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EXISTENCE OF SOLUTIONS FOR A CLASS OF FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH "MAXIMA"

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Abstract. We study a boundary value problem associated to a second-order differential inclusion with "maxima". Several existence results are obtained by using suitable fixed point theorems when the right hand side has convex or non convex values.
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1. Introduction

Differential equations with maximum have proved to be strong tools in the modelling of many physical problems: systems with automatic regulation, problems in control theory that correspond to the maximal deviation of the regulated quantity etc.. As a consequence there was an intensive development of the theory of differential equations with "maxima" [1], [6], [7], [8], [10], [12], [13], [14], [16], etc..

A classical example is the one of an electric generator ([1], [15]). In this case the mechanism becomes active when the maximum voltage variation is reached in an interval of time. The equation describing the action of the regulator has the form

$$x'(t) = ax(t) + b \max_{s\in[t-h,t]} x(s) + f(t),$$

where a, b are constants given by the system, $x(.)$ is the voltage and $f(.)$ is a perturbation given by the change of voltage.

This paper is devoted to the study of second-order functional differential inclusions of the form

$$x''(t) \in F(t, x(t), \max_{s\in[t-h_1,t]} x(s), \max_{s\in[t,t+h_2]} x(s)) \quad a.e. \ ([a, b]), \tag{1.1}$$

with "boundary conditions" of mixed type

$$x(t) = \alpha(t), \quad t \in [a - h_1, a], \quad x(t) = \beta(t), \quad t \in [b, b + h_2], \tag{1.2}$$

where $h_1, h_2 > 0$ are given, $\alpha(.) : [a-h_1, a] \to \mathbf{R}$, $\beta(.) : [b, b+h_2] \to \mathbf{R}$ are continuous mappings and $F : [a,b] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

The aim of the present paper is to extend the study in [14], where problem (1.1)-(1.2) is considered with F single-valued, to the set-valued framework and to present some existence results for problem (1.1)-(1.2). Our results are essentially based on a nonlinear alternative of Leray-Schauder type, on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Covitz and Nadler set-valued contraction principle. The methods used are known in the theory of differential inclusions, however their exposition in the framework of problems with "maxima" is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space with the corresponding norm $|.|$ and let $I \subset \mathbf{R}$ be a compact interval. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x : I \to X$ endowed with the norm $|x|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x : I \to X$ endowed with the norm $|x|_1 = \int_I |x(t)| \mathrm{d}t$.

A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

Consider $T : X \to \mathcal{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for $T(.)$ if $x \in T(x)$. $T(.)$ is said to be bounded on bounded sets if $T(B) := \cup_{x \in B} T(x)$ is a bounded subset of X for all bounded sets B in X. $T(.)$ is said to be compact if $T(B)$ is relatively compact for any bounded sets B in X. $T(.)$ is said to be totally compact if $\overline{T(X)}$ is a compact subset of X. $T(.)$ is said to be upper semicontinuous if for any $x_0 \in X$, $T(x_0)$ is a nonempty closed subset of X and if for each open set D of X containing $T(x_0)$ there exists an open neighborhood V_0 of x_0 such that $T(V_0) \subset D$. Let E a Banach space, $Y \subset E$ a nonempty closed subset and $T(.) : Y \to \mathcal{P}(E)$ a multifunction with nonempty closed values. $T(.)$ is said to be lower semicontinuous if for any open subset $D \subset E$, the set $\{y \in Y; T(y) \cap D \neq \emptyset\}$ is open. $T(.)$ is called completely continuous if it is upper semicontinuous and totally compact on X.

It is well known that a compact set-valued map $T(.)$ with nonempty compact values is upper semicontinuous if and only if $T(.)$ has a closed graph.

We recall the following nonlinear alternative of Leray-Schauder type and its consequences.

Theorem 2.1. [11] *Let D and $\overline{D}$ be open and closed subsets in a normed linear space X such that $0 \in D$ and let $T : \overline{D} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.3. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

i) the equation $x = T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

We recall that a multifunction $T : X \to \mathcal{P}(X)$ is said to be lower semicontinuous if for any closed subset $C \subset X$, the subset $\{s \in X : T(s) \subset C\}$ is closed.

If $F : [a, b] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map with compact values and $x \in C([a - h_1, b + h_2], \mathbf{R})$ we define

$$\begin{aligned} &S_F(x) := \{f \in L^1([a, b], \mathbf{R}) : \\ &f(t) \in F(t, x(t), \max_{s\in[t-h_1,t]} x(s), \max_{s\in[t,t+h_2]} x(s)) \quad a.e.\ ([a, b])\}. \end{aligned}$$

We say that F is of *lower semicontinuous type* if $S_F(.)$ is lower semicontinuous with closed and decomposable values.

Theorem 2.4. [2] *Let S be a separable metric space and $G : S \to \mathcal{P}(L^1(I, \mathbf{R}))$ be a lower semicontinuous set-valued map with closed decomposable values.*

Then G has a continuous selection (i.e., there exists a continuous mapping $g : S \to L^1(I, \mathbf{R})$ such that $g(s) \in G(s) \quad \forall s \in S$).

A set-valued map $G : I \to \mathcal{P}(\mathbf{R}^n)$ with nonempty compact convex values is said to be *measurable* if for any $x \in \mathbf{R}^n$ the function $t \to d(x, G(t))$ is measurable.

A set-valued map $F : I \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^m)$ is said to be *Carathéodory* if $t \to F(t, x)$ is measurable for all $x \in \mathbf{R}^n$ and $x \to F(t, x)$ is upper semicontinuous for almost all $t \in I$.

F is said to be L^1-*Carathéodory* if for any $l > 0$ there exists $h_l \in L^1(I, \mathbf{R})$ such that $\sup\{|v| : v \in F(t, x)\} \le h_l(t)$ a.e. I, $\forall x \in \overline{B_l(0)}$.

Theorem 2.5. [9] *Let X be a Banach space, let $F : I \times X \to \mathcal{P}(X)$ be a L^1-Carathéodory set-valued map with $S_F \neq \emptyset$ and let $\Gamma : L^1(I, X) \to C(I, X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_F : C(I, X) \to \mathcal{P}(C(I, X))$ defined by

$$(\Gamma \circ S_F)(x) = \Gamma(S_F(x))$$

has compact convex values and has a closed graph in $C(I, X) \times C(I, X)$.

Note that if $dim X < \infty$, and F is as in Theorem 2.5, then $S_F(x) \neq \emptyset$ for any $x \in C(I, X)$ (e.g., [9]).

Consider a set valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that

$$d_H(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in X.$$

The set-valued contraction principle [4] states that if X is complete, and $T : X \to \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then T has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$.

Let $I(.) : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ a set-valued map with compact convex values defined by $I(t) = [a(t), b(t)]$, where $a(.), b(.) : \mathbf{R} \to \mathbf{R}$ are continuous functions with $a(t) \leq b(t)$ $\forall t \in \mathbf{R}$. For $x(.) : \mathbf{R} \to \mathbf{R}$ continuous we define

$$(\max_I)(t) = \max_{s \in I(t)} x(s).$$

$\max\limits_I : C(\mathbf{R}, \mathbf{R}) \to C(\mathbf{R}, \mathbf{R})$ is an operator whose properties are summarized in the next lemma proved in [14].

Lemma 2.6. *If $x(.), y(.) \in C(\mathbf{R}, \mathbf{R})$, then one has*

i) $|\max\limits_{s \in I(t)} x(s) - \max\limits_{s \in I(t)} y(s)| \leq \max\limits_{s \in I(t)} |x(s) - y(s)| \ \forall t \in \mathbf{R}$.

ii) $\max\limits_{t \in K} |\max\limits_{s \in I(t)} x(s) - \max\limits_{s \in I(t)} y(s)| \leq \max\limits_{s \in \cup_{t \in K} I(t)} |x(s) - y(s)| \ \forall t \in \mathbf{R}$.

Remark 2.7. We recall that if $f(.) \in L^1([a, b], \mathbf{R})$ then the solution

$$x(.) \in C([a - h_1, b + h_2], \mathbf{R}) \cap C^2([a, b], \mathbf{R})$$

of problem $x'' = f(t) \quad t \in [a, b]$ with boundary conditions (1.2) is given by

$$x(t) = \begin{cases} \alpha(t), \quad t \in [a - h_1, a], \\ P(t) - \displaystyle\int_a^b G(t, s) f(s) ds, \quad t \in [a, b], \\ \beta(t), \quad t \in [b, b + h_2], \end{cases}$$

where

$$P(t) = \frac{t - a}{b - a} \beta(b) + \frac{b - t}{b - a} \alpha(a)$$

and

$$G(t, s) := \begin{cases} \dfrac{(s - a)(b - t)}{b - a}, \quad \text{if } s \leq t, \\ \dfrac{(t - a)(b - s)}{b - a}, \quad \text{if } s \geq t. \end{cases}$$

Obviously, $|P(t)| \leq ||\alpha|| + ||\beta||$, where

$$||\alpha|| := \max\{|\alpha(\theta)|; \ \theta \in [a - h_1, a]\}, \quad ||\beta|| := \max\{|\beta(\theta)|; \ \theta \in [b, b + h_2]\}$$

and

$$G(t, s) \leq \frac{b - a}{4}, \ \forall t, s \in I.$$

3. The main results

In what follows $I = [a, b]$ and the Banach space $C([a - h_1, b + h_2], \mathbf{R})$ is endowed with Chebyshev norm $||x(.)|| = \sup_{t \in [a-h_1, b+h_2]} |x(t)|$.

We are able now to present the existence results for problem (1.1)-(1.2). We consider first the case when F is convex valued.

Hypothesis 3.1. i) $F : I \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.

ii) There exist $\varphi \in L^1(I, \mathbf{R})$ with $\varphi(t) > 0$ a.e. I and there exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$\sup\{|v|, \ \ v \in F(t, x, y, z)\} \leq \varphi(t)\psi(\max\{|x|, |y|, |z|\}) \quad a.e.\ I, \ \forall x, y, z \in \mathbf{R}.$$

Theorem 3.2. *Assume that Hypothesis 3.1 is satisfied and there exists $r > 0$ such that*

$$r > ||\alpha|| + ||\beta|| + \frac{b-a}{4}|\varphi|_1 \psi(r). \tag{3.1}$$

Then problem (1.1)-(1.2) has at least one solution x such that $||x|| < r$.

Proof. Let $X = C([a - h_1, b + h_2], \mathbf{R})$ and consider $r > 0$ as in (3.1). It is obvious that the existence of solutions to problem (1.1)-(1.2) reduces to the existence of the solutions of the integral inclusion

$$\begin{aligned}
&x(t) = \alpha(t), \quad t \in [a - h_1, a],\\
&x(t) \in P(t) - \int_a^b G(t, s)F(s, x(s), \max_{\sigma \in [s-h_1, s]} x(\sigma), \max_{\sigma \in [s, s+h_2]} x(\sigma))ds, t \in I\\
&x(t) = \beta(t), \quad t \in [b, b + h_2]
\end{aligned} \tag{3.2}$$

Consider the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(C(I, \mathbf{R}))$ defined by

$$T(x) := \{v \in C(I, \mathbf{R});\ v(t) = P(t) - \int_a^b G(t, s)f(s)ds, \quad f \in S_F(x)\}. \tag{3.3}$$

We show that T satisfies the hypotheses of Corollary 2.2.

First, we show that $T(x) \subset C(I, \mathbf{R})$ is convex for any $x \in C(I, \mathbf{R})$. If $v_1, v_2 \in T(x)$ then there exist $f_1, f_2 \in S_F(x)$ such that for any $t \in I$ one has

$$v_i(t) = P(t) - \int_a^b G(t, s)f_i(s)ds, \quad i = 1, 2.$$

Let $0 \leq \alpha \leq 1$. Then for any $t \in I$ we have

$$(\alpha v_1 + (1 - \alpha)v_2)(t) = P(t) - \int_a^t G(t, s)[\alpha f_1(s) + (1 - \alpha)f_2(s)]ds.$$

The values of F are convex, thus $S_F(x)$ is a convex set and hence

$$\alpha v_1 + (1 - \alpha)v_2 \in T(x).$$

Secondly, we show that T is bounded on bounded sets of $C(I, \mathbf{R})$. Let $B \subset C(I, \mathbf{R})$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \leq m \ \forall x \in B$. If $v \in T(x)$

there exists $f \in S_F(x)$ such that $v(t) = P(t) - \int_a^b G(t,s)f(s)ds$. One may write for any $t \in I$

$$\begin{aligned} |v(t)| &\leq ||\alpha|| + ||\beta|| + \frac{b-a}{4}\int_a^b |f(s)|ds \\ &\leq ||\alpha|| + ||\beta|| + \frac{b-a}{4}\int_a^b \varphi(s)\psi(\max\{|x(s)|, |\max_{\sigma\in[s-h_1,s]} x(\sigma)|, |\max_{\sigma\in[s,s+h_2]} x(\sigma)|\})ds \\ &\leq ||\alpha|| + ||\beta|| + \frac{b-a}{4}\int_a^b \varphi(s)\psi(\max\{|x(s)|, \max_{\sigma\in[s-h_1,s]} |x(\sigma)|, \max_{\sigma\in[s,s+h_2]} |x(\sigma)|\})ds \\ &\leq ||\alpha|| + ||\beta|| + \frac{b-a}{4}\int_a^b \varphi(s)\psi(|x|_C))ds \leq ||\alpha|| + ||\beta|| + \frac{b-a}{4}|\varphi|_1\psi(m). \end{aligned}$$

and therefore

$$|v|_C \leq ||\alpha|| + ||\beta|| + \frac{b-a}{4}|\varphi|_1\psi(m)$$

$\forall v \in T(x)$, i.e., $T(B)$ is bounded.

We show next that T maps bounded sets into equi-continuous sets.
Let $B \subset C(I, \mathbf{R})$ be a bounded set as before and $v \in T(x)$ for some $x \in B$.
There exists $f \in S_F(x)$ such that

$$v(t) = P(t) - \int_a^b G(t,s)f(s)ds.$$

Then for any $t, \tau \in I$ we have

$$\begin{aligned} |v(t) - v(\tau)| &\leq |\int_a^b G(t,s)f(s)ds - \int_a^b G(\tau,s)f(s)ds| \\ &\leq \int_a^b |G(t,s) - G(\tau,s)|\varphi(s)\psi(m)ds. \end{aligned}$$

It follows that $|v(t) - v(\tau)| \to 0$ as $t \to \tau$. Therefore, $T(B)$ is an equi-continuous set in $C(I, \mathbf{R})$. We apply now Arzela-Ascoli's theorem we deduce that T is completely continuous on $C(I, \mathbf{R})$.

In the next step of the proof we prove that T has a closed graph. Let $x_n \in C(I, \mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in T(x_n)$ $\forall n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in T(x^*)$. Since $v_n \in T(x_n)$, there exists $f_n \in S_F(x_n)$ such that

$$v_n(t) = P(t) - \int_a^b G(t,s)f_n(s)ds.$$

Define $\Gamma : L^1(I, \mathbf{R}) \to C(I, \mathbf{R})$ by

$$(\Gamma(f))(t) := -\int_a^b G(t,s)f(s)ds.$$

One has $|v_n(.) - P(.) - (v^*(.) - P(.))|_C = |v_n - v^*|_C \to 0$ as $n \to \infty$.

We apply Theorem 2.5 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of

$f^* \in S_F(x^*)$ such that

$$v^*(t) - P(t) = -\int_a^b G(t,s)f^*(s)ds.$$

Therefore, T is upper semicontinuous and compact on $\overline{B_r(0)}$.

We apply Corollary 2.2 to deduce that either i) the inclusion $x \in T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Assume that ii) is true. With the same arguments as in the second step of our proof we get

$$r = |x|_C \leq ||\alpha|| + ||\beta|| + \frac{b-a}{4}|\varphi|_1\psi(r)$$

which contradicts (3.1). Hence only i) is valid and theorem is proved.

We consider now the case when F is not necessarily convex valued. Our first existence result in this case is based on the Leray-Schauder alternative for single valued maps and on Bressan Colombo selection theorem.

Hypothesis 3.3. i) $F : I \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has compact values, F is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ measurable and $(x,y,z) \to F(t,x,y,z)$ is lower semicontinuous for almost all $t \in I$.

ii) There exist $\varphi \in L^1(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. I and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|,\ v \in F(t,x,y,z)\} \leq \varphi(t)\psi(\max\{|x|,|y|,|z|\}) \quad a.e.\ I, \quad \forall x,y \in \mathbf{R}.$$

Theorem 3.4. *Assume that Hypothesis 3.3 is satisfied and there exists $r > 0$ such that condition (3.1) is satisfied.*

Then problem (1.1)-(1.2) has at least one solution.

Proof. We note first that if Hypothesis 3.3 is satisfied then F is of lower semicontinuous type (e.g., [5]). Therefore, we apply Theorem 2.4 to deduce that there exists $f : C(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ such that $f(x) \in S_F(x)$ $\forall x \in C(I,\mathbf{R})$.

We consider the corresponding problem

$$\begin{aligned}
&x(t) = \alpha(t), \quad t \in [a-h_1,a],\\
&x(t) \in P(t) - \int_a^b G(t,s)f(x(s))ds, \quad t \in I, \qquad (3.4)\\
&x(t) = \beta(t), \quad t \in [b,b+h_2]
\end{aligned}$$

It is clear that if $x \in C([a-h_1,b+h_2],\mathbf{R})$ is a solution of the problem (3.4) then x is a solution to problem (1.1)-(1.2).

Let $r > 0$ that satisfies condition (3.1) and define the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(C(I,\mathbf{R}))$ by

$$(T(x))(t) := P(t) - \int_a^b G(t,s)f(x(s))ds.$$

Obviously, the integral equation (3.4) is equivalent with the operator equation

$$x(t) = (T(x))(t), \quad t \in I. \qquad (3.5)$$

It remains to show that T satisfies the hypotheses of Corollary 2.3.

We show that T is continuous on $\overline{B_r(0)}$. From Hypotheses 3.3. ii) we have

$$|f(x(t))| \leq \varphi(t)\psi(\max\{|x(t)|, |\max_{s\in[t-h_1,t]} x(s)|, |\max_{s\in[t,t+h_2]} x(s)|\}) \quad a.e.\ (I)$$

for all $x \in C(I, \mathbf{R})$. Let $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then

$$|f(x_n(t))| \leq \varphi(t)\psi(r) \quad a.e.\ (I).$$

From Lebesgue's dominated convergence theorem and the continuity of f we obtain, for all $t \in I$

$$\begin{aligned}\lim_{n\to\infty}(T(x_n))(t) &= P(t) - \lim_{n\to\infty}\int_a^b G(t,s)f(x_n(s))ds \\ &= P(t) - \int_a^b G(t,s)f(x(s))ds = (T(x))(t),\end{aligned}$$

i.e., T is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.2 with corresponding modifications it follows that T is compact on $\overline{B_r(0)}$. We apply Corollary 2.3 and we find that either i) the equation $x = T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

As in the proof of Theorem 3.2 if the statement ii) holds true, then we obtain a contradiction to (3.1). Thus only the statement i) is true and problem (1.1) has a solution with $|x|_C < r$.

In order to obtain an existence result for problem (1.1)-(1.2) by using the set-valued contraction principle we introduce the following hypothesis on F.

Hypothesis 3.5. i) $F : I \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty compact values, is integrably bounded and for every $x, y, z \in \mathbf{R}$, $F(., x, y, z)$ is measurable.

ii) There exists $l_1, l_2, l_3 \in L^1(I, \mathbf{R}_+)$ such that for almost all $t \in I$,

$$d_H(F(t, x_1, y_1, z_1), F(t, x_2, y_2, z_2)) \leq l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2| + l_3(t)|z_1 - z_2|$$

$\forall\ x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbf{R}$.

iii) There exists $l \in L^1(I, \mathbf{R}_+)$ such that for almost all $t \in I$, $d(0, F(t, 0, 0, 0)) \leq l(t)$.

Theorem 3.6. *Assume that Hypothesis 3.5. is satisfied and*

$$\frac{b-a}{4}(|l_1|_1 + |l_2|_1 + |l_3|_1) < 1.$$

Then problem (1.1)-(1.2) has a solution.

Proof. We transform the problem (1.1)-(1.2) into a fixed point problem. Consider the set-valued map $T : C([a - h_1, b + h_2], \mathbf{R}) \to \mathcal{P}(C([a - h_1, b + h_2], \mathbf{R}))$ defined by

$$T(x) := \begin{cases} x(t), & \text{if } t \in [a - h_1, a], \\ x(t) \in P(t) - \displaystyle\int_a^b G(t,s)f(s)ds, \quad f \in S_F(x), & \text{if } t \in I, \\ x(t), & \text{if } t \in [b, b + h_2]. \end{cases}$$

Since the set-valued map $t \to F(t, x(t), \max_{s\in[t-h_1,t]} x(s), \max_{s\in[t,t+h_2]} x(s))$ is measurable with the measurable selection theorem (e.g., Theorem III. 6 in [3]) it admits a measurable selection $f : I \to \mathbf{R}$. Moreover, since F is integrably bounded, $f \in L^1(I, \mathbf{R})$. Therefore, $S_F(x) \neq \emptyset$.

It is clear that the fixed points of T are solutions of problem (1.1)-(1.2). We shall prove that T fulfills the assumptions of Covitz Nadler contraction principle.

First, we note that since $S_F(x) \neq \emptyset$, $T(x) \neq \emptyset$ for any $x \in C([a-h_1, b+h_2], \mathbf{R})$.

Secondly, we prove that $T(x)$ is closed for any $x \in C([a-h_1, b+h_2], \mathbf{R})$. Let $\{x_n\}_{n\geq 0} \in T(x)$ such that $x_n \to x^*$ in $C([a-h_1, b+h_2], \mathbf{R})$.
Then $x^* \in C([a-h_1, b+h_2], \mathbf{R})$ and there exists $f_n \in S_F(x)$ such that

$$x_n(t) = P(t) - \int_a^b G(t,s) f_n(s) ds, \quad t \in I.$$

Since F has compact values and Hypothesis 3.5 is satisfied we may pass to a subsequence (if necessary) to get that f_n converges to $f \in L^1(I, \mathbf{R})$ in $L^1(I, \mathbf{R})$. In particular, $f \in S_F(x)$ and for any $t \in I$ we have

$$x_n(t) \to x^*(t) = P(t) - \int_a^b G(t,s) f(s) ds,$$

i.e., $x^* \in T(x)$ and $T(x)$ is closed.

Finally, we show that T is a contraction on $C([a-h_1, b+h_2], \mathbf{R})$. Let $x_1, x_2 \in C([a-h_1, b+h_2], \mathbf{R})$ and $v_1 \in T(x_1)$. Then there exist $f_1 \in S_F(x_1)$ such that

$$v_1(t) = P(t) - \int_a^b G(t,s) f_1(s) ds, \quad t \in I.$$

Consider the set-valued map

$$\begin{aligned} H(t) := F(t, x_2(t), &\max_{s\in[t-h_1,t]} x_2(s), \max_{s\in[t,t+h_2]} x_2(s)) \cap \{x \in \mathbf{R};\ |f_1(t) - x| \\ &\leq l_1(t)|x_1(t) - x_2(t)| + l_2(t)| \max_{s\in[t-h_1,t]} x_1(s) \\ &- \max_{s\in[t-h_1,t]} x_2(s)| + l_3(t)| \max_{s\in[t,t+h_2]} x_1(s) - \max_{s\in[t,t+h_2]} x_2(s)|\}, \quad t \in I. \end{aligned}$$

From Hypothesis 3.5 one has

$$\begin{aligned} d_H(F(t, x_1(t), &\max_{s\in[t-h,t]} x_1(s), \max_{s\in[t,t+h_2]} x_1(s)), F(t, x_2(t), \max_{s\in[t-h,t]} x_2(s)|, \max_{s\in[t,t+h_2]} x_2(s))) \\ &\leq l_1(t)|x_1(t) - x_2(t)| + l_2(t)| \max_{s\in[t-h,t]} x_1(s) \\ &- \max_{s\in[t-h,t]} x_2(s)| + l_3(t)| \max_{s\in[t,t+h_2]} x_1(s) - \max_{s\in[t,t+h_2]} x_2(s)|, \end{aligned}$$

hence H has nonempty closed values. Moreover, since H is measurable, there exists f_2 a measurable selection of H. It follows that $f_2 \in S_F(x_2)$ and for any $t \in I$

$$\begin{aligned} |f_1(t) - f_2(t)| &\leq l_1(t)|x_1(t) - x_2(t)| + l_2(t)| \max_{s\in[t-h_1,t]} x_1(s) \\ &- \max_{s\in[t-h_1,t]} x_2(s)| + l_3(t)| \max_{s\in[t,t+h_2]} x_1(s) - \max_{s\in[t,t+h_2]} x_2(s)| \end{aligned}$$

Define

$$v_2(t) = P(t) - \int_a^b G(t,s) f_2(s) ds, \quad t \in I$$

Using Lemma 2.6 we have

$$\begin{aligned}
|v_1(t) - v_2(t)| &\leq \frac{b-a}{4} \int_a^t |f_1(s) - f_2(s)| ds \\
&\leq \int_a^t [l_1(s)|x_1(s) - x_2(s) + l_2(s)| \max_{s \in [t-h_1,t]} x_1(s) \\
&- \max_{s \in [t-h_1,t]} x_2(s)| + l_3(s)| \max_{s \in [t,t+h_2]} x_1(s) - \max_{s \in [t,t+h_2]} x_2(s)|] ds \\
&\leq \frac{b-a}{4} (|l_1|_1 + |l_2|_1 + |l_3|_1) |x_1 - x_2|_C.
\end{aligned}$$

So,

$$|v_1 - v_2|_C \leq \frac{b-a}{4} (|l_1|_1 + |l_2|_1 + |l_3|_1) |x_1 - x_2|_C.$$

From an analogous reasoning by interchanging the roles of x_1 and x_2 it follows

$$d_H(T(x_1), T(x_2)) \leq \frac{b-a}{4} (|l_1|_1 + |l_2|_1 + |l_3|_1) |x_1 - x_2|_C.$$

Therefore, T admits a fixed point which is a solution to problem (1.1)-(1.2).

Remark 3.7. If F is single-valued, then Theorem 3.6 yields Theorem 4.1 in [14].

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