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BEST PROXIMITY POINT THEOREMS FOR NON-SELF PROXIMAL REICH TYPE CONTRACTIONS IN COMPLETE METRIC SPACES

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Abstract. Recall from [2], that a mapping $T : X \mapsto X$ is called a Reich mapping if it satisfies for all $x, y \in X$, $d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y)$, where a,b,c are nonnegative and satisfy $a + b + c < 1$. Alternatively, one could define a Reich mapping as follows: $T : X \mapsto X$ is called a Reich mapping if there exists a nonnegative constant k with $k < \frac{1}{3}$ such that $d(Tx, Ty) \leq k[d(x, Tx)+d(y, Ty)+d(x, y)]$. In the present paper, we address the following: How do we characterize Theorem 3 [2], when T is a non-self map? We show such a characterization is given by Theorem 3.1 or Corollary 3.2 in this paper.
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1. Introduction

Fixed point theorems is concerned with solving equations of the form $Tx = x$, in which T is a self-mapping defined on a subset of some space. Let A and B be two nonempty subsets of say a metric space. If $T : A \mapsto B$ is a non-self map, the fixed point equation $Tx = x$ may not have a solution. In this case we are interested in finding an approximate solution $x \in A$ such that the error $d(x, Tx)$ is a minimum, possibly $d(x, Tx) = d(A, B)$. A point $p \in A$ is called a best proximity point of $T : A \mapsto B$, if $d(p, Tp) = d(A, B)$, where $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. A best proximity point becomes a fixed point if the underlying map is a self-mapping. This paper is organized as follows. In Section 2, some preliminary ideas and remarks that would be useful in the sequel are presented. In Section 3, we present the main results, showing the characterization of the Reich mapping theorem for non-self maps is given by Theorem 3.1 or Corollary 3.2. The concluding remarks also emphasizes this point. The last section contains an open question for further exploration.

2. Preliminaries

Notation 2.1. Let A and B be nonempty subsets of a metric space X

(*a*) $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$

(b) $A_0 = \{x \in A : d(x,y) = d(A,B) \ for \ some \ y \in B\}$
(c) $B_0 = \{y \in B : d(x,y) = d(A,B) \ for \ some \ x \in A\}$

Definition 2.2. Let $S : A \mapsto B$ and $T : B \mapsto A$ be non-self mappings. The pair (S,T) will be said to form a proximal cyclic Reich contraction if there exists a nonnegative number $k < \frac{1}{3}$ such that $d(u,Sx) = d(A,B)$ and $d(v,Ty) = d(A,B)$ implies that $d(u,v) \leq k[d(x,u) + d(y,v) + d(x,y)] + (1-3k)d(A,B)$ for all $x, u \in A$ and $y, v \in B$.

Remark 2.3. If in Definition 2.2, $A = B$ and $S = T$, then the pair (T,T) forms a Reich contraction.

Definition 2.4. A mapping $S : A \mapsto B$ will be called a proximal Reich contraction of the first kind if there exists a nonnegative number $k < \frac{1}{3}$ such that $d(u_1, Sx_1) = d(A,B)$ and $d(u_2, Sx_2) = d(A,B)$ implies that $d(u_1,u_2) \leq k[d(x_1,u_1) + d(x_2,u_2) + d(x_1,x_2)]$ for all $u_1, u_2, x_1, x_2 \in A$.

Remark 2.5. A self-mapping that is a proximal Reich contraction of the first kind is precisely a Reich contraction.

Definition 2.6. A mapping $S : A \mapsto B$ will be called a proximal Reich contraction of the second kind if there exists a nonnegative number $k < \frac{1}{3}$ such that $d(u_1, Sx_1) = d(A,B)$ and $d(u_2, Sx_2) = d(A,B)$ implies that $d(Su_1, Su_2) \leq k[d(Sx_1, Su_1) + d(Sx_2, Su_2) + d(Sx_1, Sx_2)]$ for all $u_1, u_2, x_1, x_2 \in A$.

Remark 2.7. A self-mapping that is a proximal Reich contraction of the second kind is a "Reich contraction".

Definition 2.8. Given a mapping $S : A \mapsto B$ and an isometry $g : A \mapsto A$, the mapping S is said to preserve isometric distance with respect to g if $d(Sgx_1, Sgx_2) = d(Sx_1, Sx_2)$ for all $x_1, x_2 \in A$.

Definition 2.9. An element x in A is said to be a best proximity point of the mapping $S : A \mapsto B$ if it satisfies the condition that $d(x, Sx) = d(A,B)$.

Remark 2.10. If the underlying map in Definition 2.9 is a self-mapping, then the best proximity point reduces to a fixed point.

Definition 2.11. The set B is said to be approximately compact with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x, y_n) \to d(x, B)$ for some x in A has a convergent subsequence.

3. Main results

Theorem 3.1. *Let A and B be non-void closed subsets of a complete metric space such that A_0 and B_0 are non-void. Let $S : A \mapsto B$, $T : B \mapsto A$, and $g : A \cup B \mapsto A \cup B$ satisfy the following conditions:*

(a) *S and T are proximal Reich contractions of the first kind*
(b) *$S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$*
(c) *The pair (S,T) forms a proximal cyclic Reich contraction*
(d) *g is an isometry*
(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$*

Then there exist a unique element x in A and and a unique element y in B satisfying the conditions that

$$d(gx, Sx) = d(A,B)$$

$$d(gy, Ty) = d(A,B)$$

$$d(x,y)=d(A,B).$$

Proof. Let x_0 be an element in A_0. In view of the fact that $S(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, it follows that there exists an element x_1 in A_0 such that $d(gx_1,Sx_0)=d(A,B)$. Again since $S(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, it follows that there exists an element x_2 in A_0 such that $d(gx_2,Sx_1)=d(A,B)$. Continuing, one has $d(gx_{n+1},Sx_n)=d(A,B)$ for all $n\geq 0$, since $S(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$. Since g is an isometry and T is a proximal Reich contraction of the first kind, we have,

$$\begin{aligned}d(x_n,x_{n+1})&=d(gx_n,gx_{n+1})\\&\leq k[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n-1},x_n)]\\&=2kd(x_{n-1},x_n)+kd(x_n,x_{n+1}).\end{aligned}$$

From the above, put $\alpha:=\frac{2k}{1-k}<1$. Then it follows that, $d(x_n,x_{n+1})\leq\alpha d(x_{n-1},x_n)$. Consequently $\{x_n\}$ is Cauchy, and hence converges to some element x in A. Similarly, since $T(B_0)$ is contained in A_0 and B_0 is contained in $g(B_0)$, it follows that there exist a sequence $\{y_n\}$ of elements in B_0 such that $d(gy_{n+1},Ty_n)=d(A,B)$. Since g is an isometry and T is a proximal Reich contraction of the first kind, it follows that $d(y_n,y_{n+1})\leq\alpha d(y_{n-1},y_n)$, where $\alpha:=\frac{2k}{1-k}<1$. Consequently, $\{y_n\}$ is Cauchy and hence converges to some element y in B. Since the pair (S,T) forms a proximal cyclic Reich contraction and g is an isometry, it follows that,

$$\begin{aligned}d(x_{n+1},y_{n+1})&=d(gx_{n+1},gy_{n+1})\\&\leq\alpha d(x_n,y_n)+(1-\alpha)d(A,B)\end{aligned}$$

where $\alpha:=\frac{2k}{1-k}<1$. Now going in the limit of inequality immediately above, we conclude that $d(x,y)=d(A,B)$. It follows that x is a member of A_0 and y is a member of B_0. Since $S(A_0)$ is contained in B_0 and $T(B_0)$ is contained in A_0, there exist an element u in A and an element v in B such that $d(u,Sx)=d(A,B)$ and $d(v,Ty)=d(A,B)$. Since S is a proximal Reich contraction of the first kind, then it follows that $d(u,gx_{n+1})\leq\alpha d(x,x_n)$, where $\alpha:=\frac{2k}{1-k}<1$. Thus in the limit of the inequality $d(u,gx_{n+1})\leq\alpha d(x,x_n)$, we get $u=gx$, and so $d(gx,Sx)=d(A,B)$. Similarly, it can be shown that $v=gy$, and so $d(gy,Ty)=d(A,B)$. Finally, we show uniqueness. We suppose that $d(gx^*,Sx^*)=d(A,B)$ for x^* in A and $d(gy^*,Ty^*)=d(A,B)$ for y^* in B. Since g is an isometry, S and T are proximal Reich contractions of the first kind, it follows that with $\alpha:=\frac{2k}{1-k}<1$, one has,

$$\begin{aligned}d(x,x^*)&=d(gx,gx^*)\\&\leq\alpha d(x,x^*)\\d(y,y^*)&=d(gy,gy^*)\\&\leq\alpha d(y,y^*).\end{aligned}$$

From the above two inequalities we conclude that $x=x^*$ and $y=y^*$.
If g is the identity in the above theorem, then we get the following

Corollary 3.2. *Let A and B be non-void closed subsets of a complete metric space such that A_0 and B_0 are non-void. Let $S : A \mapsto B$, $T : B \mapsto A$ satisfy the following conditions:*

(a) *S and T are proximal Reich contractions of the first kind*
(b) *$S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$*
(c) *The pair (S,T) forms a proximal cyclic Reich contraction.*

Then there exist a unique element x in A and and a unique element y in B satisfying the conditions that

$$d(x,Sx) = d(A,B)$$
$$d(y,Ty) = d(A,B)$$
$$d(x,y) = d(A,B).$$

4. Concluding remarks

If $A = B$ and $S = T$ in Corollary 3.2, then we get Theorem 3 [2]. Alternatively, if $A = B$, $S = T$ and g is the identity in Theorem 3.1 we get Theorem 3 [2].

5. Open question

How do we characterize Theorem 1(a) [1] when T is a non-self map?

References

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