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FIXED POINT THEOREMS FOR THE SUM OF TWO OPERATORS ON UNBOUNDED CONVEX SETS AND AN APPLICATION

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Abstract. In this paper, we establish new fixed point results for the sum of two operators A and B, where the operator A is assumed to be weakly compact and (ws)-compact, while B is a weakly condensing and expansive operator defined on unbounded domains under different boundary conditions as well as other additional assumptions. In addition, we get new generalized forms of the Krasnosel'skii fixed point theorem in a Banach space by using the concept of measure of weak noncompactness of De Blasi. Later on, we give an application to solve a nonlinear Hammerstein integral equation in L^1-space.
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1. Introduction

The existence of fixed points for the sum of two operators has been followed with interest for a long time. In 1958, to study the existence of solutions of nonlinear equations of the form

$$Ax + Bx = x, \ x \in M.$$

Krasnosels'skii [9] first proved that operator $A + B$ has a fixed point whenever M is a nonempty closed convex subset of a Banach space X and the operators A and B satisfy:

(i) A is continuous on M, and $A(M)$ is relatively compact,
(ii) B is a k-contraction with $k \in [0, 1)$,
(iii) $A(M) + B(M) \subset M$.

It is well known that this theorem combines the Schauder's fixed point theorem and Banach contraction mapping. In 1955, Darbo [6] extended the Schauder's fixed point theorem to the setting of noncompact operators introducing the notion of k-set contraction. Since then many interesting works has appeared. For example, in 1967, Sadovskii [13] gave a fixed point result more general than the Darbo theorem using the concept of condensing operator on a closed, bounded and convex subset of a Banach space.

In 1977, De Blasi [5] introduced the concept of measure of weak noncompactness. Emmanuele [7] established a Sadovskii type fixed point result using the concept of weakly condensing with respect to the measure of weak noncompactness, in which the weak continuity of the operator is required. In [4] Burton improved this result requiring instead of $AM + BM \subset M$, the more general condition ($x = Bx + Ay$, $y \in M$), then $x \in M$. Xiang and Yuan [14] considered that the operator B as an expansive rather than a contraction.

Recently, A. Ben Amar and J. Garciat-Falset [2] established a new version of Sadovskii type fixed point theorem for the classes of (ws)-compact operators.

In this paper, on the basis of a fixed point theorem proved by A. Ben Amar and J. Garcia-Falset in [2], we establish new fixed point results for the sum of two operators A and B, where the operator A is assumed to be weakly compact and (ws)-compact, while B is a weakly condensing and expansive operator defined on unbounded domains under different boundary conditions as well as other additional assumptions. In addition, we get new generalized forms of the Krasnosels'skii fixed point in Banach spaces by using the concept of measure of weak noncompactness of De Blasi. In the last section of the paper, we give an application to solve a nonlinear Hammerstein integral equation in L^1-space.

2. Preliminaries

We first gather together some notations and preliminary facts of some weak topology feature which will be need in our further considerations. Let $\mathfrak{B}(X)$ be the collection of all nonempty bounded subsets of a Banach space X, and let $\mathcal{W}(X)$ be the subset of $\mathfrak{B}(X)$ consisting of all weakly compact subsets of X. Also, let $\mathbf{B}_\varepsilon$ denote the closed ball in X centered in 0 and with radius ε.

De Blasi [5] introduced the map $w : \mathfrak{B}(X) \to \mathbb{R}^+$ defined by

$$w(M) = \inf\{\varepsilon > 0,\ M \subset Y + \mathbf{B}_\varepsilon,\ Y \in \mathcal{W}(X)\},\ \text{for } M \in \mathfrak{B}(X).$$

For completeness, we recall some useful properties of the function $w(.)$

Proposition 2.1. Let M_1, M_2 be in $\mathfrak{B}(X)$, then we have:

(1) $w(M_1) \leq w(M_2)$ whenever $M_1 \subset M_2$.
(2) $w(M_1) = 0$ if and only if M_1 is relatively weakly compact.
(3) $w(\overline{M_1}) = w(M_1)$.
(4) $w(conv(M_1)) = w(M_1)$, where $conv(M_1)$ refers to the convex hull of M_1.
(5) $w(\lambda M_1) = |\lambda| w(M_1)$, for all $\lambda \in \mathbb{R}$.
(6) $w(M_1 + M_2) \leq w(M_1) + w(M_2)$.
(7) $w(M_1 \cup M_2) = \max(w(M_1), w(M_2))$.

Next, we introduce the notion of (ws)-compact operators.

Definition 2.1. Let X be a Banach space and let M be a subset of X. A map $T : M \to X$ is said to be (ws)-compact if it is continuous and for any weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in M the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.

Remark 2.1. A map T is (ws)-compact if and only if it maps relatively weakly compact sets into relatively compact ones. T is (ws)-compact does not imply that T is weakly sequentially continuous, i.e., $x_n \rightharpoonup x$ implies $Tx_n \rightharpoonup Tx$; here $\rightharpoonup$ denote weak convergence in X.

Remark 2.2. Clearly, a strongly continuous operator is (ws)-compact. The converse is not true in general(even if X is reflexive), as the following example illustrates. Let $X = L^2(0,1)$ and let $T : X \to X$ be defined by

$$(Tx)(t) = \int_0^1 x^2(s)\mathrm{d}s = \|x\|_2^2.$$

Clearly T is $\|\cdot\|$-continuous and in fact compact since the range of T is $\mathbb{R}$; here, T is (ws)-compact. On the other hand, if $x_n(s) = \sin n\pi s$ then $x_n \rightharpoonup \theta$ in $L^2(0,1)$, but $Tx_n \nrightarrow \theta$ in $L^2(0,1)$ because $\|Tx_n\|_2 = \frac{1}{2}$ for all $n \geq 1$.

Let T be a nonlinear operator from X into itself, we introduce the following condition transform weakly compact sets into weakly compact ones.

(P) If $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ is a weakly convergent sequence in X, then $(Tx_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in X.

Definition 2.2. Let X be a Banach space and K be a subset of X. A continuous and bounded map $T : K \to X$ is w-k-set contractive if for any bounded set $A \subset K$, we have $w(T(A)) \leq kw(A)$, $k \in [0,1]$. T is strictly w-k-set contractive if T is w-k-set contractive and $w(T(A)) < kw(A)$ for all bounded sets $A \subset K$ with $w(A) \neq 0$. T is a w-condensing map if T is strictly w-1-set contractive.

The following theorem was proved by A. Ben Amar and J. Garcia-Falset [2].

Theorem 2.1. *Let M be a nonempty unbounded closed convex set in a Banach space X. Assume that $T : M \to M$ is (ws)-compact. In addition, suppose that $T(M)$ is bounded. If T is w-condensing map, then T has a fixed point.*

Lemma 2.1. *Let $T : D(T) \subset X \to X$ be a (ws)-compact operator and let $Q : R(T) \subset X \to X$ be continuous. Then the compound operator $Q \circ T : D(T) \subset X \to X$ is (ws)-compact.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in $D(T)$. By the hypothesis of T is (ws)-compact, $(Tx_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence, say $(Tx_{n_k})_{k\in\mathbb{N}}$. The continuity of Q implies that $(QTx_{n_k})_{k\in\mathbb{N}}$ is also strongly convergent and therefore $Q \circ T$ is (ws)-compact operator.

Definition 2.3. Let X be a Banach space. An operator $T : D(T) \to X$ is said to be weakly compact if $T(\Omega)$ is relatively weakly compact for every bounded subset $\Omega \subseteq X$.

Definition 2.4. Let (X, d) be a metric space and M be a subset of X. The mapping $T : M \to X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq hd(x, y), \ \forall x, \ y \in M.$$

Lemma 2.2. *Let $(X, \|\cdot\|)$ be a linear normed space and $M \subset X$. Assume that the mapping $T : M \to X$ is expansive with constant $h > 1$.*

Then the inverse of $F := I - T : M \to (I-T)(M)$ exists and

$$\|F^{-1}x - F^{-1}y\| \leq \frac{1}{h-1}\|x-y\|, \quad x,\ y \in F(M). \tag{2.1}$$

Proof. For each $x, y \in M$, we have

$$\|Fx - Fy\| = \|(Tx - Ty) - (x-y)\| \geq (h-1)\|x-y\|, \tag{2.2}$$

which shows that F is one-to-one, hence the inverse of $F : M \to F(M)$ exists. Now taking $x, y \in F(M)$, then $F^{-1}x, F^{-1}y \in M$, thus using $F^{-1}x, F^{-1}y$ substitute for x, y in (2.2), respectively, we obtain

$$\|F^{-1}x - F^{-1}y\| \leq \frac{1}{h-1}\|x-y\|.$$

Theorem 2.2. *Let M be a closed subset of X. Assume that the mapping $T : M \to X$ is expansive and $T(M) \supseteq M$, then there exists a unique point $x^* \in M$ such that $Tx^* = x^*$.*

Theorem 2.3. [2] *Let X be a Banach space, $\Omega \subset X$ a closed convex subset and $U \subset \Omega$ an open set (with respect to the topology of Ω) such that $\theta \in U$. Assume that $T : \overline{U} \to \Omega$ is w-condensing (ws)-compact mapping with $T(\overline{U})$ bounded. Then, either*

(A1) T has a fixed point, or

(A2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u = \lambda T(u)$.

The following fixed point result stated in [10], will be used in the next section. The proof follows from Schauder's fixed point theorem.

Theorem 2.4. *Let M be a nonempty closed convex set in a Banach space X. Assume that $T : M \to M$ is (ws)-compact. If $T(M)$ is relatively weakly compact, then there exists $x \in M$ such that $T(x) = x$.*

3. Fixed point theorems for the sum of two operators

Our purpose here is to establish a fixed point theorem for the sum of two operators defined on unbounded domains under different boundary conditions as well as other additional assumptions.

Theorem 3.1. *Let $K \subset X$ be a nonempty closed unbounded convex subset. Suppose that T and S map K into X such that*

(i) S is (ws)-compact and weakly compact.

(ii) T is expansive and w-condensing.

(iii) $z \in S(K)$ implies $T(K) + z \supset K$, where $T(K) + z = \{y + z / y \in T(K)\}$

(iv) $(I-T)^{-1}S(K)$ is bounded.

Then there exists a point $x^* \in K$ such that $Sx^* + Tx^* = x^*$.

Proof. From (ii) and (iii), for any $z \in S(K)$, we see that the mapping $T + z : K \to X$ satisfies the assumptions of Theorem 2.2.

Thus, the equation

$$Tx + z = x$$

has a unique solution in K. Therefore by Lemma 2.2, we have $x = (I-T)^{-1}z \in K$, which implies that

$$(I-T)^{-1}S(K) \subseteq K.$$

Then we have S is (ws)-compact and $(I-T)^{-1}$ continuous. Lemma 2.1 implies that $(I-T)^{-1}S$ is (ws)-compact. Moreover $(I-T)^{-1}S$ is w-condensing. Indeed, let M be a bounded subset of K, then we have

$$(I-T)^{-1}S(M) \subseteq S(M) + T(I-T)^{-1}S(M) \subseteq S(M) + T(M).$$

This implies that $w((I-T)^{-1}S(M)) \leq w(S(M)+T(M))$. Since S is weakly compact, we deduce that $w(S(M)) = 0$, which means that

$$w((I-T)^{-1}S(M)) \leq w(T(M)) < w(M).$$

By Theorem 2.1 , we conclude that there exists $x \in K$ such that $(I-T)^{-1}S(x) = x$, that is $S(x) = (I-T)(x)$, and hence $S(x) + T(x) = x$. This completes the proof.

Corollary 3.1. *Let $K \subset X$ be a nonempty closed unbounded convex subset. Suppose that T and S map K into X such that*

(i) S is (ws)-compact and weakly compact.

(ii) T is expansive and $T(M)$ is relatively weakly compact whenever M is a bounded set of K.

(iii) $z \in S(K)$ implies $T(K) + z \supset K$, where $T(K) + z = \{y + z/y \in T(K)\}$.

(iv) $(I-T)^{-1}S(K)$ is bounded.

Then there exists a point $x^* \in K$ such that $Sx^* + Tx^* = x^*$.

Proof. This is an immediate consequence of Theorem 3.1, since T is a w-condensing map.

Theorem 3.2. *Let $K \subset X$ be a nonempty closed unbounded convex subset. Suppose that T and S map K into X such that*

(i) S is (ws)-compact and $S(K)$ is relatively weakly compact.

(ii) T is expansive and w-condensing.

(iii) $z \in S(K)$ implies $T(K) + z \supset K$, where $T(K) + z = \{y + z/y \in T(K)\}$.

Then there exists a point $x^* \in K$ such that $Sx^* + Tx^* = x^*$.

Proof. From (ii) and (iii), for any $z \in S(K)$, we see that the mapping $T + z : K \to X$ satisfies the assumptions of Theorem 2.2.

Thus, the equation

$$Tx + z = x$$

has a unique solution in K. Therefore by Lemma 2.2, we have $x = (I-T)^{-1}z \in K$, which implies that

$$(I-T)^{-1}S(K) \subseteq K.$$

Then we have S is (ws)-compact and $(I-T)^{-1}$ continuous. Lemma 2.1 implies that $(I-T)^{-1}S$ is (ws)-compact. Moreover $(I-T)^{-1}S(K)$ is relatively weakly compact. Indeed, we have

$$(I-T)^{-1}S(K) \subseteq S(K) + T(I-T)^{-1}S(K).$$

If $w((I-T)^{-1}S(K)) \neq 0$, we obtain

$$w((I-T)^{-1}S(K)) \leq w(S(K)+T(I-T)^{-1}S(K)) \leq w(S(K)) + w(T((I-T)^{-1}S(K))).$$

Since $S(K)$ is relatively weakly compact, we deduce that $w(S(K)) = 0$, which means that

$$w((I-T)^{-1}S(K)) \leq w(T((I-T)^{-1}S(K))) < w((I-T)^{-1}S(K)),$$

which is a contradiction. So, $w((I-T)^{-1}S(K)) = 0$ and $(I-T)^{-1}S(K)$ is relatively weakly compact.

By Theorem 2.4, we conclude that there exists $x \in K$ such that $(I-T)^{-1}S(x) = x$. This completes the proof.

Theorem 3.3. *Let $K \subset X$ be a nonempty closed unbounded convex subset. Suppose that $T : K \to X$ and $S : K \to X$ such that:*

(i) S is (ws)-compact and weakly compact.

(ii) T is expansive and w-condensing.

(iii) $I - T$ is surjective.

(iv) $(I-T)^{-1}S(K)$ is bounded.

Then there exists a point $x^* \in K$ such that $S(x^*) + T(x^*) = x^*$.

Proof. Lemma 2.2 shows that the inverse of $I - T$ exists. Since $I - T$ is surjective, then $(I-T)^{-1}S$ map K into K. Once we prove that $(I-T)^{-1}S$ has a fixed point in K, the proof is achieved.

Obviously, by Lemma 2.1 the compound operator $(I-T)^{-1}S$ is (ws)-compact since S is (ws)-compact and $(I-T)^{-1}$ is continuous. Now we will show that $(I-T)^{-1}S$ is w-condensing. Indeed, let M be a bounded subset of K, then we have

$$(I-T)^{-1}S(M) \subseteq S(M) + T(I-T)^{-1}S(M) \subseteq S(M) + T(M).$$

Since $\overline{S(M)}^w$ is relatively weakly compact and T is w-condensing, it follows that

$$w((I-T)^{-1}S(M)) \leq w(S(M) + T(M)) < w(M),$$

By Theorem 2.1, $(I-T)^{-1}S$ has a fixed point $x^* \in K$. This finishes the proof.

Remark 3.1. $\overline{U}$ and $\partial_\Omega U$ denote the closure and boundary of U in Ω, respectively.

Theorem 3.4. *Let X be a Banach space, $\Omega \subset X$ a closed convex subset, $U \subset \Omega$ an open(with respect to topology of Ω) and such that $\theta \in U$. Assume that $T : \overline{U} \to \Omega$ is (ws)-compact, w-condensing, $T(\overline{U})$ is bounded and*

$$\|Tx\| \leq \|Tx - x\| \tag{3.1}$$

for each $x \in \partial_\Omega U$. Then T has at least one fixed point in $\overline{U}$.

Proof. We suppose that the operators T has no fixed point in $\overline{U}$. By Theorem 2.3, there exist $x_0 \in \partial_\Omega U$ et $\lambda_0 \in (0,1)$ with $x_0 = \lambda_0 T(x_0)$. That is $T(x_0) = \frac{1}{\lambda_0}x_0$. Inserting $T(x_0) = \frac{1}{\lambda_0}x_0$ into (3.1), we obtain

$$\left\| \frac{1}{\lambda_0}x_0 \right\| \leq \left\| \frac{1}{\lambda_0}x_0 - x_0 \right\|.$$

This implies

$$\frac{1}{\lambda_0}\|x_0\| \leq \left(\frac{1}{\lambda_0} - 1 \right) \|x_0\|. \tag{3.2}$$

Since $x_0 \in \partial_\Omega U$, we see $x_0 \neq 0$. Therefore, $\|x_0\| \neq 0$, by (3.2), we obtain

$$\frac{1}{\lambda_0} - 1 \geq \frac{1}{\lambda_0},$$

and this is a contradiction since $\lambda_0 \in (0,1)$. Accordingly, by Theorem 2.3 T has a fixed point in $\overline{U}$.

Theorem 3.5. *Let Ω be a convex open subset of X such that $\theta \in \Omega$. Suppose that $T : \overline{\Omega} \to X$ and $S : \overline{\Omega} \to X$ such that:*

(i) S is (ws)-compact and weakly compact operator.

(ii) T is an expansive map with constant $h > 1$ and w-condensing.

(iii) $I - T$ is surjective.

(iv) $\|Sx + T\theta\| \leq \frac{(h-1)}{2}\|x\|$ for each $x \in \overline{\Omega}$.

(v) $(I - T)^{-1}S(\overline{\Omega})$ is bounded.

Then there exists a point $x^* \in \overline{\Omega}$ such that $Sx^* + Tx^* = x^*$.

Proof. From the proof of Theorem 3.1, we obtain $(I - T)^{-1}S := GS : \overline{\Omega} \to \overline{\Omega}$ is a (ws)-compact and w-condensing map.

Now for each $x \in \overline{\Omega}$, we see that there exists $y \in \overline{\Omega}$ such that

$$y - T(y) = S(x). \tag{3.3}$$

It is remained to check that(3.1) holds. Indeed, for each $x \in \overline{\Omega}$, from (3.3), we obtain

$$T(GSx) + Sx = GSx$$

which implies that

$$\|T(GSx) - T\theta\| \leq \|GSx\| + \|Sx + T\theta\|. \tag{3.4}$$

On the other hand, we have

$$\|T(GSx) - T\theta\| \geq h\|GSx\|. \tag{3.5}$$

From (3.4) and (3.5), we deduce that

$$\|GSx\| \leq \frac{1}{h-1}\|Sx + T\theta\|. \tag{3.6}$$

For any $x \in \overline{\Omega}$, from (3.6) and (iv), we derive that

$$\|GSx\|^2 - (\|GSx\| - \|x\|)^2 = \|x\|(2\|GSx\| - \|x\|) \leq \|x\| \left(\frac{2}{h-1}\|Sx + T\theta\| - \|x\| \right) \leq 0,$$

which implies (3.1). This completes the proof.

Lemma 3.1. [1] *Let X be a Banach space. Assume that a mapping $B : X \to X$ is a k-Lipshitzian mapping, that is*

$$\forall x, y \in X, \ \|Bx - By\| \leq k\|x - y\|.$$

In addition, suppose that B verifies (P). Then for each bounded subset S of X, we have $w(B(S)) \leq k\ w(S)$.

Next, we extend the above results to a large class of mappings in some sense.

Theorem 3.6. *Let $K \subset X$ be a nonempty closed unbounded convex subset. Suppose that $T : K \to X$ and $S : K \to X$ such that:*

(i) S is a w-k-set contractive and (ws)-compact map.

(ii) T is an expansive map with constant $h > k+1$, $I-T$ is surjective and $(I-T)^{-1}$ satisfies (P).

(iii) $z \in S(K)$ implies $K \subset z + T(K)$, where $T(K) + z = \{y + z | y \in T(K)\}$.

(iv) $(I - T)^{-1}S(K)$ is bounded.

Then there exists a point $x^* \in K$ with $Sx^* + Tx^* = x^*$.

Proof. The result follows immediately from Theorem 2.1 and Theorem 3.1, once we show that $(I-T)^{-1}S : K \to K$ is a w-condensing map. To see this, let M be a bounded subset in K. From Lemma 2.2, we have,

$$\|(I-T)^{-1}(x) - (I-T)^{-1}(y)\| \leq \frac{1}{h-1}\|x-y\|.$$

Since $(I-T)^{-1}$ satisfies (P), then from Lemma 2.2, we have

$$w((I-T)^{-1}(S(M)) \leq \frac{1}{h-1}w(S(M)).$$

Notice that S is w-k-set contractive, consequently if $w(M) \neq 0$,

$$w((I-T)^{-1}S(M)) \leq \frac{k}{h-1}w(M) < w(M),$$

which illustrates that $(I-T)^{-1}S : K \to K$ is a w-condensing map. This finishes the proof.

The next lemma holds easily.

Lemma 3.2. [3] *When $y > 1$ and $\beta > 0$, the following inequality holds:*

$$(y-1)^{\beta+1} < y^{\beta+1} - 1.$$

According to the above lemma, we can prove the following result.

Theorem 3.7. *Let Ω be a closed convex set in a Banach space X, $0 \in int(\Omega)$. Assume that $T : \Omega \to X$ is (ws)-compact and w-condensing mapping which satisfies that $T(\Omega)$ is bounded. In addition, assume that there exists $\gamma > 0$ such that*

$$\|T(x) - x\|^{\gamma+1} \geq \|T(x)\|^{\gamma+1} - \|x\|^{\gamma+1}, \tag{3.7}$$

for all $x \in \partial\Omega$. Then T has at least one fixed point in Ω.

Proof. We suppose that T has no fixed point in Ω. Then there exist $x_0 \in \partial\Omega$ and $\lambda_0 \in (0,1)$ with $\lambda_0 T(x_0) = x_0$. That is $T(x_0) = \frac{1}{\lambda_0}x_0$. Inserting $T(x_0) = \frac{1}{\lambda_0}x_0$ into (3.7), we obtain

$$\left\|\frac{1}{\lambda_0}x_0 - x_0\right\|^{\gamma+1} \geq \left\|\frac{1}{\lambda_0}x_0\right\|^{\gamma+1} - \|x_0\|^{\gamma+1}.$$

This implies

$$\left(\frac{1}{\lambda_0} - 1\right)^{\gamma+1}\|x_0\|^{\gamma+1} \geq \left(\frac{1}{\lambda_0^{\gamma+1}} - 1\right)\|x_0\|^{\gamma+1}. \tag{3.8}$$

Since $x_0 \in \partial\Omega$, we see $x_0 \neq 0$. Therefore, $\|x_0\| \neq 0$ and by (3.8), we obtain

$$\left(\frac{1}{\lambda_0} - 1\right)^{\gamma+1} \geq \frac{1}{\lambda_0^{\gamma+1}} - 1,$$

and this contradicts Lemma 3.2, since $\frac{1}{\lambda_0} \in (1,\infty)$. Accordingly, by Theorem 2.3 T has a fixed point in Ω.

Remark 3.2. Theorem 3.7 is a generalization of Altman's fixed point theorem in the case of (ws)-compact and w-condensing mapping.

Corollary 3.2. *Let Ω be a closed convex set in a Banach space X, $0 \in int(\Omega)$. Assume that $T : \Omega \to X$ is (ws)-compact and w-condensing mapping wich satisfies that $T(\Omega)$ is bounded. If one of the following condition is satisfied:*

(i) $\|T(x)\| \leq \|x\|$, *for all* $x \in \partial\Omega$ *(the condition of Rothe type).*

(ii) $\|x - T(x)\| \geq \|T(x)\|$, *for all* $x \in \partial\Omega$ *(the condition of Petryshyn type).*

(iii) $\|T(x) - x\|^2 \geq \|T(x)\|^2 - \|x\|^2$, *for all* $x \in \partial\Omega$ *(the condition of Altman type).*

Then T *has at least one fixed point in* Ω.

Theorem 3.8. *Let* $K \subset X$ *be a convex open subset. Suppose that* T *and* S *map* $\overline{K}$ *into* X *such that:*

(i) S *is a w-k-set contractive and (ws)-compact map.*

(ii) T *is an expansive map with constant* $h > k + 1$.

(iii) $I - T$ *is surjective and* $(I - T)^{-1}$ *satisfy (P).*

(iv) $\|Sx + T\theta\| \leq \frac{h-1}{2}\|x\|$, $\forall x \in \partial K$.

(v) $(I - T)^{-1}S(\overline{K})$ *is bounded.*

Then there exist $x^* \in \overline{K}$ *such that* $Sx^* + Tx^* = x^*$.

Proof. From the proof of Theorem 3.3, we obtain $GS : (I - T)^{-1}S : \overline{K} \to \overline{K}$ is a (ws)-compact mapping. Moreover $(I - T)^{-1}S$ is w-condensing. Indeed, let A be a bounded subset in $\overline{K}$. By Lemma 3.1 and (2.1), we obtain

$$w((I - T)^{-1}(S(A))) \leq \frac{1}{h-1} w(S(A)).$$

Notice that S is w-k-set contractive, consequently if $w(A) \neq 0$, we have

$$w((I - T)^{-1}(S(A))) \leq \frac{k}{h-1} w(A) < w(A),$$

which implies that $(I - T)^{-1}S := GS : \overline{K} \to \overline{K}$ is w-condensing.

Now for each $x \in \overline{K}$, we see that there exist $y \in \overline{K}$ such that

$$S(x) = y - T(y).$$

In what follows, we check that the conditions of Corollary 3.2 are satisfied. In fact, by (3.6), we have

$$\|GSx\| \leq \frac{1}{h-1}\|Sx + T\theta\|.$$

Since $\|Sx + T\theta\| \leq \frac{h-1}{2}\|x\|$, then we have

$$\|GSx\| \leq \|x\|.$$

Thus from (3.6) and (iv), we derive that

$$\|x - GSx\|^2 - \|GSx\|^2 \geq \|x\| \left(\|x\| - \frac{2}{h-1}\|Sx + T\theta\| \right) \geq 0.$$

Now, we have

$$\|GSx - x\|^2 - \|GSx\|^2 + \|x\|^2 \geq \|x\|^2 + \|GSx\|^2 - 2\|x\|\|GSx\| - \|GSx\|^2 + \|x\|^2 \geq 0.$$

Thus, the result follows from (i) or (ii) or (iii) of Corollary 3.2.

Theorem 3.9. [2] *Let* X *be a Banach space,* Ω *a nonempty unbounded closed convex subset of* X *and* $U \subseteq \Omega$ *an open set and let* z *be an element of* U*. Assume that* $T : \overline{U} \to \Omega$ *is w-condensing and (ws)-compact mapping which satisfies that* $T(\overline{U})$ *is bounded.*

(L-S) $\lambda T(u) \neq u - (1 - \lambda)z$ *for any* $u \in \partial_\Omega U$, $0 < \lambda < 1$.

Then T *has at least one fixed point in* $\overline{U}$.

Theorem 3.10. *Let $\theta \in K \subset X$ be a nonempty unbounded closed convex subset of X, $U \subset K$ an open set and $z \in U$. Suppose that $T : K \to X$ and $S : \overline{U} \to X$ such that:*

(i) S is w-k-set contractive and (ws)-compact map.

(ii) T is an expansive map with constant $h > k+1$.

(iii) $(I-T)$ is surjective and $(I-T)^{-1}$ satisfy (P).

(iv) $(I-T)^{-1}S(\overline{U})$ is bounded.

Additionally, if $Sx \neq (I-T)\left(\frac{1}{\lambda}x - \frac{(1-\lambda)}{\lambda}z\right)$ for all $x \in \partial_K U$, $\lambda > 1$, then there exists $x^* \in \overline{U}$ with $Sx^* + Tx^* = x^*$.

Proof. From the proof of Theorem 3.3, we obtain $(I-T)^{-1}S : \overline{U} \to K$ is a (ws)-compact mapping. Now, let M be a bounded subset in $\overline{U}$, then from Lemma 3.1 we have

$$w((I-T)^{-1}S(M)) \leq \frac{1}{h-1} w(S(M)).$$

Since S is w-k-set contractive map, then

$$w((I-T)^{-1}S(M)) \leq \frac{k}{h-1} w(M) < w(M) \quad (if \quad w(M) \neq 0),$$

which implies that $(I-T)^{-1}S : \overline{U} \to K$ is a w-condensing map. It is easy to see that the condition(L-S) holds. The result follows from Theorem 3.9.

Lemma 3.3. *Let $(X, \|.\|)$ be a linear normed space, $M \subset X$. Assume that the mapping $T : M \to X$ is contractive with constant $\alpha < 1$, then the inverse of $F := I - T : M \to (I-T)(M)$ exists and*

$$\|F^{-1}x - F^{-1}y\| \leq \frac{1}{1-\alpha}\|x-y\|, \quad x, \ y \in F(M). \tag{3.9}$$

Proof. For each $x, \ y \in M$, we have

$$\|Fx - Fy\| \geq (1-\alpha)\|x-y\|,$$

which shows that F is one-to-one, thus the inverse of $F : M \to F(M)$ exists. Now we set

$$G := F^{-1} - I : F(M) \to X.$$

From the identity

$$I = F \circ F^{-1} = (I-T) \circ (I+G) = I + G - T \circ (I+G),$$

We obtain that

$$G = T \circ (I+G).$$

Hence,

$$\|Gx - Gy\| \leq \alpha(\|x-y\| + \|Gx - Gy\|).$$

Therefore

$$\|Gx - Gy\| \leq \frac{\alpha}{1-\alpha}\|x-y\|, \quad x, y \in F(M)$$

and so

$$\|F^{-1}x - F^{-1}y\| \leq \|Gx - Gy\| + \|x-y\| \leq \frac{1}{1-\alpha}\|x-y\|.$$

The classical Krasnosel'skii fixed-point theorem can be easily extended due to Sadovskii fixed point theorem. For the purpose of completeness, we include the generalized Krasnosel'skii fixed point theorem as follows.

Theorem 3.11. *Let $\theta \in K \subset X$ be a nonempty closed convex subset and $U \subseteq K$ an open set and let z be an element of U. Suppose that $T : K \to X$ and $S : \overline{U} \to X$ such that:*

(i) T is a contraction with constant $\alpha < 1$.

(ii) S is a strictly w-(1-α)−set contractive and (ws)-compact map.

(iii) $(I - T)^{-1}$ satisfy (P) and $I - T$ is surjective.

(iv) $(I - T)^{-1}S(\overline{U})$ bounded.

Additionally, if $Sx \neq (I - T)\left(\frac{1}{\lambda}x - \frac{1-\lambda}{\lambda}z\right)$, for all $x \in \partial_K U$, $\lambda > 1$. Then there exist $x^* \in \overline{U}$ such that $Sx^* + Tx^* = x^*$.

Proof. By Lemma 3.3, we have the inverse of $I - T$ exists. Then, since $I - T$ is surjective, we have $(I - T)^{-1}S$ map $\overline{U}$ into K. Again by Lemma 2.1, one can easily know that the mapping $(I-T)^{-1}S$ is (ws)-compact. Now, let M be a bounded subset of $\overline{U}$, then by Lemma 3.3 and (i), we have

$$\|(I - T)^{-1}x - (I - T)^{-1}y\| \leq \frac{1}{1 - \alpha}\|x - y\|, \; x, y \in (I - T)^{-1}(K).$$

Since $(I - T)^{-1}$ satisfy (P) and S is a strictly w-(1-α)-set contractive map, then Lemma 3.1 implies that

$$w((I - T)^{-1}S(M)) \leq \frac{1}{1 - \alpha}w(S(M)) < w(M).$$

Then $(I - T)^{-1}S$ is w-condensing. The condition of (L-S) is holds. Thus Theorem 3.9 implies that $(I - T)^{-1}S$ has a fixed point in $\overline{U}$.

4. Application to Hammerstein integral equation in L^1 space

Let Ω be a bounded subset of $\mathbb{R}^N$. A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function if

(i) for any fixed $x \in \mathbb{R}$, the function $t \to f(t, x)$ is measurable from Ω to $\mathbb{R}$;

(ii) for almost any $x \in \Omega$, the function $f(t, .) : \mathbb{R} \to \mathbb{R}$ is continuous;

(iii) for each $r > 0$, there exists $\mu_r \in L([0, 1]), \; \mathbb{R})$ such that $\|z\| \leq r$ implies $\|f(t, z)\| \leq \mu_r(t)$ for almost all $t \in [0, 1]$.

Let $m(\Omega, \mathbb{R})$ be the set of all measurable functions $\psi : \Omega \to \mathbb{R}$. If f is a Carathéodory function, then f defined a mapping $N_f : m(\Omega, \mathbb{R}) \to m(\Omega, \mathbb{R})$ by $N_f(\psi)(t) := f(t, \psi(t))$. This mapping is called the Nemytskii operator associated to f.

The following result due to Lucchetti and Patrone [12] is an extension to separable Banach spaces of a previous one due to Krasnosel'skii about Nemytskii operators for scalar valued functions [8].

Lemma 4.1. *Let X and Y be two separable Banach spaces. If f is a Carathéodory function, then the Nemytskii operator N_f maps $L^1(\Omega, X)$ into $L^1(\Omega, Y)$ if and only if there exist a constant $\eta > 0$ and a function $\zeta(.) \in L^1_+(\Omega)$ such that*

$$\|f(t, x)\|_Y \leq \zeta(t) + \eta\|x\|_X,$$

where $L^1_+(\Omega)$ *denotes the positive cone of the space* $L^1(\Omega)$.

We shall use the following characterization of the relatively weak compactness in $L^1(\Omega)$ which is a consequence of Dunford-Pettis theorem plus the de la Vallée-Poussin theorem and of Lemma 1 of [11].

Lemma 4.2. *Let* X, Y *be two finite dimensional Banach spaces and let* Ω *be a bounded domain in* $\mathbb{R}^N$. *If* $f : \Omega \times X \to Y$ *is a Carathéodory function and* N_f *maps* $L^1(\Omega, X)$ *into* $L^1(\Omega, Y)$, *then* N_f *satisfies* (A_2).

Theorem 4.1. *A sequence* (g_n) *of* $L^1(\Omega)$ *is relatively weakly compact if and only if there exists* $j \in J_0 := \{j : \mathbb{R} \to [0, \infty],$ *convex, L.S.C,* $j(0) = 0\}$ *such that*

$$\lim_{r\to\infty} \frac{j(r)}{r} = +\infty \quad \textit{and} \quad \int_\Omega j(|g_n|) \le 1.$$

Next, we are concerned with the solvability of the following variant of Hammerstein's integral equation:

$$u(t) = B(u(t)) + \lambda \int_\Omega \xi(t,s) f(s, u(s)) ds, \quad \textit{in} \quad L^1(\Omega). \tag{4.1}$$

Notice that eq(4.1) may be written in the form

$$u(t) = (Au)(t) + (Bu)(t)$$

where A is given by

$$\begin{aligned} A : L^1(\Omega) &\to L^1(\Omega) \\ u \mapsto Au(t) :=& \quad \lambda \textstyle\int_\Omega \xi(t,s) f(s,u(s)) ds. \end{aligned}$$

Suppose that ξ and f satisfy the following conditions:

(1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that
$$|f(s,x)| \le a(s) + b|x|$$
where $a \in L^1_+(\Omega)$ and $b \ge 0$

(2) The function $\xi : \Omega \times \Omega \to \mathbb{R}$ is strongly measurable and $\int_\Omega \xi(.,s) u(s) ds \in L^1(\Omega)$ whenever $u \in L^1(\Omega)$.

(3) There exists a function $\tau : \Omega \to \mathbb{R}$, belonging to $L^\infty(\Omega, \mathbb{R})$ such that
$$|\xi(t,s)| \le \tau(t)$$
for all $(t,s) \in \Omega \times \Omega$.

(4) B is w-condensing and $(I - B)^{-1} A(L^1(\Omega))$ is bounded.

(5) $(I - B)$ is surjective.

Theorem 4.2. *Assume that conditions* $(1) - (4)$ *are satisfied; then problem* (4.1) *has at least one solution in* $L^1(\Omega)$.

Proof. First A is a (ws)-compact map. In fact, A is a continuous operator. Indeed,

$$\|Au - Av\|_1 = \int_\Omega |Au(t) - Av(t)| dt = \int_\Omega |\lambda \int_\Omega \xi(t,s)[f(s,u(s)) - f(s,v(s))] ds| dt.$$

By Lemma 4.1, we have $N_f(u)(s) = f(s,u(s))$, acts from $L^1(\Omega)$ into $L^1(\Omega)$ and is continuous.

Since, by Hypothesis 3, there exists a positive function $\tau \in L^1(\Omega)$ such that for every $s \in \Omega, |\xi(t,s)| \leq \tau(t)$, then

$$\|Au - Av\|_1 \leq |\lambda| \int_\Omega \tau(t)dt \int_\Omega |f(s,u(s)) - f(s,v(s))|ds = |\lambda| \|\tau\|_1 \|N_f(u) - N_f(v)\|_1,$$

which implies the continuity of A, since the operator N_f is continuous in $L^1(\Omega)$.
Let $(x_n)_n$ be a weakly convergent sequence of $L^1(\Omega)$. According to Lemma 4.2, $(N_f(x_n))$ has a weakly convergent subsequence, say $(N_f(x_{n_k}))$. Thus there exists $x \in L^1(\Omega)$ such that $N_f(x_{n_k}) \rightharpoonup x$.
Therefore, since for every $t \in \Omega$ the mapping $\xi(t,.) : \Omega \to \mathbb{R}$ is bounded, we have

$$A(x_{n_k})(t) = \lambda \int_\Omega \xi(t,s) f(s, x_{n_k}(s))ds \to \lambda \int_\Omega \xi(t,s)x(s)ds.$$

Now, by the assumption on N_f, since Ω is a bounded domain, we can apply the dominated convergence theorem to conclude that the sequence

$$(A(x_{n_k}) = \lambda \int_\Omega \xi(.,s) f(s, x_{n_k}(s))ds)$$

is convergent in $L^1(\Omega)$.
Let C be a bounded sets of $L^1(\Omega$; then there exists $m > 0$ such that $\|g\|_1 \leq m$, for all $g \in C$.
On the other hand, we have

$$\begin{aligned} |A(g)(t)| &\leq |\lambda| \int_\Omega |\xi(t,s)||f(s,g(s))|ds \\ &\leq |\lambda| \int_\Omega \tau(t)(a(s) + b|g(s)|)ds \\ &\leq |\lambda| \tau(t)(\|a\|_1 + b\ m). \end{aligned}$$

We consider $j : \mathbb{R} \to [0, \infty]$ defined by

$$j(r) = \frac{r^2}{|\lambda|^2(\|a\|_1 + bM)^2 \|\tau^2\|_1}$$

It is clear that j satisfies:

- j belongs to J_0
- It is an increasing function for $r \geq 0$
- $\lim_{r\to\infty} \frac{j(r)}{r} = +\infty$

These facts imply that j under the condition of Theorem 4.1. Furthermore,

$$j(|A(g)(t)|) \leq j(|\lambda|\tau(t)(\|a\|_1 + b\ m)).$$

Consequently

$$\int_\Omega j(|A(g)(t)|)dt \leq \int_\Omega j(|\lambda|\tau(t)(\|a\|_1 + b\ m))dt = \int_\Omega \frac{|\lambda|^2\tau^2(t)(\|a\|_1 + b\ m)^2}{|\lambda|^2(\|a\|_1 + b\ m)^2\|\tau^2\|_1} dt = 1$$

Now, by using Theorem 4.1, we may conclude that $A(C)$ is a relatively weakly compact subset of $L^1(\Omega)$. The above arguments show that A and B satisfy the assumptions of Theorem 3.3, which allows us to affirm that eq (4.1) has a solution.

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