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ON GENERALIZED COUPLED FIXED POINTS WITH APPLICATIONS TO THE SOLVABILITY OF COUPLED SYSTEMS OF NONLINEAR QUADRATIC INTEGRAL EQUATIONS

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Abstract. In this paper, we introduce the notion of generalized coupled fixed points and, as a consequence of Darbo's fixed point theorem associated to an abstract measure of noncompactness, we present a result about the existence of this class of coupled fixed points in Banach spaces. As an application, we investigate the existence of solutions for a class of coupled systems of nonlinear quadratic integral equations of Volterra type.

Key Words and Phrases: Measures of noncompactness, generalized coupled fixed point, coupled system, integral equation, fixed point theorem.

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1. Introduction

Quadratic integral equations describe numerous problems and events of the real world. For example, quadratic integral equations are often applicable in kinetic theory of gases, in the theory of radiative transfer, in the traffic theory and in the theory of neutron transport, see for instance the book [10] by Chandrasekhar and the research papers of Banaś *et al.* [6, 7], Darwish *et al.* [12, 13, 14, 15, 16, 17], Hu *et al.* [18], Kelley [20], Leggett [22] and Stuart [26], and the references therein.

The concept of coupled fixed point appears in the theory of fixed point in metric spaces [4, 8, 19, 21, 23, 24, 25]. Its definition is the following.

Definition 1.1. Let (X,d) be a metric space and $G : X \times X \to X$ a mapping. An element $(x,y) \in X \times X$ is called a coupled fixed point of G if $G(x,y) = x$ and $G(y,x) = y$.

Recently, the authors in [1] proved the existence of coupled fixed points of a mapping G by using measures of noncompactness.

Next, we recollect some basic facts about measures of noncompactness.

Assume that E is a real Banach space with the norm $\|.\|$ and the zero element θ. By $B(x, r)$ we denote the closed ball in E centered at x with radius r. By B_r we denote the ball $B(\theta, r)$. If X is a nonempty subset of E then the symbols $\overline{X}$ and Conv$X$ denote the closure and closed convex hull of X, respectively. Moreover, by $\mathfrak{M}_E$ we will denote the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets.

Through this paper, we will accept the following definition of measure of noncompactness which appears in [5].

Definition 1.2. A mapping $\mu : \mathfrak{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° The family ker$\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and ker$\mu \subset \mathfrak{N}_E$.
2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3° $\mu(\overline{X}) = \mu(X)$.
4° $\mu(\text{Conv}X) = \mu(X)$.
5° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\, \mu(X) + (1-\lambda)\, \mu(Y)$ for $\lambda \in [0, 1]$.
6° If (X_n) is a sequence of closed subsets of $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then the intersection set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family kerμ appearing in 1° is called the kernel of the measure of noncompactness μ. Notice that the set X_∞ appearing in 6° belongs to kerμ. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$ and this means that $X_\infty \in$ kerμ.

In [11], Darbo proved the following fixed point theorem which is a version of the classical Banach contraction principle in the context of measures of noncompactness.

Theorem 1.3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ satisfying*

$$\mu(TX) \leq k\mu(X),$$

for any nonempty subset X of Ω, where μ is a measure of noncompactness in E.

Then T has a fixed point in Ω.

The following generalization of Darbo's fixed point theorem appears in [2].

Theorem 1.4. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator satisfying*

$$\mu(TX) \leq \varphi(\mu(X)),$$

for any nonempty and noncompact subset X of Ω, where μ is a measure of noncompactness in E and $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$, where φ^n denotes the $n-$iteration of φ.

Then T has a fixed point in Ω.

The following result appears in [3].

Theorem 1.5. *Suppose that $\mu_1, \mu_2, \ldots, \mu_n$ are measures of noncompactness in the Banach spaces $E_1, E_2, \ldots, E_n$, respectively. Let $F : [0, \infty)^n \to [0, \infty)$ be a mapping such that F is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$.*

Then

$$\tilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n)),$$

where X is a nonempty and bounded subset of $E_1 \times E_2 \times \ldots \times E_n$, defines a measure compactness in $E_1 \times E_2 \times \ldots \times E_n$, being X_i the natural projection of X into E_i, for $i = 1, 2, \ldots, n$.

Remark 1.6. Let μ be a measure of noncompactness in a Banach space E and let $F : [0, \infty) \times [0, \infty) \to [0, \infty)$ be the mapping defined by $F(x, y) = \max(x, y)$. It is easily seen that F is convex and $F(x, y) = 0$ if and only if $x = y = 0$ and, therefore, by Theorem 1.5, the function $\tilde{\mu}(X) = \max\{\mu(X_1),\ \mu(X_2)\}$ defined on $\mathfrak{M}_{E_1 \times E_2}$ defines a measure of noncompactness in the space $E \times E$. Similarly, if we take $F(x, y) = x + y$ then it follows that $\tilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ defines a measure of noncompactness in the space $E \times E$.

The main result of [1] is the following.

Theorem 1.7. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and μ a measure of noncompactness in E. Suppose that $G : \Omega \times \Omega \to \Omega$ is a continuous mapping satisfying*

$$\mu(G(X_1 \times X_2)) \leq \varphi\left(\frac{\mu(X_1) + \mu(X_2)}{2}\right)$$

for any X_1 and X_2 subsets of Ω, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing and upper semicontinuous function such that $\varphi(t) < t$ for any $t > 0$.

Then G has at least a coupled fixed point.

The main aim of this paper is to present a generalization of the concept of coupled fixed point and to prove a result about the existence of such points. Finally, we will apply our result to the solvability of a general class of coupled systems of quadratic nonlinear integral equations of Volterra type in the space of the real functions defined and continuous on the interval $[0, 1]$.

As our solutions are placed in the space $C[0, 1] = \{x : [0, 1] \to \mathbb{R},\ x \text{ is continuous}\}$ with the usual supremum norm, i.e., $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$ for $x \in C[0, 1]$, we will present the measure of noncompactness in $C[0, 1]$ which will be used in our study. To do this, let us fix $X \in \mathfrak{M}_{C[0,1]}$ and $\varepsilon > 0$. For $x \in X$, we denote by $\omega(x, \varepsilon)$ the modulus of continuity of x, i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, 1],\ |t - s| \leq \varepsilon\}.$$

Put

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon).$$

In [5], it is proved that $\omega_0(X)$ is a measure of noncompactness in $C[0, 1]$.

2. Main results

Our starting point in this section is the following definition.

Definition 2.1. Let X be a nonempty set and $G : X \times X \to X$ a mapping. Suppose that $F, H : X \times X \to X$ are two mappings. An element $(x, y) \in X \times X$ is said to be a F-H coupled fixed point of G if $G(x, y) = x$ and $G(F(x, y), H(x, y)) = y$.

Remark 2.2. Notice that a coupled fixed point of a mapping $G : X \times X \to X$ is a F-H coupled fixed point of G, where $F(x, y) = y$ and $H(x, y) = x$. Therefore, Definition 2.1 is a generalization of the concept of coupled fixed point.

Next, we present the following result about the existence of F-H coupled fixed point of a mapping G.

Theorem 2.3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E, μ be a measure of noncompactness in E and $\varphi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Suppose that $G : \Omega \times \Omega \to \Omega$ is a continuous mapping and $F, H : \Omega \times \Omega \to \Omega$ two continuous mappings such that*

$$\mu(F(X_1 \times X_2)) \leq \max(\mu(X_1), \mu(X_2)) \tag{2.1}$$

and

$$\mu(H(X_1 \times X_2)) \leq \max(\mu(X_1), \mu(X_2)) \tag{2.2}$$

for any nonempty subsets X_1 and X_2 of Ω.

Assume that

$$\mu(G(X_1 \times X_2)) \leq \varphi(\max(\mu(X_1), \mu(X_2))) \tag{2.3}$$

for any nonempty subsets X_1 and X_2 of Ω.

Then G has at least a F-H coupled fixed point.

Proof. Taking into account Remark 1.6, the function $\tilde{\mu}(X) = \max(\mu(X_1), \mu(X_2))$, where X_i $(i = 1, 2)$ denote the natural projections of X, is a measure of noncompactness in the space $E \times E$.

Now, we consider the operator $\tilde{G} : \Omega \times \Omega \to \Omega \times \Omega$ defined by

$$\tilde{G}(x, y) = (G(x, y), G(F(x, y), H(x, y))).$$

Since G, F and H are continuous mappings, it is clear that $\tilde{G}$ is continuous on $\Omega \times \Omega$.

In the sequel, we will prove that $\tilde{G}$ satisfies the contractive condition appearing in Theorem 1.4.

To do this, we take a nonempty subset X of $\Omega \times \Omega$. Then

$$\begin{aligned} \tilde{\mu}(\tilde{G}(X)) &= \tilde{\mu}(G(X_1 \times X_2), G(F(X_1 \times X_2) \times H(X_1 \times X_2))) \\ &= \max\{\mu(G(X_1 \times X_2)), \mu(G(F(X_1 \times X_2) \times H(X_1 \times X_2)))\}. \end{aligned} \tag{2.4}$$

From (2.3), it follows that

$$\mu(G(X_1 \times X_2)) \leq \varphi(\max(\mu(X_1), \mu(X_2))) \tag{2.5}$$

and

$$\mu(G(F(X_1 \times X_2) \times H(X_1 \times X_2))) \leq \varphi(\max(\mu(F(X_1 \times X_2)), \mu(H(X_1 \times X_2)))).$$

Taking into account (2.1), (2.2) and the fact that φ is nondecreasing, from the last inequality we infer

$$\mu(G(F(X_1 \times X_2) \times H(X_1 \times X_2))) \leq \varphi(\max(\mu(X_1), \mu(X_2))). \tag{2.6}$$

Now, taking into account (2.5) and (2.6), from (2.4) we deduce

$$\tilde{\mu}(\tilde{G}(X)) \leq \varphi(\max(\mu(X_1), \mu(X_2))) = \varphi(\tilde{\mu}(X)).$$

This proves that $\tilde{G}$ satisfies the conditions of Theorem 1.4 and, consequently, $\tilde{G}$ has at least a fixed point in $\Omega \times \Omega$, i.e., there exists $(x, y) \in \Omega \times \Omega$ such that $\tilde{G}(x, y) = (x, y)$. Since $\tilde{G}(x, y) = (G(x, y), G(F(x, y), H(x, y)))$, we get $G(x, y) = x$ and $G(F(x, y), H(x, y)) = y$. This means that (x, y) is a F-H coupled fixed point of G. This finishes the proof. □

As a consequence of Theorem 2.3, we have the following result about the existence of coupled fixed point.

Corollary 2.4. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E, μ be a measure of noncompactness in E and $\varphi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Suppose that $G : \Omega \times \Omega \to \Omega$ is a continuous operator satisfying*

$$\mu(G(X_1 \times X_2)) \leq \varphi(\max(\mu(X_1), \mu(X_2)))$$

for any nonempty subsets X_1 and X_2 of Ω. Then G has at least a coupled fixed point.

Proof. By Remark 2.2, a coupled fixed point is a F-H coupled fixed point with $F(x, y) = y$ and $H(x, y) = x$. Since

$$\mu(F(X_1 \times X_2)) = \mu(X_2) \leq \max(\mu(X_1), \mu(X_2))$$

and

$$\mu(G(X_1 \times X_2)) = \mu(X_1) \leq \max(\mu(X_1), \mu(X_2)),$$

all the conditions of Theorem 2.3 are satisfied. Consequently, G has a F-H coupled fixed point with $F(x, y) = y$ and $H(x, y) = x$, or, equivalently, G has a coupled fixed point. This completes the proof. □

Corollary 2.4 is a similar result to Theorem 2.5 of [1].

Remark 2.5. An example of operator $F : \Omega \times \Omega \to \Omega$ satisfying (2.1) is the following

$$F(x, y) = \lambda x + (1 - \lambda)y, \ \lambda \in [0, 1].$$

Notice that F is well defined since Ω is convex. Moreover, taking into account the properties of a measure of noncompactness,

$$\begin{aligned} \mu(F(X_1 \times X_2)) &= \mu(\lambda X_1 + (1 - \lambda)X_2) \\ &\leq \lambda\mu(X_1) + (1 - \lambda)\mu(X_2) \\ &\leq \lambda \max(\mu(X_1), \mu(X_2)) + (1 - \lambda) \max(\mu(X_1), \mu(X_2)) \\ &= \max(\mu(X_1), \mu(X_2)). \end{aligned}$$

3. Applications

In this section, as an application of our results, we will study the existence of solutions for the following class of systems of nonlinear integral equations of Volterra type

$$\begin{cases} x(t) = a(t) + T(x,y)(t) \int\limits_0^t g(t,s) f(s, x(s), y(s))\, ds \\ y(t) = a(t) + T(F(x,y)(t), H(x,y)(t)) \int\limits_0^t g(t,s) f(s, F(x,y)(s), H(x,y)(s))\, ds. \end{cases} \tag{3.1}$$

We consider the following general assumptions:

(A_1) $a \in C[0,1]$.
(A_2) $g : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function.
(A_3) $T : C[0,1] \times C[0,1] \to C[0,1]$ is a continuous operator such that if $X, Y \in \mathfrak{M}_{C[0,1]}$ then

$$\omega_0(T(X \times Y)) \leq \varphi(\max(\omega_0(X), \omega_0(Y))),$$

where ω_0 is the measure of noncompactness in $C[0,1]$ appearing in Section 1. Moreover, $T(B_r \times B_r) \subset B_r$ for any $r > 0$, where B_r is the ball in $C[0,1]$ centered at θ and with radius r.
(A_4) There exist nonnegative constants c and d such that

$$\|T(x,y)\| \leq c + d\ \max(\|x\|, \|y\|)$$

for any $x, y \in C[0,1]$.
(A_5) $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$|f(t,u,v) - f(t,u_1,v_1)| \leq \max(|u - u_1|, |v - v_1|)$$

for any $t \in [0,1]$ and for any $u, v, u_1, v_1 \in \mathbb{R}$.
Notice that since f is continuous there exists $\sup\{|f(t,0,0)| : t \in [0,1]\}$. Put $M = \sup\{|f(t,0,0)| : t \in [0,1]\}$.
(A_6) $F, H : C[0,1] \times C[0,1] \to C[0,1]$ are two continuous operators satisfying

$$\max(\omega_0(F(X \times Y), \omega_0(H(X \times Y)) \leq \max(\omega_0(X), \omega_0(Y))),$$

for any $X, Y \in \mathfrak{M}_{C[0,1]}$. Moreover, $F(B_r \times B_r) \subset B_r$ and $H(B_r \times B_r) \subset B_r$ for any $r > 0$.
(A_7) There exists $r_0 > 0$ such that

$$\|a\| + (c + dr_0) Q(r_0 + M) \leq r_0,$$

where $Q = \sup\{|g(t,s)| : t, s \in [0,1]\}$, (the existence of Q is guaranteed by assumption (A_2)). Moreover, $Q(r_0 + M) \leq 1$.

Now, we are ready to formulate the main result of this section.

Theorem 3.1. *Under assumptions (A_1) − (A_7), the system (3.1) has at least one solution (x, y) in the space $C[0,1] \times C[0,1]$.*

Proof. We consider the space $C[0,1] \to C[0,1]$ with the measure of noncompactness given by

$$\tilde{\omega}_0(X \times Y) = \max(\omega_0(X), \omega_0(Y))$$

for any $X, Y \in \mathfrak{M}_{C[0,1]}$ (see Remark 1.6).

Now, let G be the operator defined on $C[0,1] \to C[0,1]$ by

$$G(x,y)(t) = a(t) + T(x,y)(t) \int_0^t g(t,s) f(s,x(s),y(s))\, ds$$

for any $x, y \in C[0,1]$ and $t \in [0,1]$.
First, we will prove that $G(x,y) \in C[0,1]$ for any $x, y \in C[0,1]$. In fact, in virtue of (A_1) and (A_3), it is sufficient to prove that $L(x,y) \in C[0,1]$, where $L(x,y)$ is defined by

$$L(x,y)(t) = \int_0^t g(t,s) f(s,x(s),y(s))\, ds,\ t \in [0,1].$$

In order to prove this, we will see that $\lim_{\varepsilon\to 0} \omega(L(x,y),\varepsilon) = 0$. Fix $\varepsilon > 0$ and take $t_1, t_2 \in [0,1]$ such that $|t_1 - t_2| \le \varepsilon$. Without loss of generality, we can suppose that $t_2 > t_1$. Then, we have

$$\begin{aligned}
&|L(x,y)(t_2) - L(x,y)(t_1)| \\
&= \left| \int_0^{t_2} g(t_2,s) f(s,x(s),y(s))\, ds - \int_0^{t_1} g(t_1,s) f(s,x(s),y(s))\, ds \right| \\
&= \left| \int_0^{t_1} g(t_2,s) f(s,x(s),y(s))\, ds + \int_{t_1}^{t_2} g(t_2,s) f(s,x(s),y(s))\, ds \right. \\
&\quad \left. - \int_0^{t_1} g(t_1,s) f(s,x(s),y(s))\, ds \right| \\
&= \left| \int_0^{t_1} (g(t_2,s) - g(t_1,s)) f(s,x(s),y(s))\, ds + \int_{t_1}^{t_2} g(t_2,s) f(s,x(s),y(s))\, ds \right| \\
&\le \int_0^{t_1} |g(t_2,s) - g(t_1,s)|\, |f(s,x(s),y(s))|\, ds + \int_{t_1}^{t_2} |g(t_2,s)|\, |f(s,x(s),y(s))|\, ds \\
&\le \int_0^{t_1} \omega_g(\varepsilon) |f(s,x(s),y(s))|\, ds + \int_{t_1}^{t_2} |g(t_2,s)|\, |f(s,x(s),y(s))|\, ds, \qquad (3.2)
\end{aligned}$$

where

$$\omega_g(\varepsilon) = \sup\{|g(t,s) - g(t',s)| : t, t' \in [0,1],\ |t - t'| \le \varepsilon\}.$$

Since f and g are continuous, there exist the following quantities

$$P = \sup\{|f(t,u,v)| : t \in [0,1],\ u \in [-\|x\|, \|x\|],\ v \in [-\|y\|, \|y\|]\}$$

and

$$Q = \sup\{|g(t,s)| : t, s \in [0,1]\}$$

(see assumption (A_7)). Therefore, from (3.2) it follows that

$$\begin{aligned}
|L(x,y)(t_1) - L(x,y)(t_2)| &\le \omega_g(\varepsilon) P t_1 + QP(t_2 - t_1) \\
&\le \omega_g(\varepsilon) P + QP\varepsilon.
\end{aligned}$$

This gives us

$$\omega(L(x,y),\varepsilon) \le \omega_g(\varepsilon) P + QP\varepsilon. \qquad (3.3)$$

Since g is uniformly continuous on the compact $[0,1]\times[0,1]$, we have that $\omega_g(\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, from (3.3) we infer $\lim\limits_{\varepsilon\to 0}\omega(L(x,y),\varepsilon)=0$. This proves our claim. Therefore, $G(x,y)\in C[0,1]$ for any $x,y\in C[0,1]$, i.e., $G: C[0,1]\times C[0,1]\to C[0,1]$. Moreover, taking into account our assumptions, we derive the following estimate

$$
\begin{aligned}
|G(x,y)(t)| &\le |a(t)| + |T(x,y)(t)|\int_0^t |g(t,s)|\,|f(s,x(s),y(s))|\,ds\\
&\le \|a\| + \|T(x,y)\|Q\int_0^t [|f(s,x(s),y(s)) - f(s,0,0)| + |f(s,0,0)|]\,ds\\
&\le \|a\| + [c + d\max(\|x\|,\|y\|)]Q\int_0^t [\max(|x(s)|,|y(s)|) + M]\,ds\\
&\le \|a\| + [c + d\max(\|x\|,\|y\|)]Q(\max(\|x\|,\|y\|) + M)\,t\\
&\le \|a\| + [c + d\max(\|x\|,\|y\|)]Q(\max(\|x\|,\|y\|) + M),
\end{aligned}
$$

(recall that $M=\sup\{|f(t,0,0)| : t\in[0,1]\}$, see assumption (A_5)).
Taking into account assumption (A_7), we deduce that G applies $B_{r_0}\times B_{r_0}$ into B_{r_0}.

Next, we will prove that G is a continuous operator on the ball B_{r_0}. To do this, we take sequences $(x_n),(y_n)\subset B_{r_0}$ and $x,y\in B_{r_0}$ with $x_n\to x$ and $y_n\to y$ and we have to prove that $G(x_n,y_n)\to G(x,y)$. In fact, for $t\in[0,1]$, we get

$$
\begin{aligned}
&|G(x_n,y_n)(t) - G(x,y)(t)|\\
&= \left|T(x_n,y_n)(t)\int_0^t g(t,s)f(s,x_n(s),y_n(s))ds - T(x,y)(t)\int_0^t g(t,s)f(s,x(s),y(s))ds\right|\\
&\le \left|T(x_n,y_n)(t)\int_0^t g(t,s)f(s,x_n(s),y_n(s))ds - T(x,y)(t)\int_0^t g(t,s)f(s,x_n(s),y_n(s))ds\right|\\
&+ \left|T(x,y)(t)\int_0^t g(t,s)f(s,x_n(s),y_n(s))ds - T(x,y)(t)\int_0^t g(t,s)f(s,x(s),y(s))ds\right|\\
&\le |T(x_n,y_n)(t) - T(x,y)(t)|\int_0^t |g(t,s)|\,[|f(s,x_n(s),y_n(s)) - f(s,0,0)| + |f(s,0,0)|]\,ds\\
&+ |T(x,y)(t)|\int_0^t |g(t,s)||f(s,x_n(s),y_n(s)) - f(s,x(s),y(s))|ds\\
&\le \|T(x_n,y_n) - T(x,y)\|Q\int_0^t [\max(|x_n(s)|,|y_n(s)|) + M]\,ds\\
&+ \|T(x,y)\|Q\int_0^t \max(|x_n(s)-x(s)|,|y_n(s)-y(s)|)ds\\
&\le \|T(x_n,y_n) - T(x,y)\|Q\,[\max(\|x_n\|,\|y_n\|) + M]\\
&+ [c + d\max(\|x\|,\|y\|)]Q\max(\|x_n - x\|,\|y_n - y\|)\\
&\le \|T(x_n,y_n) - T(x,y)\|Q(r_0 + M) + (c + dr_0)Q\max(\|x_n - x\|,\|y_n - y\|).
\end{aligned}
$$

Since T is a continuous operator, $\|T(x_n,y_n) - T(x,y)\|\to 0$ when $n\to\infty$, and, therefore, from the last inequality, we infer that $\|G(x_n,y_n)(t) - G(x,y)(t)\|\to 0$ when $n\to\infty$. This proves the continuity of the operator G on $B_{r_0}\times B_{r_0}$.

In the sequel, we will prove that G satisfies condition (2.3) appearing in Theorem 2.3. To do this, we fix $\varepsilon > 0$ and take two nonempty subsets X_1 and X_2 of B_{r_0}. Let $t_1, t_2 \in [0,1]$ be such that $|t_1 - t_2| \leq \varepsilon$ and suppose that $t_1 < t_2$. Then for $x \in X_1$ and $y \in X_2$, and, taking into account our assumptions, we get

$$
\begin{aligned}
&|G(x,y)(t_2) - G(x,y)(t_1)| \\
&= \left| a(t_2) + T(x,y)(t_2) \int_0^{t_2} g(t_2,s) f(s,x(s),y(s)) ds \right. \\
&\left. -a(t_1) - T(x,y)(t_1) \int_0^{t_1} g(t_1,s) f(s,x(s),y(s))\, ds \right| \\
&\leq |a(t_2) - a(t_1)| + \left| T(x,y)(t_2) \int_0^{t_2} g(t_2,s) f(s,x(s),y(s))\, ds \right. \\
&\left. -T(x,y)(t_1) \int_0^{t_2} g(t_2,s) f(s,x(s),y(s))\, ds \right| \\
&+ \left| T(x,y)(t_1) \int_0^{t_2} g(t_2,s) f(s,x(s),y(s)) ds - T(x,y)(t_1) \int_0^{t_2} g(t_1,s) f(s,x(s),y(s)) ds \right| \\
&+ \left| T(x,y)(t_1) \int_0^{t_2} g(t_1,s) f(s,x(s),y(s)) ds - T(x,y)(t_1) \int_0^{t_1} g(t_1,s) f(s,x(s),y(s)) ds \right| \\
&\leq \omega(a,\varepsilon) + |T(x,y)(t_2) - T(x,y)(t_1)| \int_0^{t_2} |g(t_2,s)|\, |f(s,x(s),y(s))| ds \\
&+ |T(x,y)(t_1)| \int_0^{t_2} |g(t_2,s) - g(t_1,s)|\, |f(s,x(s),y(s))| ds \\
&+ |T(x,y)(t_1)| \int_{t_1}^{t_2} |g(t_1,s)|\, |f(s,x(s),y(s))| ds \\
&\leq \omega(a,\varepsilon) + \omega(T(x,y),\varepsilon) Q \int_0^{t_2} [|f(s,x(s),y(s)) - f(s,0,0)| + |f(s,0,0)|] ds \\
&+ \|T(x,y)\| \int_0^{t_2} \omega_g(\varepsilon) [|f(s,x(s),y(s)) - f(s,0,0)| + |f(s,0,0)|] ds \\
&+ \|T(x,y)\| Q P_{r_0} (t_2 - t_1) \\
&\leq \omega(a,\varepsilon) + \omega(T(x,y),\varepsilon) Q [\max(\|x\|, \|y\|) + M] \\
&+ [c + d \max(\|x\|, \|y\|)] \omega_g(\varepsilon) [\max(\|x\|, \|y\|) + M] + [c + d \max(\|x\|, \|y\|)] Q P_{r_0} \varepsilon \\
&\leq \omega(a,\varepsilon) + \omega(T(x,y),\varepsilon) Q (r_0 + M) + (c + d r_0) \omega_g(\varepsilon)(r_0 + M) + (c + d r_0) Q P_{r_0} \varepsilon,
\end{aligned}
$$

where

$$\omega_g(\varepsilon) = \sup\{|g(t,s) - g(t',s)| : t, t' \in [0,1],\ |t - t'| \leq \varepsilon\}$$

and

$$P_{r_0} = \sup\{|f(t,u,v)| : t \in [0,1],\ u, v \in [-r_0, r_0]\}.$$

Therefore, we infer that

$$\begin{aligned}\omega(G(X_1\times X_2),\varepsilon) &\leq \omega(a,\varepsilon)+\omega(T(X_1\times X_2),\varepsilon)Q(r_0+M)\\ &\quad +(c+dr_0)\omega_g(\varepsilon)(r_0+M)+(c+dr_0)QP_{r_0}\varepsilon.\end{aligned}$$

Since $\omega_g(\varepsilon)\to 0$ as $\varepsilon\to 0$ by uniform continuity of the function g on the compact $[0,1]\times[0,1]$ and $\omega(a,\varepsilon)\to 0$ as $\varepsilon\to 0$ by the continuity of a on $[0,1]$, from the last inequality, it follows that

$$\omega_0(G(X_1\times X_2))\leq \omega_0(T(X_1\times X_2))Q(r_0+M).$$

Now, taking into account assumptions (A_3), we have

$$\omega_0(G(X_1\times X_2))\leq Q(r_0+M)\varphi(\max(\omega_0(X_1),\omega_0(X_2)).$$

Since $Q(r_0+M)\leq 1$, the function $\varphi_1=Q(r_0+M)\varphi$ is nondecreasing and satisfies $\varphi_1^{(n)}(t)\to 0$ for $t>0$. This proves that G satisfies condition (2.3) of Theorem 2.3. Since the operators $F,H:C[0,1]\times C[0,1]\to C[0,1]$ satisfy $F,H:B_{r_0}\times B_{r_0}\to B_{r_0}$ and

$$\max(\omega_0(F(X\times Y)),\omega_0(H(X\times Y)))\leq \max(\omega_0(X),\omega_0(Y))$$

for any $X,Y\in\mathfrak{M}_{C[0,1]}$ (assumption (A_6)), the conditions of Theorem 2.3 are satisfies and, by this theorem, G has at least a F-H coupled fixed point in $B_{r_0}\times B_{r_0}$. This means that there exists $(x,y)\in B_{r_0}\times B_{r_0}$ such that $G(x,y)=x$ and $G(F(x,y),H(x,y))=y$ or, equivalently,

$$\begin{aligned}x(t) &= a(t)+T(x,y)(t)\int_0^t g(t,s)f(s,x(s),y(s))\,ds,\\ y(t) &= a(t)+T(F(x,y),H(x,y))(t)\int_0^t g(t,s)f(s,F(x,y)(s),H(x,y)(s))\,ds.\end{aligned}$$

This is the desired result.
This finishes the proof. □

Since the operators $F,H:C[0,1]\times C[0,1]\to C[0,1]$ defined by

$$F(x,y)=y \text{ and } H(x,y)=x$$

satisfy assumption (A_6) of Theorem 3.1, we have the following corollary.

Corollary 3.2. *Suppose the following system of nonlinear integral equations*

$$\begin{cases} x(t)=a(t)+T(x,y)(t)\int\limits_0^t g(t,s)f(s,x(s),y(s))\,ds\\ y(t)=a(t)+T(y,x)(t)\int\limits_0^t g(t,s)f(s,y(s),x(s))\,ds.\end{cases}\tag{3.4}$$

Under assumptions $(A_1)-(A_5)$ *and* (A_7) *of Theorem* 3.1, *the system (3.4) has at least one solution* (x,y) *in the space* $C[0,1]\times C[0,1]$.

Now, we present some examples of operators defined on $C[0,1]\times C[0,1]$ satisfying assumption (A_6) of Theorem 3.1.

Example 3.3. Suppose that $\varphi_i : [0,1] \to [0,1]$ $(i = 1,2)$ are two continuous functions with bounded derivatives by K_i $(i = 1,2)$. Consider the operator F_{φ_1,φ_2} defined on $C[0,1] \times C[0,1]$ by

$$F_{\varphi_1,\varphi_2}(x,y)(t) = \frac{1}{2}(x(\varphi_1(t)) + y(\varphi_2(t))) \text{ for } t \in [0,1].$$

It is clear that F_{φ_1,φ_2} applies $C[0,1] \times C[0,1]$ into $C[0,1]$. Moreover, if $\|x\| \le r$ and $\|y\| \le r$ for $r > 0$ then

$$\begin{aligned}
\|F_{\varphi_1,\varphi_2}(x,y)\| &= \sup\{|F_{\varphi_1,\varphi_2}(x,y)(t)| : t \in [0,1]\} \\
&\le \sup\left\{\frac{1}{2}[|x(\varphi_1(t)) + |y(\varphi_2(t))|] : t \in [0,1]\right\} \\
&\le \frac{1}{2}(\|x\| + \|y\|) \\
&\le \max(\|x\|, \|y\|) \le r.
\end{aligned}$$

Therefore, $F_{\varphi_1,\varphi_2}(B_r \times B_r) \subset B_r$ for any $r > 0$.
Next, fix $x, y \in C[0,1]$ and for $\varepsilon > 0$ and $t_1, t_2 \in [0,1]$ with $|t_2 - t_1| \le \varepsilon$, we have

$$\begin{aligned}
&|F_{\varphi_1,\varphi_2}(x,y)(t_2) - F_{\varphi_1,\varphi_2}(x,y)(t_1)| \\
&\qquad \le \frac{1}{2}[|x(\varphi_1(t_2)) - x(\varphi_1(t_1))| + |y(\varphi_2(t_2)) - y(\varphi_2(t_1))|].
\end{aligned}$$

Since φ_1 and φ_2 have bounded derivatives, by using the Mean Value Theorem, we have

$$|\varphi_1(t_2) - \varphi_1(t_1)| \le K_1|t_2 - t_1| \le K_1\varepsilon$$

and

$$|\varphi_2(t_2) - \varphi_2(t_1)| \le K_2|t_2 - t_1| \le K_2\varepsilon.$$

Then, from the last inequality, it follows that

$$\omega(F_{\varphi_1,\varphi_2}(x,y),\varepsilon) \le \frac{1}{2}[\omega(x, K_1\varepsilon) + \omega(y, K_2\varepsilon)]$$

and, therefore, for any $X, Y \in \mathfrak{M}_{C[0,1]}$, we have

$$\omega(F_{\varphi_1,\varphi_2}(X \times Y),\varepsilon) \le \frac{1}{2}[\omega(X, K_1\varepsilon) + \omega(Y, K_2\varepsilon)].$$

Letting $\varepsilon \to 0$, we infer

$$\omega_0(F_{\varphi_1,\varphi_2}(X \times Y)) \le \frac{1}{2}[\omega_0(X) + \omega_0(Y)] \le \max(\omega_0(X), \omega_0(Y)).$$

Therefore, F_{φ_1,φ_2} satisfies assumption (A_6) of Theorem 3.1.
Example 3.4. Suppose that $\varphi_i : [0,1] \to [0,1]$ $(i = 1,2)$ are two continuous functions. Consider the operator F^{φ_1,φ_2} defined on $C[0,1] \times C[0,1]$ by

$$F^{\varphi_1,\varphi_2}(x,y)(t) = \frac{1}{2}[x(t)\varphi_1(t) + y(t)\varphi_2(t)] \text{ for } t \in [0,1].$$

It is clear that F^{φ_1,φ_2} applies $C[0,1]\times C[0,1]$ into $C[0,1]$. Moreover, if $\|x\|\le r$ and $\|y\|\le r$ for $r>0$ then, for $t\in[0,1]$, we have

$$\begin{aligned}|F^{\varphi_1,\varphi_2}(x,y)(t)| &\le \frac{1}{2}[|\varphi_1(t)|\,|x(t)|+|\varphi_2(t)|\,|y(t)|]\\ &\le \frac{1}{2}(\|x\|+\|y\|)\\ &\le \max(\|x\|,\|y\|)\\ &\le r.\end{aligned}$$

Therefore, $F^{\varphi_1,\varphi_2}(B_r\times B_r)\subset B_r$ for any $r>0$.
Next, fix $x,y\in C[0,1]$ and, for $\varepsilon>0$ and $t_1,t_2\in[0,1]$ with $|t_2-t_1|\le\varepsilon$, we have

$$\begin{aligned}&|F^{\varphi_1,\varphi_2}(x,y)(t_2)-F^{\varphi_1,\varphi_2}(x,y)(t_1)|\\ &=\left|\frac{1}{2}[x(t_2)\varphi_1(t_2)+y(t_2)\varphi_2(t_2)]-\frac{1}{2}[x(t_1)\varphi_1(t_1)+y(t_1)\varphi_2(t_1)]\right|\\ &\le\left|\frac{1}{2}[x(t_2)\varphi_1(t_2)-x(t_1)\varphi_1(t_1)]\right|+\left|\frac{1}{2}[y(t_2)\varphi_2(t_2)-y(t_1)\varphi_2(t_1)]\right|\\ &\le\frac{1}{2}\left[|x(t_2)\varphi_1(t_2)-x(t_1)\varphi_1(t_2)|+|x(t_1)\varphi_1(t_2)-x(t_1)\varphi_1(t_1)|\right]\\ &\quad+\frac{1}{2}\left[|y(t_2)\varphi_2(t_2)-y(t_1)\varphi_2(t_2)|+|y(t_1)\varphi_2(t_2)-y(t_1)\varphi_2(t_1)|\right]\\ &\le\frac{1}{2}\left[|x(t_2)-x(t_1)|\,|\varphi_1(t_2)|+|x(t_1)|\,|\varphi_1(t_2)-\varphi_1(t_1)|\right]\\ &\quad+\frac{1}{2}\left[|y(t_2)-y(t_1)|\,|\varphi_2(t_2)|+|y(t_1)|\,|\varphi_2(t_2)-\varphi_2(t_1)|\right]\\ &\le\frac{1}{2}[\omega(x,\varepsilon)+\|x\|\omega(\varphi_1,\varepsilon)]+\frac{1}{2}[\omega(y,\varepsilon)+\|y\|\omega(\varphi_2,\varepsilon)],\end{aligned}$$

where we have used the fact that $\|\varphi_i\|\le 1$ $(i=1,2)$. Therefore, we have

$$\omega(F^{\varphi_1,\varphi_2}(x,y),\varepsilon)\le\frac{1}{2}[\omega(x,\varepsilon)+\|x\|\omega(\varphi_1,\varepsilon)]+\frac{1}{2}[\omega(y,\varepsilon)+\|y\|\omega(\varphi_2,\varepsilon)].$$

From this, we infer that, for any $X,Y\in\mathfrak{M}_{C[0,1]}$, we have

$$\omega(F^{\varphi_1,\varphi_2}(X\times Y),\varepsilon)\le\frac{1}{2}[\omega(X,\varepsilon)+\|X\|\omega(\varphi_1,\varepsilon)]+\frac{1}{2}[\omega(Y,\varepsilon)+\|Y\|\omega(\varphi_2,\varepsilon)],$$

where for $A\in\mathfrak{M}_{C[0,1]}$, the symbol $\|A\|$ denotes the quantity $\|A\|=\sup\{\|a\|:a\in A\}$. In virtue of the continuity of φ_1 and φ_2, we have that $\omega(\varphi_1,\varepsilon),\omega(\varphi_2,\varepsilon)\to 0$ when $\varepsilon\to 0$. Therefore, from the last inequality, we get

$$\begin{aligned}\omega_0(F^{\varphi_1,\varphi_2}(X\times Y)) &\le \frac{1}{2}[\omega_0(X)+\omega_0(Y)]\\ &\le \max(\omega_0(X),\omega_0(Y))\end{aligned}$$

and, consequently, F^{φ_1,φ_2} satisfies assumption (A_6) of Theorem 3.1.
Example 3.5. Let $\phi_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ be nonexpansive mappings, i.e.,

$$|\phi_i(t)-\phi_i(t')|\le|t-t'|$$

for any $t, t' \in \mathbb{R}$ $(i = 1, 2)$ and, moreover, $\phi_i(0) = 0$ $(i = 1, 2)$. Consider the operator H_{ϕ_1,ϕ_2} defined on $C[0,1] \times C[0,1]$ by

$$H_{\phi_1,\phi_2}(x, y)(t) = \frac{1}{2}[\phi_1(x(t)) + \phi_2(y(t))] \text{ for } t \in [0,1].$$

Since nonexpansive mappings are continuous, it is clear that H_{ϕ_1,ϕ_2} applies $C[0,1] \times C[0,1]$ into $C[0,1]$. Moreover, if $\|x\| \le r$ and $\|y\| \le r$ for $r > 0$ then, for $t \in [0,1]$, we have

$$\begin{aligned}
|H_{\phi_1,\phi_2}(x, y)(t)| &\le \frac{1}{2}[|\phi_1(x(t))| + |\phi_2(y(t))|] \\
&= \frac{1}{2}[|\phi_1(x(t)) - \phi_1(0)| + |\phi_2(y(t)) - \phi_2(0)|] \\
&\le \frac{1}{2}(|x(t)| + |y(t)|) \\
&\le \frac{1}{2}(\|x\| + \|y\|) \\
&\le \max(\|x\|, \|y\|) \\
&\le r,
\end{aligned}$$

where we have used the facts that $\phi_i(0) = 0$ $(i = 1, 2)$ and ϕ_i are nonexpansive mappings $(i = 1, 2)$. Therefore, $H_{\phi_1,\phi_2}(B_r \times B_r) \subset B_r$ for any $r > 0$.
Moreover, for $x, y \in C[0,1]$ and, for $\varepsilon > 0$ and $t_1, t_2 \in [0,1]$ with $|t_2 - t_1| \le \varepsilon$, we have

$$\begin{aligned}
|H_{\phi_1,\phi_2}(x, y)(t_2) - H_{\phi_1,\phi_2}(x, y)(t_1)| &\le \frac{1}{2}[|\phi_1(x(t_2)) - \phi_1(x(t_1))| + |\phi_2(y(t_2)) - \phi_2(y(t_1))|] \\
&\le \frac{1}{2}[|x(t_2) - x(t_1)| + |y(t_2) - y(t_1)|] \\
&\le \frac{1}{2}[\omega(x, \varepsilon) + \omega(y, \varepsilon)].
\end{aligned}$$

Therefore,

$$\omega(H_{\phi_1,\phi_2}(x, y), \varepsilon) \le \frac{1}{2}[\omega(x, \varepsilon) + \omega(y, \varepsilon)]$$

and, therefore, for any $X, Y \in \mathfrak{M}_{C[0,1]}$, we deduce

$$\begin{aligned}
\omega_0(H_{\phi_1,\phi_2}(X \times Y)) &\le \frac{1}{2}[\omega_0(X) + \omega_0(Y)] \\
&\le \max(\omega_0(X), \omega_0(Y)).
\end{aligned}$$

This proves that H_{ϕ_1,ϕ_2} satisfies assumption (A_6) of Theorem 3.1.
Remark 3.6. Examples of functions ϕ_i satisfying conditions of Example 3.5 are $\phi(t) = \sin t$, $\phi(t) = \arctan t$ and $\phi(t) = \ln(1 + t)$.
Example 3.7. In [9], the authors proved that the operator Q defined on $C[0,1]$ by

$$(Qx)(t) = \max_{0 \le \tau \le t} |x(\tau)|$$

satisfies

(*a*) $Q : C[0,1] \to C[0,1]$
(*b*) $\omega(Qx, \varepsilon) \le \omega(x, \varepsilon)$ for any $x \in C[0,1]$ and $\varepsilon > 0$
(*c*) Q is continuous

(*d*) $\|Qx\| \leq \|x\|$.

Taking into account this, it is easily checked that the operator H_Q defined on $C[0,1] \times C[0,1]$ by

$$H_Q(x,y)(t) = \frac{1}{2}((Qx)(t) + (Qy)(t))$$

satisfies assumption (A_6) of Theorem 3.1.

Example 3.8. Consider the operator K defined on $C[0,1]$ by

$$(Kx)(t) = \int_0^t x(s)\, ds.$$

It is clear that K applies $C[0,1]$ into itself. Moreover, K is continuous and

$$\|Kx\| \leq \|x\|.$$

On the other hand, for $x \in C[0,1]$, $\varepsilon > 0$ and $t_1, t_2 \in [0,1]$ with $|t_2 - t_1| \leq \varepsilon$ and $t_1 < t_2$, we have

$$\begin{aligned} |(Kx)(t_2) - (Kx)(t_1)| &= \left| \int_0^{t_2} x(s)\, ds - \int_0^{t_1} x(s)\, ds \right| \\ &= \left| \int_{t_1}^{t_2} x(s)\, ds \right| \\ &\leq \|x\|(t_2 - t_1) \\ &\leq \|x\|\varepsilon, \end{aligned}$$

and, from this, it is easily proved that $\omega_0(X) = 0$ for any $X \in \mathfrak{M}_{C[0,1]}$.

Therefore, using the same argument that in Example 3.7, the operator H_K defined on $C[0,1] \times C[0,1]$ by

$$H_K(x,y)(t) = \frac{1}{2}\left(\int_0^t x(s)\, ds + \int_0^t y(s)\, ds\right)$$

satisfies $\omega_0(H_K(x,y)) = 0$ for any $X, Y \in \mathfrak{M}_{C[0,1]}$. Therefore, H_K satisfies assumption (A_6) of Theorem 3.1.

Remark 3.9. Notice that the arguments used in Examples 3.3, 3.4, 3.5, 3.7 and 3.8 also work for the operators on $C[0,1] \times C[0,1]$ defined by

$$F^{\lambda}_{\varphi_1,\varphi_2}(x,y)(t) = \lambda(x(\varphi_1(t))) + (1-\lambda)(y(\varphi_2(t))),$$

$$F^{\varphi_1,\varphi_2}_{\lambda}(x,y)(t) = \lambda x(t)\varphi_1(t) + (1-\lambda)y(t)\varphi_2(t),$$

$$H^{\lambda}_{\phi_1,\phi_2}(x,y)(t) = \lambda\phi_1(x(t)) + (1-\lambda)\phi_2(y(t)),$$

$$H^{\lambda}_{Q}(x,y)(t) = \lambda\left(\max_{0\leq\tau\leq t}|x(\tau)| + (1-\lambda)\max_{0\leq\tau\leq t}|y(\tau)|\right),$$

and

$$H^{\lambda}_{K}(x,y)(t) = \lambda\int_0^t x(s)\, ds + (1-\lambda)\int_0^t y(s)\, ds,$$

where $\lambda \in [0,1]$, and, therefore, these operators are also examples of operators satisfying assumption (A_6) of Theorem 3.1.

In order to illustrate our results, we present the following example.

Example 3.10. Consider the following system of integral equations

$$\begin{cases} x(t) = \alpha e^{-t} + \left(\lambda \int\limits_0^t x(s)ds + (1-\lambda)\int\limits_0^t y(s)ds\right) \\ \qquad\qquad \times \int\limits_0^t \sin(t+s)\left(s + \frac{1}{2}\arctan x(s) + \frac{1}{2}y(s)\right)ds \\ y(t) = \alpha e^{-t} + \left(\lambda \int\limits_0^t y(s)ds + (1-\lambda)\int\limits_0^t x(s)ds\right) \\ \qquad\qquad \times \int\limits_0^t \sin(t+s)\left(s + \frac{1}{2}\arctan y(s) + \frac{1}{2}x(s)\right)ds \end{cases} \tag{3.5}$$

where $\lambda \in [0,1]$ and $\alpha \geq 0$.
Notice that this system is a particular case of system (3.1) with $a(t) = \alpha e^{-t}$,

$$T(x,y)(t) = \lambda \int\limits_0^t y(s)ds + (1-\lambda)\int\limits_0^t x(s)ds,$$

$$g(t,s) = \sin(s+t)$$

and

$$f(t,u,v) = t + \frac{1}{2}\arctan u + \frac{1}{2}v.$$

It is clear that assumptions (A_1) and (A_2) of Theorem 3.1 are satisfied.
Notice that the operator T is the operator H_K^λ appearing in Remark 3.9 and, therefore, $\omega_0(T(X\times Y)) = \omega_0(H_K^\lambda(X\times Y)) = 0$ and $T(B_r\times B_r) \subset B_r$ for any $r > 0$ (see Example 3.8). Consequently, assumption (A_3) of Theorem 3.1 is satisfied. Moreover, we have

$$\begin{aligned} |T(x,y)(t)| &\leq \lambda\|x\| + (1-\lambda)\|y\| \\ &\leq \max(\|x\|, \|y\|) \end{aligned}$$

for any $t \in [0,1]$ and, therefore,

$$\|T(x,y)\| \leq \max(\|x\|, \|y\|).$$

This proves that assumption (A_4) of Theorem 3.1 is satisfied with $c = 0$ and $d = 1$.
For assumption (A_5) of Theorem 3.1, we have, for any $t \in [0,1]$ and for any $u, v, u_1, v_1 \in \mathbb{R}$,

$$\begin{aligned} |f(t,u,v) - f(t,u_1,v_1)| &= \left|\frac{1}{2}\arctan u + \frac{1}{2}v - \frac{1}{2}\arctan u_1 - \frac{1}{2}v_1\right| \\ &\leq \frac{1}{2}|\arctan u - \arctan u_1| + \frac{1}{2}|v - v_1|. \end{aligned}$$

Since $|\arctan u - \arctan u_1| \leq |u - u_1|$ (by the Mean Value Theorem), from the last inequality, it follows that

$$\begin{aligned} |f(t,u,v) - f(t,u_1,v_1)| &\leq \frac{1}{2}|u-u_1| + \frac{1}{2}|v-v_1| \\ &\leq \max(|u-u_1|, |v-v_1|), \end{aligned}$$

and this proves that assumption (A_5) of Theorem 3.1 is satisfied. Moreover, it is clear that

$$M = \sup\{|f(t,0,0)| : t \in [0,1]\} = \sup\{|t : \ t \in [0,1]\} = 1.$$

In this particular case, the operators F and H appearing in Theorem 3.1 are defined by $F(x,y) = y$ and $H(x,y) = x$ and it is clear that these operators satisfy assumption (A_6) of Theorem 3.1.

Finally, the inequality appearing in assumption (A_7) of Theorem 3.1 takes the form

$$\begin{aligned} r_0 &\geq \|a\| + (c + dr_0)Q(r_0 + M) \\ &= \alpha + r_0(r_0 + 1)\sin 2, \end{aligned}$$

where we have used that $\|a\| = \alpha$ and

$$\begin{aligned} Q &= \sup\{|g(t,s)| : t, s \in [0,1]\} \\ &= \sup\{|\sin(t+s)| : s \in [0,1]\} \\ &= \sin 2. \end{aligned}$$

This gives us

$$r_0^2 \sin 2 + (\sin 2 - 1)r_0 + \alpha \leq 0.$$

This inequality has a positive solution for $\alpha \leq \frac{(\sin 2 - 1)^2}{4 \sin 2}$ which is

$$0 < r_0 = \frac{(1 - \sin 2) - \sqrt{(\sin 2 - 1)^2 - 4\alpha \sin 2}}{2 \sin 2}.$$

Moreover, it is easily seen that $Q(r_0 + M) \leq 1$. Therefore, assumptions of Theorem 3.1 are satisfied and this proves that, for $0 < \alpha \leq \frac{(\sin 2 - 1)^2}{4 \sin 2}$, the system (3.5) has a solution $(x,y) \in C[0,1] \times C[0,1]$ with

$$\|x\|, \|y\| \leq r_0 = \frac{(1 - \sin 2) - \sqrt{(\sin 2 - 1)^2 - 4\alpha \sin 2}}{2 \sin 2}.$$

Notice that the same argument can be used to prove that the following systems, for example,

$$\begin{cases} x(t) = \alpha e^{-t} + \left(\lambda \int\limits_0^t x(s)ds + (1-\lambda) \int\limits_0^t y(s)ds \right) \\ \qquad \times \int\limits_0^t \sin(t+s) \left(s + \frac{1}{2} \arctan x(s) + \frac{1}{2} y(s) \right) ds \\ y(t) = \alpha e^{-t} + \left(\lambda \int\limits_0^t \frac{s^2(x(s)+y(s))}{2} ds + (1-\lambda) \int\limits_0^t \frac{s(x(s)+y(s))}{2} ds \right) \\ \qquad \times \int\limits_0^t \sin(t+s) \left(s + \frac{1}{2} \arctan \frac{s^2(x(s)+y(s))}{2} + \frac{1}{2} \frac{s(x(s)+y(s))}{2} \right) ds \end{cases} \tag{3.6}$$

and

$$\begin{cases} x(t) = \alpha e^{-t} + \left(\lambda \int\limits_0^t x(s)ds + (1-\lambda)\int\limits_0^t y(s)ds\right) \\ \qquad\qquad \times \int\limits_0^t \sin(t+s)\left(s + \frac{1}{2}\arctan x(s) + \frac{1}{2}y(s)\right) ds \\ y(t) = \alpha e^{-t} + \left(\lambda \int\limits_0^t \frac{1}{2}\left(\max\limits_{0\le\tau\le s}|x(\tau)| + \max\limits_{0\le\tau\le s}|y(\tau)|\right) ds + (1-\lambda)\int\limits_0^t x(s)\, ds\right) \\ \quad \times \int\limits_0^t \sin(t+s)\left(s + \frac{1}{2}\arctan\left(\frac{1}{2}\left(\max\limits_{0\le\tau\le s}|x(\tau)| + \max\limits_{0\le\tau\le s}|y(\tau)|\right)\right) + \frac{1}{2}x(s)\right) ds \end{cases} \tag{3.7}$$

have a solution $(x, y) \in C[0,1] \times C[0,1]$ with

$$\|x\|, \|y\| \le r_0 = \frac{(1-\sin 2) - \sqrt{(\sin 2 - 1)^2 - 4\alpha \sin 2}}{2\sin 2}.$$

This is due to that the operators $F, H : C[0,1]\times[0,1] \to C[0,1]$ appearing in Theorem 3.1 must satisfy only assumption (A_6) of Theorem 3.1 and, in the system (3.6), these operators are defined as

$$F(x,y)(t) = \frac{1}{2}\left[t^2(x(t)+y(t))\right]$$

$$H(x,y)(t) = \frac{1}{2}\left[t(x(t)+y(t))\right]$$

which satisfy assumption (A_6) of Theorem 3.1 (see Example 3.4).

In the system (3.7), these operators are

$$F(x,y)(t) = \frac{1}{2}\left(\max_{0\le\tau\le t}|x(\tau)| + \max_{0\le\tau\le t}|y(\tau)|\right)$$

and

$$H(x,y)(t) = x(t)$$

and they satisfy assumption (A_6) of Theorem 3.1 (see Example 3.5).

This proves that our Theorem 3.1 is applicable to a great number of similar systems of nonlinear integral equations to (3.1).

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