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α-TYPE FUZZY $\mathcal{H}$-CONTRACTIVE MAPPINGS IN FUZZY METRIC SPACES

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Abstract. We introduce a new concept of α-fuzzy $\mathcal{H}$-contractive mapping which is essentially weaker than the class of fuzzy contractive mapping and stronger than the concept of α-ϕ-fuzzy contractive mapping. For this type of contractions, the existence and uniqueness of fixed point in fuzzy M-complete metric spaces is also established.

Key Words and Phrases: Contractive mapping, fixed point, t-norm, fuzzy metric space, strong fuzzy metric space.
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1. Introduction

The concept of fuzzy metric spaces was obtained in different ways [9, 12, 29]. To obtain a Hausdorff topology of fuzzy metric spaces, [13, 14, 18, 19] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [31] and showed that every ordinary metric induces a fuzzy metric in the sense of George and Veeramani. Later, several authors have made investigation about various topological aspect as well as variety of applications of this metric. For instance, [19]-[21], [23]-[26], [38, 42].

The fixed point theory of the fuzzy metric spaces was started by Grabiec [17], where a fuzzy metric (in the sense of Kramosil and Michalek [31]) version of the Banach contraction principle was proved. (However, it is important to note that no method is available to obtain metric Banach contraction from Grabiec fuzzy contraction). In 2002, Gregori and Sapena [18], introduced the notion of fuzzy contractive mappings and established Banach contraction theorem in various classes of complete fuzzy metric spaces in the sense of George and Veeramani [13], Kramosil and Michalek

[31] and Grabiec [17]. Soon after, Mihet [32], proposed a fixed point theorem for a (weak) Banach contraction in M-complete fuzzy metric space. In this direction, Mihet [34] further extended the fixed point theory for contractive mappings in fuzzy metric spaces besides introducing some new types such as: Edelstein fuzzy contractive mappings, fuzzy contractive mappings of (ϵ-λ) type, fuzzy ψ-contractive mappings etc. For more references on the development of fixed point theory in fuzzy metric spaces, see [2]-[8], [10, 11], [15]-[18], [27, 28], [32]-[39], [41], [43]-[47].

Recently, Wardowski [45] introduced a new concept of a fuzzy $\mathcal{H}$-contractive mapping and established some interesting fixed point theorems for such contraction (see also [22]). On the other hand, inspired from [39], Gopal et al. [16], proposed the notion of α-ϕ-fuzzy contractive mapping and proved some interesting fixed point theorems in G-complete fuzzy metric spaces in the sense of Grabiec [17]. In continuation of these work we propose the notion of α-fuzzy $\mathcal{H}$-contractive mapping and establish some fixed point results for such mappings. Our work extend, generalize and improve several corresponding results given in the literature.

2. Preliminaries

Consistent with Wardowski [45] and Gregori and Minana [22], the following definitions and results will be needed in the sequel.

Definition 2.1. (Schweizer and Sklar [40]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (t-norm) if the following conditions hold:

(i) $a * 1 = a,\ a \in [0,1]$,
(ii) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d,\ a, b, c, d \in [0,1]$,
(iii) $*$ is associative and commutative.

Three basic examples of continuous t-norms are:

$$a * b = \min\{a, b\}, \quad a * b = ab, \quad a * b = \max\{a + b - 1, 0\}$$

(Lukasiewicz t-norm, we will denote it by $*_L$). For $a_1, a_2, ..., a_n \in [0,1]$ and $n \in \mathbb{N}$, the product $a_1 * a_2 * \cdots * a_n$ will be denoted by $\prod_{i=1}^{n} a_i$.

A t-norm $*$ is said to be positive, if $a * b > 0,\ a, b \in (0,1]$.

A t-norm is said to be nilpotent, if $a * b$ is continuous and for each $a \in (0,1)$ there exists $n \in \mathbb{N}$ such that $\prod_{i=1}^{n} a_i = 0$. For example, consider the Lukasiewicz t-norm for which we have $a * a * \cdots * a = 0,\ a \in (0,1)$. For the details concerning t-norms we also refer [30].

In the present paper, we will use the following definitions of fuzzy metric spaces.

Definition 2.2. (George and Veeramani [13]). A triple $(X, M, *)$ is called a fuzzy metric space if X is a Lukasiewicz nonempty set, $*$ is a continuous t-norm and $M : X^2 \times (0, \infty) \to [0,1]$ is a fuzzy set satisfying the following conditions:

(GV1) $M(x, y, t) > 0,\ x, y \in X,\ t > 0$,
(GV2) $M(x, y, t) = 1,\ t > 0,\ \Longleftrightarrow\ x = y$,
(GV3) $M(x, y, t) = M(y, x, t),\ x, y \in X,\ t > 0$,
(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s),\ x, y, z \in X,\ t, s > 0$,
(GV5) $M(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous for every $x, y \in X$.

If in the above definition, the triangular inequality (GV4) is replaced by the following condition:

$$\text{(SFM): } M(x,z,t) \geq M(x,y,t) * M(y,z,t),\ x,y,z \in X,\ t>0,$$

then the triple $(X, M, *)$ is called strong fuzzy metric space [21].

Definition 2.3. ([13, 17, 42]). Let $(X, M, *)$ be a fuzzy metric space. Then

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ converge to $x \in X$ (i.e. $\lim_{n\to+\infty} x_n = x$), if

$$\lim_{n\to+\infty} M(x_n, x, t) = 1,\ t>0.$$

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called M-Cauchy if for each $\varepsilon \in (0,1)$ and $t>0$ there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > 1-\varepsilon,\ m,n \geq n_0.$$

(iii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called G-Cauchy if

$$\lim_{n\to+\infty} M(x_n, x_{n+m}, t) = 1,\ m \in \mathbb{N}, t>0.$$

A fuzzy metric space $(X, M, *)$ is called M-complete (G-complete) if every M-Cauchy (G-Cauchy) sequence is convergent.

Definition 2.4. (Gregori and Sapena [18]). Let $(X, M, *)$ be a fuzzy metric space. $T : X \to X$ is called a fuzzy contractive mapping if there exist $k \in (0,1)$ such that:

$$\left(\frac{1}{M(Tx,Ty,t)} - 1\right) \leq k\left(\frac{1}{M(x,y,t)} - 1\right),\ x,y \in X,\ t>0. \tag{2.1}$$

Then, k is called the contractive constant of T.

Definition 2.5. (Mihet [34, 35]). Let Ψ be the class of all mappings $\psi : (0,1] \to (0,1]$ such that ψ is continuous, nondecreasing and $\psi(t) > t$ for all $t \in (0,1)$. Let $\psi \in \Psi$. A mapping $T : X \to X$ is said to be fuzzy ψ-contractive mapping if:

$$M(Tx,Ty,t) \geq \psi(M(x,y,t)),\ x,y \in X,\ t>0. \tag{2.2}$$

Definition 2.6. (Wardowski [45]). Denoted by $\mathcal{H}$ the family of mappings $\eta : (0,1] \to [0,\infty)$ satisfying the following two conditions:

(H1) η transforms $(0,1]$ onto $[0,\infty)$;

(H2) η is strictly decreasing.

Note that *(H1)* and *(H2)* implies that $\eta(1) = 0$.

Definition 2.7. Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if there exists $k \in (0,1)$ satisfying the following condition:

$$\eta\left(M(Tx,Ty,t)\right) \leq k\eta(M(x,y,t)),\ x,y \in X,\ t>0. \tag{2.3}$$

Note that for a mapping $\eta \in \mathcal{H}$ of the form $\eta(t) = \frac{1}{t} - 1$, $t \in (0,1]$, Definition 2.7 reduces to Definition 2.4.

Remark 2.8. It has been shown in [22] that the class of fuzzy $\mathcal{H}$-contractive mappings are included in the class of ψ-contractive mappings.

Proposition 2.9. *Let $(X, M, *)$ be a fuzzy metric space and let $\eta \in \mathcal{H}$. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is M-Cauchy if and only if for every $\varepsilon > 0$ and $t > 0$ there exist $n_0 \in \mathbb{N}$ such that:*

$$\eta\left(M(x_m, x_n, t)\right) < \varepsilon,\ m, n \geq n_0.$$

Proposition 2.10. *Let $(X, M, *)$ be a fuzzy metric space and let $\eta \in \mathcal{H}$. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is convergent to $x \in X$ if and only if $\lim_{n\to\infty} \eta(M(x_n, x, t)) = 0$, $t > 0$.*

By Φ is denoted the family of all right continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(r) < r$, $r > 0$.

Definition 2.11. (Gopal and Vetro [16]). Let $(X, M, *)$ be a fuzzy metric space. $T : X \to X$ is called an α-ϕ-fuzzy contractive mapping if there exists two functions $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ and $\phi \in \Phi$ such that

$$\alpha(x, y, t)\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \phi\left(\frac{1}{M(x, y, t)} - 1\right),\ x, y \in X,\ t > 0. \tag{2.4}$$

Remark 2.12. If $\alpha(x, y, t) = 1$, $x, y \in X$, $t > 0$, and for some $k \in (0, 1)$ is $\phi(r) = kr$, $r > 0$, then Definition 2.11 reduces to the Definition 2.4 but converse is not necessarily true (see [16]).

Theorem 2.13. *(Wardowski [45]). Let $(X, M, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a fuzzy $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ such that*

(a) $\prod_{i=1}^{k} M(x, Tx, t_i) \neq 0$, $x \in X$, $k \in \mathbb{N}$ *and any sequence* $\{t_i\} \subset (0, \infty), t_i \searrow 0$;
(b) $r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$, $r, s \in \{M(x, Tx, t) : x \in X, t > 0\}$;
(c) $\eta(M(x, Tx, t_i) : i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $\{t_i\}_{n\in\mathbb{N}} \subset (0, \infty), t_i \searrow 0$.

Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

3. Fixed point

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space. We say that $f : X \to X$ is an α-fuzzy-$\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ if there exists a function $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ such that

$$\alpha(x, y, t)\eta\left(M(fx, fy, t)\right) \leq k\eta\left(M(x, y, t)\right),\ x, y \in X,\ t > 0. \tag{3.1}$$

Remark 3.2. If $\alpha(x, y, t) = 1, x, y \in X, t > 0$, then Definition 3.1 reduces to the Definition 2.7 but converse is not necessarily true (see Example 3.5 given bellow).

Definition 3.3. Let $(X, M, *)$ be a fuzzy metric space. We say that $f : X \to X$ is α-admissible if there exists a function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ such that:

$$\alpha(x, y, t) \geq 1,\ x, y \in X,\ t > 0 \Longrightarrow \alpha(fx, fy, t) \geq 1,\ x, y \in X,\ t > 0.$$

Now, we are ready to state and prove our first result.

Theorem 3.4. *Let $(X, M, *)$ be a M-complete fuzzy metric space, where $*$ is positive. Let $f : X \to X$ be an α-fuzzy-$\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ satisfying the following conditions:*

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0, t) \geq 1,\ t > 0,$
(ii) f *is* α*-admissible,*
(iii) $\eta(r * s) \leq \eta(r) + \eta(s),\ r, s \in (0, 1],$
(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}, t) \geq 1$, $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = x$, *then* $\alpha(x_n, x, t) \geq 1$, $n \in \mathbb{N}$, $t > 0$.

Then, f *has a fixed point* $x^* \in X$. *Moreover, the sequence* $\{f^n x_0\}_{n\in N}$ *converges to* x^*.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$, $t > 0$. Define the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_{n+1} = fx_n$, $n \in \mathbb{N} \cup \{0\}$. If $x_{n+1} = x_n$, for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of f.

So, assume that $x_n \neq x_{n+1}$, $n \in \mathbb{N}$. Since f is α-admissible, we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, fx_0, t) \geq 1,\ t > 0 \Longrightarrow \alpha(fx_0, fx_1, t) = \alpha(x_1, x_2, t) \geq 1,\ t > 0.$$

By induction, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1,\ n \in \mathbb{N},\ t > 0. \tag{3.2}$$

Now, applying (3.1) and using (3.2), we obtain

$$\begin{aligned}
\eta\left(M(x_{n+1}, x_{n+2}, t)\right) &= \eta\left(M(fx_n, f_{n+1}, t)\right) \\
&\leq \alpha(x_n, x_{n+1}, t)\eta\left(M(fx_n, f_{n+1}, t)\right) \\
&\leq k\eta\left(M(x_n, x_{n+1}, t)\right) \\
&\leq k\alpha(x_{n-1}, x_n, t)\eta\left(M(fx_{n-1}, fx_n, t)\right) \\
&\leq kk\eta\left(M(x_{n-2}, x_{n-1}, t)\right) \\
&\leq \cdots\cdots\cdots \\
&\leq k^{n+1}\eta\left(M(x_0, x_1, t)\right),\ t > 0.
\end{aligned}$$

Since $k \in (0, 1)$ and η is strictly decreasing we have that

$$\eta\left(M(x_{n+1}, x_{n+2}, t)\right) < \eta\left(M(x_0, x_1, t)\right),\ t > 0,$$

and

$$M(x_{n+1}, x_{n+2}, t) \geq M(x_0, x_1, t) > 0,\ n \in N,\ t > 0. \tag{3.3}$$

Now, let us consider any $m, n \in N$, $m < n$, and let $\{a_i\}_{i\in N}$ be a strictly decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i = 1$. From (GV-4), (GV-2) and positivity of $*$, we get

$$M(x_m, x_n, t) \geq M\left(x_m, x_m, t - \sum_{i=m}^{n-1} a_i t\right) * M\left(x_m, x_n, \sum_{i=m}^{n-1} a_i t\right) = M\left(x_m, x_n, \sum_{i=m}^{n-1} a_i t\right)$$

$$\geq M\left(x_m, x_{m+1}, a_m t\right) * M\left(x_{m+1}, x_{n+2}, a_{m+1} t\right) * \cdots * M\left(x_{n-1}, x_n, a_{n-1} t\right),\ t > 0.$$

By condition *(iii)* and (3.3) we get

$$\eta\left(M(x_m, x_n, t)\right) \leq \eta\left(\prod_{i=m}^{n-1} M(x_i, x_{i+1}, a_i t)\right)$$

$$\leq \sum_{i=m}^{n-1} \eta\left(M(x_i, x_{i+1}, a_i t)\right) \leq \sum_{i=m}^{n-1} k^i \eta\left(M(x_0, x_1, t)\right), \ m, n \in N, \ m < n, \ t > 0.$$

The above sum is finite, i.e. for any $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that:

$$\eta\left(M(x_m, x_n, t)\right) \leq \sum_{i=m}^{n-1} k^i \eta\left(M(x_0, x_1, t)\right) < \varepsilon, \ m, n \geq n_0, m < n, \ t > 0.$$

Thus, by Proposition 2.9 follows that $\{x_n\}_{n\in\mathbb{N}}$ is a M-Cauchy sequence in X. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Due to Proposition 2.10,

$$\lim_{n\to\infty} \eta(M(x_n, x^*, t)) = 0, \ t > 0.$$

Now, applying (iv) and (3.1), we obtain

$$\eta\left(M(x_{n+1}, fx^*, t)\right) = \eta\left(M(fx_n, fx^*, t)\right)$$
$$\leq \alpha(x_n, x^*, t)\eta\left(M(fx_n, fx^*, t)\right) \leq k\eta\left(M(x_n, x^*, t)\right), \ t > 0,$$

which implies that

$$\lim_{n\to\infty} \eta\left(M(x_{n+1}, fx^*, t)\right) = 0, \ t > 0,$$

i.e.

$$fx^* = \lim_{n\to\infty} x_{n+1} = x^*.$$

So, x^* is a fixed point of f.

The following examples shows the usefulness of our work.

Example 3.5. Let $X = \mathbb{R}$, $a * b = \min\{a, b\}$, $a, b \in [0, 1]$ and

$$M(x, y, t) = \frac{t}{t + |x - y|}, \ x, y \in X, \ t > 0.$$

Clearly, $(X, M, *)$ is a M-complete fuzzy metric space.

Define the mapping $f : X \to X$ by

$$f(x) = \begin{cases} \frac{x^2}{4}, & \text{if } x \in [0, 1], \\ 2, & \text{otherwise.} \end{cases}$$

Also, define $\eta(s) = \frac{1}{s} - 1$, $s \in (0, 1]$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, f is an α-fuzzy-$\mathcal{H}$-contractive mapping with $k = \frac{1}{2}$.

Now, let $x, y \in X$ such that $\alpha(x, y, t) \geq 1, t > 0$, this implies that $x, y \in [0, 1]$ and by the definitions of f and α, we have

$$f(x) = \frac{x^2}{4} \in [0, 1], \ f(y) = \frac{y^2}{4} \in [0, 1] \text{ and } \alpha(fx, fy, t) = 1, \ t > 0,$$

i.e., f is α-admissible. Further, there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$, $t > 0$, indeed for any $x_0 \in [0, 1]$, we have $\alpha(x_0, fx_0, t) = 1$, $t > 0$. Finally, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. By the definition of the function α, it follows that $x_n \in [0,1]$, $n \in \mathbb{N}$ and hence $x \in [0,1]$. Therefore, $\alpha(x_n, x, t) = 1$, $n \in \mathbb{N}$. So, all the hypothesis of Theorem 3.4 are satisfied. Here, 0 and 2 are two fixed point of f.

However, f is not a fuzzy $\mathcal{H}$-contractive mapping [45]. To see this, consider $x = 2$ and $y = 1$. Then, since $k \in (0,1)$ we have

$$\eta\left(M(fx, fy, t)\right) = \frac{7}{4t} > \frac{k}{t} = k\eta(M(x, y, t)),\ t > 0.$$

Now, we give a sufficient condition to obtain the uniqueness of the fixed point in the previous theorem. Precisely, we consider the following hypothesis.

(U): for all $x, y \in X$ and for all $t > 0$ there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.

Theorem 3.6. *Adding the condition (U) to the hypothesis of Theorem 3.4, we obtain the uniqueness of the fixed point of f.*

Proof. Suppose that x^* and y^* are two fixed points of f. If $\alpha(x^*, y^*, t) \geq 1$, for some $t > 0$, then by (3.1), we conclude easily that $x^* = y^*$.

Assume $\alpha(x^*, y^*, t) < 1, t > 0$. Then, by (U), there exists $z \in X$ such that

$$\alpha(x^*, z, t) \geq 1 \text{ and } \alpha(y^*, z, t) \geq 1,\ t > 0. \tag{3.4}$$

Since f is α-admissible, and by (3.4), we get

$$\alpha(x^*, f^n z, t) \geq 1 \text{ and } \alpha(y^*, f^n z, t) \geq 1,\ n \in \mathbb{N}, t > 0. \tag{3.5}$$

Now, applying (3.1) and (3.5), we have

$$M(x^*, f^n z, t) = M(fx^*, f(f^{n-1}z), t)$$

and

$$\begin{aligned}\eta\left(M(x^*, f^n z, t)\right) &= \eta\left(M(fx^*, f(f^{n-1}z), t)\right) \leq \alpha(x^*, f^{n-1}z, t)\eta\left(M(fx^*, f(f^{n-1}z), t)\right)\\ &\leq k\eta\left(M(x^*, f^{n-1}z, t)\right) \leq \cdots \leq k^n \eta\left(M(x^*, z, t)\right),\ n \in \mathbb{N}, t > 0.\end{aligned}$$

By letting $n \to \infty$ in last relation we get

$$\lim_{n\to\infty} \eta\left(M(x^*, f^n z, t)\right) = 0,\ t > 0,$$

and

$$\lim_{n\to\infty} f^n z = x^*.$$

Analogous,

$$\lim_{n\to\infty} f^n z = y^*.$$

Finally, the uniqueness of the above limits gives us $x^* = y^*$.

The assumption that $*$ is positive can be further relaxed in Theorem 3.4. In fact, we can prove the following:

Theorem 3.7. *Let $(X, M, *)$ be a M-complete strong fuzzy metric space for a nilpotent t-norm $*_L$, and let $f : X \to X$ be an α-fuzzy-$\mathcal{H}$-contractive mapping with respect to. $\eta \in H$ satisfying the following conditions:*

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0, t) \geq 1,\ t > 0$,

(ii) f *is* α*-admissible,*

(iii) $\eta(r * s) \leq \eta(r) + \eta(s),\ r, s \in \{M(x, fx, t) : x \in X, t > 0\}$,

(iv) *each subsequence* $\{x_{n_k}\}_{k\in\mathbb{N}}$, *of a sequence* $\{x_n\}_{n\in\mathbb{N}} = \{f^n x_0\}_{n\in\mathbb{N}}$, *has a following property*

$$\alpha(x_{n_k}, x_{n_l}, t) \geq 1,\ k, l \in \mathbb{N},\ k > l,\ t > 0;$$

(v) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}, t) \geq 1,\ n \in \mathbb{N}, t > 0$, *and* $\lim_{n\to\infty} x_n = x$, *then* $\alpha(x_n, x, t) \geq 1,\ n \in \mathbb{N}, t > 0$.

Then f *has a fixed point* $x^* \in X$. *Moreover, the sequence* $\{f^n x_0\}_{n\in\mathbb{N}}$ *converges to* x^*.

Proof. Let $x_0 \in X$ and $\alpha(x_0, fx_0, t) \geq 1$, $t > 0$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n = fx_{n-1} = f^n x_0$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of f.

So, assume $x_n \neq x_{n-1}$, $n \in \mathbb{N}$. Since, f is α-admissible, we have

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1,\ t > 0 \Rightarrow \alpha(fx_0, fx_1, t) = \alpha(x_1, x_2, t) \geq 1,\ t > 0.$$

By induction, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1,\ n \in \mathbb{N}, t > 0.$$

By (3.1), we have

$$\begin{aligned}\eta(M(x_1, x_2, t)) &= \eta(M(fx_0, fx_1, t)) \\ &\leq \alpha(x_0, x_1, t)\eta(M(fx_0, fx_1, t)) \\ &\leq k\eta(M(x_0, x_1, t)),\ t > 0.\end{aligned}$$

Inductively,

$$\eta(M(x_n, x_{n+1}, t)) \leq k\eta(M(x_{n-1}, x_n, t)) \leq \cdots \leq k^n \eta(M(x_0, x_1, t)),\ n \in \mathbb{N}, t > 0. \tag{3.6}$$

Since η is strictly decreasing, and $k \in (0, 1)$, we have that

$$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t),\ n \in \mathbb{N}, t > 0.$$

So, for every $t > 0$, sequence $\{M(x_n, x_{n+1}, t)\}_{n\in\mathbb{N}}$ is nondecreasing and bounded, therefore it is convergent, i.e.

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = p,\ t > 0.$$

Let us prove by contradiction that $p = 1$. Suppose that $p < 1$. Letting $n \to \infty$ in (3.6), and since η is continuous, we have

$$\lim_{n\to\infty} \eta(M(x_n, x_{n+1}, t)) \leq k \lim_{n\to\infty} \eta(M(x_{n-1}, x_n, t)),\ t > 0.$$

So, we obtain a contradiction $\eta(p) \leq k\eta(p)$ and conclude that $p = 1$, i.e.

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1,\ t > 0. \tag{3.7}$$

Let us prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose the contrary. Then there exists $\varepsilon > 0, t_0 > 0$ and $s_0 \in \mathbb{N}$, such that:

for each $s \in \mathbb{N}$, $s \geq s_0$, there exist $m(s), n(s) \in \mathbb{N}$, $m(s) > n(s) \geq s$ such that

$$\eta\left(M(x_{m(s)}, x_{n(s)}, t_0)\right) \geq \varepsilon,$$

and, by (iv),

$$\alpha(x_{m(s)-1}, x_{n(s)-1}, t_0) \geq 1.$$

Let, for each s, $m(s)$ be the least positive integer exceeding $n(s)$ satisfying the above property, i.e. $\eta\left(M(x_{m(s)-1}, x_{n(s)}, t_0)\right) < \varepsilon$ and $\eta\left(M(x_{m(s)}, x_{n(s)}, t_0)\right) \geq \varepsilon$, $s \in \mathbb{N}$. Since η is continuous, there exists $0 < \varepsilon_1 < 1$ such that $\eta(\varepsilon_1) = \varepsilon$ i.e.

$$M(x_{m(s)-1}, x_{n(s)}, t_0) > \varepsilon_1,\ s \in \mathbb{N}. \tag{3.8}$$

Then,

$$\begin{aligned}\varepsilon \leq \eta\left(M(x_{m(s)}, x_{n(s)}, t_0)\right) &\leq \alpha(x_{m(s)-1}, x_{n(s)-1}, t_0)\, \eta\left(M(x_{m(s)}, x_{n(s)}, t_0)\right)\\ &\leq k\, \eta\left(M(x_{m(s)-1}, x_{n(s)-1}, t_0)\right),\ s \in \mathbb{N}.\end{aligned} \tag{3.9}$$

Since fuzzy metric is strong we obtain

$$\begin{aligned}M(x_{m(s)-1}, x_{n(s)-1}, t_0) &\geq *_L \left\{M(x_{m(s)-1}, x_{n(s)}, t_0), M(x_{n(s)}, x_{n(s)-1}, t_0)\right\}\\ &= \max\left\{M(x_{m(s)-1}, x_{n(s)}, t_0) + M(x_{n(s)}, x_{n(s)-1}, t_0) - 1, 0\right\},\\ &\quad s \in \mathbb{N}.\end{aligned} \tag{3.10}$$

Take ε_1 defined in (3.8). Then, by (3.7) there exist $s_0 \in \mathbb{N}$ such that

$$M(x_{n(s)}, x_{n(s)-1}, t_0) > 1 - \varepsilon_1,\ s > s_0. \tag{3.11}$$

Now, by (3.8) and (3.11), we get

$$M(x_{m(s)-1}, x_{n(s)}, t_0) + M(x_{n(s)}, x_{n(s)-1}, t_0) > 1,\ s > s_0. \tag{3.12}$$

So, applying (3.9), (3.10), (3.12) and (iii) we get

$$\begin{aligned}\varepsilon &\leq \eta(M(x_{m(s)}, x_{n(s)}, t_0))\\ &\leq k\eta\left(M(x_{m(s)-1}, x_{n(s)-1}, t_0)\right)\\ &\leq k\left[\eta\left(M(x_{m(s)-1}, x_{n(s)}, t)\right) + \eta\left(M(x_{n(s)}, x_{n(s)-1}, t)\right)\right],\ s > s_0.\end{aligned}$$

Letting $s \to \infty$ in the above expression, we get $\varepsilon \leq k\,\varepsilon < \varepsilon$. So, we get a contradiction. Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Rest of the proof follows similar lines to Theorem 3.4.

Remark 3.8. In the paper of Wardowski ([45]) one could find the following open question "Can the condition (a) in Theorem 2.13 be omitted for nilpotent t-norms?". If $\alpha(x, y, t) = 1$, $x, y \in X$, $t > 0$, in Theorem 3.7 a partial answer to this question is obtained. Namely, in narrowed space (strong fuzzy metric space) we could expand the class of t-norms i.e. in that case Theorem 3.7 holds for nilpotent t-norm $* = *_L$.

Open Problem. Can the assumption of strong fuzzy metric in Theorem 3.7 be omitted/further relaxed?

Conflicts of Interest. The authors declare that there is no conflicts of interest regarding the publication of this manuscript.

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