Fixed Point Theory, 19(2018), No. 2, 515-526
DOI: 10.24193/fpt-ro.2018.2.41
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINT RESULTS FOR ADMISSIBLE $\mathcal{Z}$-CONTRACTIONS

MARIJA CVETKOVIĆ*, ERDAL KARAPINAR** AND VLADIMIR RAKOČEVIĆ*

*University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia
E-mails: marijac@pmf.ni.ac.rs, vrakoc@sbb.rs

**Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey
E-mail: erdalkarapinar@gmail.com

Abstract. In this paper, we present some fixed point results in the setting of a complete metric spaces by defining a new contractive condition via admissible mapping embedded in simulation function.

Key Words and Phrases: Contractive mapping, simulation function, admissible mapping.
2010 Mathematics Subject Classification: 46T79, 47H10, 54H25.

1. Introduction and preliminaries

Modifications of contractive condition for a mapping on a complete metric space lead to a various fixed point results and different generalizations of famous fixed point theorems. Recently were published many papers in the field of fixed point theory and applications that include simulation functions (see [2, 7, 12, 14]).

Definition 1.1. (See [7]) A *simulation function* is a mapping $\zeta : [0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0)=0$;
(ζ_2) $\zeta(t,s)<s-t$ for all $t,s>0$;
(ζ_3) if $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n,s_n)<0. \tag{1.1}$$

Let $\mathcal{Z}$ denote the family of all simulation functions $\zeta : [0,\infty)\times[0,\infty)\to\mathbb{R}$. Due to the axiom ($\zeta_2$), we have

$$\zeta(t,t)<0 \text{ for all } t>0,\ \zeta\in\mathcal{Z}. \tag{1.2}$$

Example 1.2. (See e.g. [1, 7, 12]) Let $\zeta_i : [0,\infty)\times[0,\infty)\to\mathbb{R}$, $i\in\{1,2,3\}$, be mappings defined by

(i) $\zeta_1(t,s)=\psi(s)-\phi(t)$ for all $t,s\in[0,\infty)$, where $\phi,\psi : [0,\infty)\to[0,\infty)$ are two continuous functions such that $\psi(t)=\phi(t)=0$ if, and only if, $t=0$, and $\psi(t)<t\le\phi(t)$ for all $t>0$.

(*ii*) $\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)}t$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty) \to (0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.

(*iii*) $\zeta_3(t,s) = s - \varphi(s) - t$ for all $t, s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if, and only if, $t = 0$.

(*iv*) If $\varphi : [0,\infty) \to [0,1)$ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$, and we define

$$\zeta_T(t,s) = s\,\varphi(s) - t \qquad \text{for all } s,t \in [0,\infty),$$

then ζ_T is a simulation function.

(*v*) If $\eta : [0,\infty) \to [0,\infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, and we define

$$\zeta_{BW}(t,s) = \eta(s) - t \qquad \text{for all } s,t \in [0,\infty),$$

then ζ_{BW} is a simulation function.

(*vi*) If $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^\varepsilon \phi(u)du$ exists and $\int_0^\varepsilon \phi(u)du > \varepsilon$, for each $\varepsilon > 0$, and we define

$$\zeta_K(t,s) = s - \int_0^t \phi(u)du \qquad \text{for all } s,t \in [0,\infty),$$

then ζ_K is a simulation function.

One can find more interesting examples of simulation functions in [1, 7, 12].

Definition 1.3. ([7]) Suppose (X,d) is a metric space, T is a self-mapping on X and $\zeta \in \mathcal{Z}$. A mapping T is a *$\mathcal{Z}$-contraction with respect to* ζ, if

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0 \qquad \text{for all } x,y \in X.$$

Since (ζ_2) holds, we have the following inequality

$$x \neq y \implies d(Tx,Ty) \neq d(x,y).$$

Thus, we conclude that that T cannot be an isometry whenever T is a $\mathcal{Z}$-contraction. In other words, if a $\mathcal{Z}$-contraction T in a metric space has a fixed point, then it is necessarily unique.

Theorem 1.4. ([7]) *Every $\mathcal{Z}$-contraction on a complete metric space has a unique fixed point. In fact, every Picard sequence converges to its unique fixed point.*

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(*i*) ψ is nondecreasing;

(*ii*) there exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

Such functions are discussed in the literature densely, see e.g. [3, 9, 10, 13] and they are called as (c)-comparison functions.

Lemma 1.5. (See e.g. [13]) *If $\psi \in \Psi$, then the following hold:*

(i) $\psi^n(t) \to 0$, *as* $n \to \infty$, *for all* $t \in \mathbb{R}^+$;
(ii) $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$;
(iii) ψ *is continuous at* 0;
(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

Recently, Samet *et al.* [15] suggested a new contraction type self-mapping to unify several existing results in the literature by auxiliary functions.

Definition 1.6. Let $\alpha : X \times X \to [0, \infty)$. A self-mapping $T : X \to X$ is called $\alpha-$admissible if the condition

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1,$$

is satisfied for all $x, y \in X$.

Definition 1.7. Let T be a self-mapping defined on a metric space (X, d). Then, T is called an $\alpha - \psi$ contractive mapping if there exist two auxiliary mappings $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } x, y \in X.$$

Clearly, any contractive mapping, that is, a mapping satisfying Banach contraction, is an $\alpha - \psi$ contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, $k \in (0, 1)$. A number of examples of such type mappings are considered in [15]. Modifications of a contractive condition for $\alpha - \psi$ contractive mapping lead to a wider class of contractive mappings defined in [6].

Definition 1.8. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a *generalized* $\alpha - \psi$ *contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$,

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)), \tag{1.3}$$

where $M(x, y) = \max\left\{d(x, y), \dfrac{d(x, Tx) + d(y, Ty)}{2}, \dfrac{d(x, Ty) + d(y, Tx)}{2}\right\}$.

In [15], and respectively [6], are presented sufficient conditions for existence of a fixed point for $\alpha - \psi$ contractive and generalized $\alpha - \psi$ contractive self-mapping T on a complete metric space (X, d)

Theorem 1.9. ([15]) *Let $T : X \to X$ be an $\alpha - \psi$ contractive mapping where (X, d) is a complete metric space. Suppose that*

(i) *T is $\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *either, T is continuous, or*
$(iii)'$ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then, there exists $z \in X$ such that $Tz = z$.

Theorem 1.10. ([6]) *Let $T : X \to X$ be a generalized $\alpha - \psi$ contractive mapping where (X, d) is a complete metric space. Suppose that*

(i) *T is $\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *either, T is continuous, or*
(iii)′ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then, there exists $z \in X$ such that $Tz = z$.

By adding the condition:

(U) For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$,

to the statement of Theorem 1.9, we obtain a unique fixed point of an $\alpha - \psi$ contraction, as proved in [15].

2. Main results

The main goal will be to show existence of a fixed point for a class of α-admissible mapping when contractive condition includes a simulation function. We start with the following definition.

Definition 2.1. Let T be a self-mapping defined on a metric space (X, d). If there exist functions $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\zeta(\alpha(x, y)d(Tx, Ty), M(x, y)) \geq 0 \qquad \text{for all } x, y \in X, \tag{2.1}$$

where $M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$, then we say that T is a *generalized α-admissible $\mathcal{Z}$-contraction of type (A) with respect to ζ.*

Popescu [8] proposed the concept of triangular α-orbital admissible as a refinement of the triangular alpha-admissible notion, defined in [5].

Definition 2.2. [8] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-orbital admissible if

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

Furthermore, T is called a triangular α-orbital admissible if T is α-orbital admissible and

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1.$$

It is clear that each α-admissible (respectively, triangular α-admissible) mapping is an α-orbital admissible (respectively, triangular α-orbital admissible) mapping. For more details and distinctive examples, see e.g. [4, 8].

In what follows we recollect the following lemma for determining whether the given sequence is Cauchy.

Lemma 2.3. (See e.g. [11]) *Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n)$ is nonincreasing and that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:*

$$d(x_{m_k}, x_{n_k}), d(x_{m_k+1}, x_{n_k+1}), d(x_{m_k-1}, x_{n_k}), d(x_{m_k}, x_{n_k-1})$$

Now, we shall state the main results of this paper.

Theorem 2.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be generalized α-admissible $\mathcal{Z}$-contraction of type (A) with respect to ζ. Suppose that*

(*i*) *T is triangular α-orbital admissible;*
(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(*iii*) *T is continuous.*

Then there exists $z \in X$ such that $Tz = z$.

Proof. Choose $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ since the condition (ii) guarantees the existence of such x_0, and define an iterative sequence $\{x_n\}$ in X where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.
If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$, then $u = x_{n_0}$ is a fixed point of T. Consequently, we shall assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$, thus

$$d(x_n, x_{n+1}) > 0, \text{ for all } n \in \mathbb{N}_0.$$

Regarding that T is $\alpha-$admissible, we derive

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Recursively,

$$\alpha(x_n, x_{n+1}) \geq 1, \ n \in \mathbb{N}_0, \tag{2.2}$$

and, since T is triangular α-orbital admissible mapping,

$$\alpha(x_n, x_m) \geq 1, \ n, m \in \mathbb{N}_0, n \neq m. \tag{2.3}$$

From (2.1) and (2.2), it follows that for all $n \geq 1$, we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}), M(x_n, x_{n-1})) \\ &= \zeta(\alpha(x_n, x_{n-1})d(x_{n+1}, x_n), M(x_n, x_{n-1})) \\ &< M(x_n, x_{n-1}) - \alpha(x_n, x_{n-1})d(x_{n+1}, x_n). \end{aligned} \tag{2.4}$$

Therefore,

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) < M(x_{n-1}, x_n), \ n \in \mathbb{N}. \tag{2.5}$$

Observe that

$$\begin{aligned} &M(x_{n-1},x_n)\\ &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}, \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2}\right\}\\ &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}, \frac{d(x_{n-1},x_{n+1})}{2}\right\}\\ &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\}, \end{aligned} \tag{2.6}$$

and combining this inequality with (2.5), it follows that $M(x_{n-1},x_n)=d(x_{n-1},x_n)$. The sequence $\{d(x_n,x_{n+1})\}$ is monotonically decreasing and bounded below with zero. Thus, it is convergent, that is, there is a $L\geq 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1})=L$. Remark that, from (2.5), $\lim_{n\to\infty}\alpha(x_{n-1},x_n)d(x_n,x_{n+1})=L$. We assert that $L=0$. Suppose, on the contrary, that $L>0$. Then, by (ζ_3),

$$\limsup \zeta(\alpha(x_{n-1},x_n)d(x_n,x_{n+1}), d(x_n,x_{n+1}))<0,$$

which contradicts the condition (2.1). Hence, we conclude that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0.$$

We shall prove that $\{x_n\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Thus, there exist $\varepsilon>0$ such that, for any $k\in\mathbb{N}$, there exist $m_k>n_k>k$ and $d(x_{n_k},x_{m_k})\geq\varepsilon$ with an additional condition that m_k is the smallest possible.

By Lemma 2.3, we have:

$$\lim_{n\to\infty} d(x_{n_k},x_{m_k+1})=\lim_{n\to\infty} d(x_{n_k+1},x_{m_k})=\varepsilon$$

and

$$\lim_{n\to\infty} d(x_{n_k+1},x_{m_k+1})=\lim_{n\to\infty}\alpha(x_{n_k},x_{m_k})d(x_{n_k+1},x_{m_k+1})=\varepsilon.$$

From (1.1), from previous observations, it follows

$$\limsup_{n\to\infty} \zeta(\alpha(x_{n_k},x_{m_k})d(x_{n_k+1},x_{m_k+1}), d(x_{n_k},x_{m_k}))<0,$$

which contradicts the condition (2.1). By *reductio ad absurdum*, we conclude that $\{x_n\}$ is a Cauchy sequence.

Since (X,d) is a complete metric space, denote with z the limit of a sequence $\{x_n\}$. The continuity of T implies $Tz=z$. □

Remark 2.5. If u is another fixed point of T, then $\alpha(z,u)<1$ and $\alpha(u,z)<1$.

Theorem 2.6. *Let (X,d) be a complete metric space and let $T:X\to X$ be a generalized α-admissible $\mathcal{Z}$-contraction of type (A) with respect to ζ. Suppose that*

(*i*) *T is triangular α-orbital admissible;*

(*ii*) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;*

(*iii*) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x) \geq 1$ *for all* k.

Then there exists $z \in X$ *such that* $Tz = z$.

Proof. Choose $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define a sequence $\{x_n\}$ where $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. Assume also that $x_n \neq x_{n+1}$, $n \in \mathbb{N}_0$. Since the only difference between this theorem and Theorem 2.4 is a condition (*iii*), from the proof of Theorem 2.4 we get that the sequence $\{x_n\}$ converges to some $z \in X$. From (2.2) and condition (*iii*), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, z) \geq 1$, $k \in \mathbb{N}$. Applying (2.1), for all $k \in \mathbb{N}$, we get that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n_k}, z)d(Tx_{n_k}, Tz), M(x_{n_k}, z)) \\ &= \zeta(\alpha(x_{n_k}, z)d(x_{n_k+1}, Tz), M(x_{n_k}, z)) \\ &< M(x_{n_k}, z) - \alpha(x_{n_k}, z)d(x_{n_k+1}, Tz), \end{aligned}$$

for

$$M(x_{n_k}, z) = \max\left\{d(x_{n_k}, z), \frac{d(x_{n_k}, x_{n_k+1}) + d(z, Tz)}{2}, \frac{d(x_{n_k}, Tz) + d(x_{n_k+1}, z)}{2}\right\},$$

and

$$0 \leq d(x_{n_k+1}, Tz) \leq \alpha(x_{n_k}, z)d(x_{n_k+1}, Tz) < M(x_{n_k}, z).$$

Letting $k \to \infty$, we have $d(z, Tz) = 0$, i.e., $Tz = z$. □

Due to previously made remark, for the uniqueness of a fixed point of a α-admissible $\mathcal{Z}$-contraction with respect to ζ, we shall suggest the following hypothesis.

(U) For all $x, y \in X$, if $\alpha(x, y) < 1$, then $\alpha(x, x_0) \geq 1$ and $\alpha(y, x_0) \geq 1$.

Theorem 2.7. *Adding condition* (U) *to the hypotheses of Theorem 2.4 (resp. Theorem 2.6), we obtain that* z *is the unique fixed point of* T.

Proof. Let us assume that the sequence $\{x_n\}$, defined as in proof of Theorem 2.4, converges to a fixed point z and that $Ty = y$ for some $y \in X \setminus \{z\}$.
Previous remark allows us to assume that $\alpha(z, y) < 1$. Otherwise, since $d(z, y) \neq 0$,

$$d(z, y) \leq \alpha(z, y)d(z, y) < M(z, y) = \max\{d(z, y), 0\} = d(z, y),$$

leads to a contradiction.
Thus,$\alpha(y, x_0) \geq 1$ and $\alpha(y, x_n) \geq 1$, $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} d(y, x_{n+1}) &\leq \alpha(y, x_n)d(y, x_{n+1}) < M(y, x_n) \\ &= \max\left\{d(y, x_n), \frac{d(x_n, x_{n+1})}{2}, \frac{d(y, x_n) + d(y, x_{n+1})}{2}\right\}. \end{aligned}$$

Observe that $y \neq x_n$ since we have assumed that y is a fixed point of T, but $x_n \neq x_{n+1}$, so strong inequality holds due to contractive condition (2.1). Assuming existence of a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$, such that $M(y, x_{n_k}) = d(x_{n_k}, x_{n_k+1})$, we get that

$d(y,z) = \lim_{n\to\infty} d(y, x_{n_k}) = 0$, but $z \neq y$. Thus we may assume that, starting from some $n_0 \in \mathbb{N}$,

$$M(y,x_n) = \max\left\{d(y,x_n), \frac{d(y,x_n)+d(y,x_{n+1})}{2}\right\}.$$

If $M(y,x_n) = \dfrac{d(y,x_n)+d(y,x_{n+1})}{2}$, then

$$d(y,x_n) \leq M(x_n,y) = d(y,x_{n+1} < d(y,x_n),$$

so

$$M(y,x_n) = d(y,x_n),\ n \geq n_0.$$

Therefore,

$$d(y,z) = \lim_{n\to\infty} \alpha(y,x_n)d(y,x_{n+1}) = \lim_{n\to\infty} M(y,x_n),$$

and, by (ζ_3),

$$\zeta(\alpha(y,x_n)d(y,x_{n+1}), M(y,x_n)) < 0.$$

However, this inequality contradicts to (2.1), so our assumption is incorrect, z is a unique fixed point of T. $\square$

Definition 2.8. Let T be a self-mapping defined on a metric space (X,d). If there exist $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to [0,\infty)$ such that

$$\zeta(\alpha(x,y)d(Tx,Ty), N(x,y)) \geq 0 \qquad \text{for all } x,y \in X, \tag{2.7}$$

where $N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$. Then, we say that T is a *generalized α-admissible $\mathcal{Z}$-contraction of type **(B)** with respect to ζ.*

Theorem 2.9. *Let (X,d) be a complete metric space and let $T : X \to X$ be generalized α-admissible $\mathcal{Z}$-contraction of type (B) with respect to ζ. Suppose that*

(i) T is triangular α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.

Then there exists $z \in X$ such that $Tz = z$.

Proof. Analogously as in the proof of Theorem 2.4, (2.2) and (2.3) could be obtained, and similarly as in (2.4), it follows

$$d(x_n,x_{n+1}) \leq \alpha(x_{n-1},x_n)d(x_n,x_{n+1}) < N(x_{n-1},x_n),\ n \in \mathbb{N}. \tag{2.8}$$

Observe that

$$\begin{aligned} N(x_{n-1},x_n) &= \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1})\} & (2.9)\\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}, & (2.10)\end{aligned}$$

and combining this inequality with (2.8), it follows that

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n).$$

The sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing and bounded below with zero, thus convergent and, similarly as in the proof of Theorem 2.4, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let us assume that the sequence $\{x_n\}$ isn't a Cauchy sequence, i.e.,

$$\lim_{n,m\to\infty} d(x_n, x_m) \neq 0$$

and choose $\varepsilon > 0$ such that for any $k \in \mathbb{N}$ there exist $m_k \geq n_k \geq k$ and $d(x_{n_k}, x_{m_k}) \geq \varepsilon$ with an additional condition that m_k is the smallest possible. Easily follows that

$$\lim_{n\to\infty} d(x_{n_k}, x_{m_k}) = \lim_{n\to\infty} d(x_{n_k}, x_{m_k+1}) = \lim_{n\to\infty} d(x_{n_k+1}, x_{m_k}) = \varepsilon.$$

Based on (2.1), we have

$$\begin{aligned} d(x_{n_k}, x_{m_k}) - d(x_{n_k}, x_{n_k+1}) - d(x_{m_k}, x_{m_k+1}) &\leq d(x_{n_k+1}, x_{m_k+1}) \\ &\leq \alpha(x_{n_k}, x_{m_k}) d(x_{n_k+1}, x_{m_k+1}) \\ &< N(x_{n_k}, x_{m_k}), \end{aligned}$$

where

$$N(x_{n_k}, x_{m_k}) = \max\left\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1})\right\}.$$

Choose $k_0 \in \mathbb{N}$ such that $d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}) < \varepsilon$, for any $k \geq k_0$, so, for such k,

$$N(x_{n_k}, x_{m_k}) = d(x_{n_k}, x_{m_k}),$$

and $\lim_{n\to\infty} N(x_{n_k}, x_{m_k}) = \varepsilon$.

Moreover, $\lim_{n\to\infty} \alpha(x_{n_k}, x_{m_k}) d(x_{n_k+1}, x_{m_k+1}) = \varepsilon$ and, by (1.1),

$$\limsup_{n\to\infty} \zeta(\alpha(x_{n_k}, x_{m_k}) d(x_{n_k+1}, x_{m_k+1}), d(x_{n_k}, x_{m_k})) < 0,$$

that conflicts condition (2.1). Therefore, $\{x_n\}$ is a Cauchy sequence $\lim_{n\to\infty} x_n = z$ for some $z \in X$. The continuity of T implies $Tz = z$. □

Theorem 2.10. *Let (X, d) be a complete metric space and let $T : X \to X$ be a generalized α-admissible $\mathcal{Z}$-contraction of type (B) with respect to ζ. Suppose that*

(*i*) *T is triangular α-orbital admissible;*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(*iii*) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.*

Then there exists $z \in X$ such that $Tz = z$.

Proof. For $x_0 \in X$ determined by (*ii*) define an iterative sequence $\{x_n\}$, $x_n = Tx_{n_1}$, $n \in \mathbb{N}$. Evidently, the sequence $\{x_n\}$ is a Cauchy sequence and it converges to some $z \in X$. Then, (2.2) and (*iii*) guarantee existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, z) \geq 1$, $k \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n_k}, z) d(Tx_{n_k}, Tz), N(x_{n_k}, z)) \\ &= \zeta(\alpha(x_{n_k}, z) d(x_{n_k+1}, Tz), N(x_{n_k}, z)) \\ &< N(x_{n_k}, z) - \alpha(x_{n_k}, z) d(x_{n_k+1}, Tz), \end{aligned}$$

which is equivalent to

$$d(x_{n_k+1}, Tz) = d(Tx_{n(k)}, Tz) \leq \alpha(x_{n_k}, z)d(Tx_{n(k)}, Tz) < N(x_{n_k}, z),$$

where

$$N(x_{n_k}, z) = \max\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), d(z, Tz)\}.$$

Letting $k \to \infty$, we get $d(z, Tz) = 0$, i.e., $Tz = z$. □

Theorem 2.11. *Adding condition* (U) *to the hypotheses of Theorem 2.9 (resp. Theorem 2.10), we obtain that* z *is the unique fixed point of* T.

Proof. Since $N(y, x_n) = \{d(y, x_n), d(x_n, x_{n+1})\}$, the proof goes analogously as for Theorem 2.7. □

Observe that generalized α-admissible $\mathcal{Z}$-contractions of type (A) and (B) imply another type of generalized α-admissible $\mathcal{Z}$-contractions with respect to ζ.

Definition 2.12. Let T be a self-mapping defined on a metric space (X, d). If there exist $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\zeta(\alpha(x, y)d(Tx, Ty), K(x, y)) \geq 0, \text{ for all } x, y \in X, \tag{2.11}$$

where

$$K(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$

then we say that T is a *generalized α-admissible $\mathcal{Z}$-contraction of type **(C)** with respect to ζ*.

Combining Theorems 2.4 and 2.9, respectively Theorems 2.6 and 2.10, we obtain following results.

Theorem 2.13. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be generalized* α*-admissible* $\mathcal{Z}$*-contraction of type (C) with respect to* ζ. *Suppose that*

(i) *T is triangular α-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then there exists $z \in X$ such that $Tz = z$.

Theorem 2.14. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a generalized* α*-admissible* $\mathcal{Z}$*-contraction of type (C) with respect to* ζ. *Suppose that*

(i) *T is triangular α-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$, $k \in \mathbb{N}$.*

Then there exists $z \in X$ such that $Tz = z$.

Remark 2.15. Accordingly to the previous observations, by adding a condition (U) to the hypotheses of Theorem 2.13 and Theorem 2.14, uniqueness of the fixed point is obtained.

3. Consequences

In this section, we shall illustrate that several existing fixed point results in the literature can be derived from our main results by observing Example 1.2.

If $\psi \in \Psi$ and we define

$$\zeta_E(t,s) = \psi(s) - t \qquad \text{for all } s,t \in [0,\infty),$$

then ζ_{BW} is a simulation function (cf. Example 1.2 (v)).

We conclude that the main result of Karapınar and Samet [6] can be expressed as a corollary of our main result.

Theorem 3.1. *Theorem 1.9 is a consequence of Theorem 2.7.*

Proof. Taking $\zeta_E(t,s) = \psi(s) - t$ for all $s,t \in [0,\infty)$ in Theorem 2.7, we get that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)), \text{ for all } x,y \in X. \qquad \square$$

Theorem 3.2. *Theorem 2.8. of* [7] *is a consequence of Theorem 2.4.*

Hence, all consequences, including the famous fixed point theorem of Banach, can be expressed easily from the above theorem as in [6]. By changing the definition of $M(x,y)$ in (2.1), we get another class of contractions.

Competing interests. The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions. All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

References

[1] H.H. Alsulami, E. Karapınar, F. Khojasteh, A.F. Roldán-López-de-Hierro, *A proposal to the study of contractions in quasi-metric spaces*, Discrete Dynamics in Nature and Society 2014, Article ID 269286, 10 pages.

[2] H. Argoubi, B. Samet, C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order*, J. Nonlinear Sci. Appl., **8**(2015), 1082–1094.

[3] R.M. Bianchini, M. Grandolfi, *Transformazioni di tipo contracttivo generalizzato in uno spazio metrico*, Atti Acad. Naz. Lincei, VII. Ser., Rend., Cl. Sci. Fis. Mat. Natur., **45**(1968), 212-216.

[4] E. Karapınar, H.H. Alsulami, M. Noorwali, *Some extensions for Geragthy type contractive mappings*, J. Inequalities Appl., **303**(2015).

[5] E. Karapınar, P. Kuman, P. Salimi, *On $\alpha - \psi$-Meri-Keeler contractive mappings*, Fixed Point Theory Appl., **94**(2013).

[6] E. Karapınar, B. Samet, *Generalized α-ψ-contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal. (2012), Article ID 793486.

[7] F. Khojasteh, S. Shukla, S. Radenović, *A new approach to the study of fixed point theorems via simulation functions*, Filomat, **29**(6)(2015), 1189-1194.

[8] O. Popescu, *Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces*, Fixed Point Theory Appl., **190**(2014).

[9] P.D. Proinov, *A generalization of the Banach contraction principle with high order of convergence of successive approximations*, Nonlinear Anal., **67**(2007), 2361-2369.

[10] P.D. Proinov, *New general convergence theory for iterative processes and its applications to Newton Kantorovich type theorems*, J. Complexity, **26**(2010), 3-42.

[11] S. Radenović, Z. Kadelburg, D. Jandrlić, A. Jandrlić, *Some results on weak contraction maps*, Bull. Iranian Math. Society, **38**(2012), no. 3, 625-645.

[12] A.F. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro, J. Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math., **275**(2015), 345-355.

[13] I.A. Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001.

[14] B. Samet, *Best proximity point results in partially ordered metric spaces via simulation functions*, Fixed Point Theory Appl., **232**(2015).

[15] B. Samet, C. Vetro, P. Vetro, *Fixed point theorem for $\alpha-\psi$ contractive type mappings*, Nonlinear Anal., **75**(2012), 2154-2165.

Received: February 25, 2016; Accepted: April 10, 2016.