(x) - f^n(y)\| \leq \frac{2}{\sqrt{n+1}}\sqrt{n+1}\|x-y\| = 2\|x-y\|. \qquad \square$$

The above propositions could suggest that f is 2-uniformly Lipchitzian on K. Then the following is in some sense surprising.

Theorem 5.1. *The mapping f is not uniformly Lipschitzian on K.*

Proof. Let us consider the sequences $(u_n)_{n=3}^{\infty}$,$(v_n)_{n=3}^{\infty}$, where

$$\begin{cases} u_n := \left(\sqrt{\frac{3}{2(n+1)}},\ \sqrt{\frac{2n-1}{2(n+1)(n^2-1)}},\ \overset{(n^2-1)}{\ldots},\ \sqrt{\frac{2n-1}{2(n+1)(n^2-1)}},\ 0,\ 0,\ \ldots\right) \\ \\ v_n := \left(\sqrt{\frac{2}{3(n+1)}},\ \sqrt{\frac{3n+1}{3(n+1)(n^2-1)}},\ \overset{(n^2-1)}{\ldots},\ \sqrt{\frac{3n+1}{3(n+1)(n^2-1)}},\ 0,\ 0,\ \ldots\right) \end{cases}$$

Indeed, for $n \geq 3$, $u_n, v_n \in K_S$:

$$\begin{cases} \| u_n \|^2 = \frac{3}{2(n+1)} + (n^2-1)\frac{2n-1}{2(n+1)(n^2-1)} = \frac{3+2n-1}{2(n+1)} = 1 \\ \| v_n \|^2 = \frac{2}{3(n+1)} + (n^2-1)\frac{3n+1}{3(n+1)(n^2-1)} = \frac{2+3n+1}{3(n+1)} = 1. \end{cases}$$

Moreover, if $n \geq 3$,

$$\begin{cases} 3(n^2-1) \geq 2n-1 \\ 2(n^2-1) \geq 3n+1. \end{cases}$$

and therefore

$$\begin{cases} \sqrt{\frac{3}{2(n+1)}} \geq \sqrt{\frac{2n-1}{2(n+1)(n^2-1)}} \\ \sqrt{\frac{2}{3(n+1)}} \geq \sqrt{\frac{3n+1}{3(n+1)(n^2-1)}}. \end{cases}$$

Now we calculate $\|u_n - v_n\|$ and $\| f^n(u_n) - f^n(v_n) \|^2$.

$$\begin{aligned} \|u_n - v_n\|^2 &= \left(\sqrt{\frac{3}{2(n+1)}} - \sqrt{\frac{2}{3(n+1)}}\right)^2 \\ &\quad +(n^2-1)\left(\sqrt{\frac{2n-1}{2(n+1)(n^2-1)}} - \sqrt{\frac{3n+1}{3(n+1)(n^2-1)}}\right)^2 \\ &= \frac{1}{n+1}\left(\sqrt{\frac{3}{2}} - \sqrt{\frac{2}{3}}\right)^2 + \frac{1}{n+1}\left(\sqrt{n-\frac{1}{2}} - \sqrt{n+\frac{1}{3}}\right)^2 \\ &= \frac{\left(\sqrt{\frac{3}{2}} - \sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{n-\frac{1}{2}} - \sqrt{n+\frac{1}{3}}\right)^2}{n+1}. \end{aligned}$$

According to Proposition 5.1.

$$\begin{cases} f^n(u_n) = \dfrac{\left(\sqrt{\frac{3}{2(n+1)}}, \overset{(n+1)}{\cdots}, \sqrt{\frac{3}{2(n+1)}}, \sqrt{\frac{2n-1}{2(n+1)(n^2-1)}}, \overset{(n^2-1)}{\cdots}, \sqrt{\frac{2n-1}{2(n+1)(n^2-1)}}, 0, 0, \ldots\right)}{\sqrt{n\frac{3}{2(n+1)}+1}} \\ f^n(v_n) = \dfrac{\left(\sqrt{\frac{2}{3(n+1)}}, \overset{(n+1)}{\cdots}, \sqrt{\frac{2}{3(n+1)}}, \sqrt{\frac{3n+1}{3(n+1)(n^2-1)}}, \overset{(n^2-1)}{\cdots}, \sqrt{\frac{3n+1}{3(n+1)(n^2-1)}}, 0, 0, \ldots\right)}{\sqrt{n\frac{2}{3(n+1)}+1}}. \end{cases}$$

It follows that

$$\begin{aligned} \| f^n(u_n) - f^n(v_n) \|^2 &= (n+1)\left(\frac{\sqrt{\frac{3}{2(n+1)}}}{\sqrt{\frac{3n}{2(n+1)}+1}} - \frac{\sqrt{\frac{2}{3(n+1)}}}{\sqrt{\frac{2n}{3(n+1)}+1}}\right)^2 \\ &\quad +(n^2-1)\left(\frac{\sqrt{\frac{2n-1}{2(n+1)(n^2-1)}}}{\sqrt{\frac{3n}{2(n+1)}+1}} - \frac{\sqrt{\frac{3n+1}{3(n+1)(n^2-1)}}}{\sqrt{\frac{2n}{3(n+1)}+1}}\right)^2 \\ &= (n+1)\left(\frac{\sqrt{3}}{\sqrt{5n+2}} - \frac{\sqrt{2}}{\sqrt{5n+3}}\right)^2 \\ &\quad +(n^2-1)\left(\frac{\sqrt{\frac{2n-1}{n^2-1}}}{\sqrt{5n+2}} - \frac{\sqrt{\frac{3n+1}{n^2-1}}}{\sqrt{5n+3}}\right)^2 \\ &= (n+1)\left(\frac{\sqrt{3}}{\sqrt{5n+2}} - \frac{\sqrt{2}}{\sqrt{5n+3}}\right)^2 + \left(\sqrt{\frac{2n-1}{5n+2}} - \sqrt{\frac{3n+1}{5n+3}}\right)^2. \end{aligned}$$

Bearing in mind that

$$\lim_{n\to\infty}\left(\sqrt{n-\frac{1}{2}}-\sqrt{n+\frac{1}{3}}\right)^2 = \lim_{n\to\infty}\left(\frac{\frac{1}{2}+\frac{1}{3}}{\sqrt{n-\frac{1}{2}}+\sqrt{n+\frac{1}{3}}}\right)^2 = 0,$$

and that

$$\lim_{n\to\infty}(n+1)\left(\frac{\sqrt{3}}{\sqrt{5n+2}}-\frac{\sqrt{2}}{\sqrt{5n+3}}\right)^2 = \lim_{n\to\infty}(n+1)\frac{(\sqrt{15n+9}-\sqrt{10n+4})^2}{(5n+2)(5n+3)}$$

$$= \lim_{n\to\infty}\frac{(n+1)\left(\frac{5n+5}{\sqrt{15n+9}+\sqrt{10n+4}}\right)^2}{(5n+3)(5n+2)} = \lim_{n\to\infty}\frac{25(n+1)^3}{(5n+2)(5n+3)(\sqrt{15n+9}+\sqrt{10n+4})^2}$$
$$= \frac{1}{5(\sqrt{3}+\sqrt{2})^2},$$

we finally obtain

$$\lim_{n\to\infty}\frac{\| f^n(u_n)-f^n(v_n)\|^2}{\| u_n-v_n\|^2} = \lim_{n\to\infty}\frac{(n+1)\left(\frac{\sqrt{3}}{\sqrt{5n+2}}-\frac{\sqrt{2}}{\sqrt{5n+3}}\right)^2+\left(\sqrt{\frac{2n-1}{5n+2}}-\sqrt{\frac{3n+1}{5n+3}}\right)^2}{\frac{(\sqrt{\frac{3}{2}}-\sqrt{\frac{2}{3}})^2+(\sqrt{n-\frac{1}{2}}-\sqrt{n+\frac{1}{3}})^2}{n+1}}$$

$$= \lim_{n\to\infty}(n+1)\left[\frac{(n+1)\big(\frac{\sqrt{3}}{\sqrt{5n+2}}-\frac{\sqrt{2}}{\sqrt{5n+3}}\big)^2+\big(\sqrt{\frac{2n-1}{5n+2}}-\sqrt{\frac{3n+1}{5n+3}}\big)^2}{\big(\sqrt{\frac{3}{2}}-\sqrt{\frac{2}{3}}\big)^2+\big(\sqrt{n-\frac{1}{2}}-\sqrt{n+\frac{1}{3}}\big)^2}\right] = +\infty \qquad \square$$

Corollary 5.1. *If $|\cdot|$ is an equivalent renorming of $(\ell_2, \|\cdot\|)$, then f is not $|\cdot|$-nonexpansive on K.*

6. Asymptotic behaviour of f

A sourprising information about the behaviour of the iterates of f is given in the following theorem. We first need to refine the Dunkl-Williams inequality.

Lemma 6.1. *Let H be a (real) Hilbert space. Then, $\forall x, y \in H\setminus\{0_H\}$,*

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| = \frac{\sqrt{4\|x-y\|^2-2(1+\frac{\langle x,y\rangle}{\|x\|\|y\|})(\|x\|-\|y\|)^2}}{\|x\|+\|y\|}.$$

Proof.

$$4\|x-y\|^2-(\|x\|+\|y\|)^2\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2$$
$$= 4(\|x\|^2+\|y\|^2-2\langle x,y\rangle)-2(\|x\|^2+\|y\|^2+2\|x\|\|y\|)\left(1-\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)$$
$$= (\|x\|^2+\|y\|^2)\left[4-2\left(1-\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)\right]-8\langle x,y\rangle-4\|x\|\|y\|\left(1-\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)$$
$$= (\|x\|^2+\|y\|^2)\left(2+2\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)-8\langle x,y\rangle-4\|x\|\|y\|\left(1-\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)$$

$$= 2[(\|x\|^2 + \|y\|^2)\left(1 + \frac{\langle x, y\rangle}{\|x\|\|y\|}\right) - 2\|x\|\|y\| - 2\langle x, y\rangle]$$

$$= 2\left(1 + \frac{\langle x, y\rangle}{\|x\|\|y\|}\right)(\|x\| - \|y\|)^2. \qquad \square$$

Theorem 6.1. *For $x, y \in K$, one has*

$$\lim_{n\to\infty} \|f^n(x) - f^n(y)\| \ = \ 0.$$

Proof. From the above lemma,

$$\|f^n(x) - f^n(y)\| = \left\| \frac{g_n(x)}{\|g_n(x)\|} - \frac{g_n(y)}{\|g_n(y)\|} \right\|$$

$$= \frac{\sqrt{4[n(\psi(x)-\psi(y))^2+\|x-y\|^2]-2(1+\frac{n\psi(x)\psi(y)+\langle x,y\rangle}{\sqrt{n\psi(x)^2+\|x\|^2}\sqrt{n\psi(y)^2+\|y\|^2}})(\sqrt{n\psi(x)^2+\|x\|^2}-\sqrt{n\psi(y)^2+\|y\|^2})^2}}{\sqrt{n\psi(x)^2+\|x\|^2}+\sqrt{n\psi(y)^2+\|y\|^2}}$$

$$= \frac{\sqrt{4[(\psi(x)-\psi(y))^2+\frac{\|x-y\|^2}{n}]-2(1+\frac{n\psi(x)\psi(y)+\langle x,y\rangle}{\sqrt{n\psi(x)^2+\|x\|^2}\sqrt{n\psi(y)^2+\|y\|^2}})\frac{(\sqrt{n\psi(x)^2+\|x\|^2}-\sqrt{n\psi(y)^2+\|y\|^2})^2}{n}}}{\sqrt{\psi(x)^2+\frac{\|x\|^2}{n}}+\sqrt{\psi(y)^2+\frac{\|y\|^2}{n}}}$$

$$= \frac{\sqrt{4[(\psi(x)-\psi(y))^2+\frac{\|x-y\|^2}{n}]-2\left(1+\frac{\psi(x)\psi(y)+\frac{\langle x,y\rangle}{n}}{\sqrt{\psi(x)^2+\frac{\|x\|^2}{n}}\sqrt{\psi(y)^2+\frac{\|y\|^2}{n}}}\right)(\sqrt{\psi(x)^2+\frac{\|x\|^2}{n}}-\sqrt{\psi(y)^2+\frac{\|y\|^2}{n}})^2}}{\sqrt{\psi(x)^2+\frac{\|x\|^2}{n}}+\sqrt{\psi(y)^2+\frac{\|y\|^2}{n}}}$$

Letting $n \to \infty$ it follows the desired result. $\square$

7. Further remarks

About the domain of f. Although the Lipschitz constant of the mapping f on K is 2, we have pointed out above that this constant is just $\sqrt{2}$ on the set

$$K_S := \{x \in K : \|x\| = 1\}.$$

This set K_S is f-invariant but, of course, is not convex. Then, the following question naturaly arises.

Question 1. *Does there exist an f-invariant closed convex subset of K, say K_0, such that the Lipschitz constant of f en K_0 is strictly less than 2?*

Suppose that K_0 is such a set. If $x \in K_0$, consider the orbit

$$O(x) := (x, f(x), f^2(x), \ldots) \subset K_0.$$

According to [7, Fact 1], if $f^{n+1}(x) = \sum_{n=1}^{\infty} a_n e_n$ then

$$a_1 = a_2 = \ldots = a_n \leq \frac{1}{\sqrt{n}}.$$

This implies that the sequence $(f^n(x))$ is weakly convergent to 0_{ℓ_2}. Since K_0 is weakly compact, then $0_{\ell_2} \in K_0$, which in turn implies that the orbit $O(0_{\ell_2})$ is contained in K_0. From 5.1 one has

$$f^n(0_{\ell_2}) = \frac{1}{\sqrt{n}}(e_1 + \ldots + e_n) \in K_0.$$

Therefore, every closed convex f invariant subset of K must contain the orbit $O(0_{\ell_2})$, and hence the closed convex hull of $O(0_{\ell_2})$, namely

$$K_3 := \mathrm{cl}(\mathrm{conv}(O(0_{\ell_2})) = \mathrm{cl}\left(\mathrm{conv}\left(\{0_{\ell_2}\} \cup \left\{\frac{1}{\sqrt{n}}(e_1 + \ldots + e_n) : n = 1, 2, \ldots\right\}\right)\right).$$

This leads to the following.

Question 2. *Is the set K_3 f-invariant and hence minimal? If not, find a minimal subset for f in K.*

Here 'minimal' means that it does not exist any closed, convex, f-invariant subset of K_1.

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