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FIXED POINTS AND COMPACT WEIGHTED COMPOSITION OPERATORS ON BANACH WEIGHTED HARDY SPACES

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Abstract. We consider weighted composition operators $C_{\psi,\varphi}$ acting on Banach weighted Hardy spaces in the open unit disk such that the norms of the kernel functions for the appropriate order derivatives tend to infinity as one approaches the boundary. We investigate the relation between the compactness of $C_{\psi,\varphi}$, the angular derivatives and the fixed points of φ, and we will see that compactness of $C_{\psi,\varphi}$ for some weight functions ψ forces φ to have a fixed point inside the open unit disk.

Key Words and Phrases: Banach weighted Hardy spaces, bounded point evaluation, weighted composition operator, Denjoy-Wolff point, fixed point.
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1. Introduction

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 \le p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series. Let $H^p(\beta)$ denotes the space of

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such formal power series. These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$ and the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$ where $\frac{1}{p}+\frac{1}{q}=1$ and $\beta^{p/q}=\{\beta(n)^{p/q}\}_n$ ([22]). Also if

$$g(z)=\sum_{n=0}^{\infty}\hat{g}(n)z^n\in H^q(\beta^{p/q}),$$

then

$$\|g\|^q=\sum_{n=0}^{\infty}|\hat{g}(n)|^q\beta(n)^p.$$

The Hardy, Bergman and Dirichlet spaces can be viewed in this way when $p=2$ and respectively $\beta(n)=1$, $\beta(n)=(n+1)^{-1/2}$ and $\beta(n)=(n+1)^{1/2}$. Let $\hat{f}_k(n)=\delta_k(n)$. So $f_k(z)=z^k$ and then $\{f_k\}_k$ is a basis for $H^p(\beta)$ such that $\|f_k\|=\beta(k)$. Clearly M_z, the operator of multiplication by z on $H^p(\beta)$ shifts the basis $\{f_k\}_k$. The spaces $H^p(\beta)$ are also called as weighted Hardy spaces. We denote the set of multipliers

$$\{\psi\in H^p(\beta):\ \psi H^p(\beta)\subseteq H^p(\beta)\}$$

by $H^p_\infty(\beta)$ and the linear operator of multiplication by ψ on $H^p(\beta)$ by M_ψ.

Remember that a complex number λ is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at λ, e_λ, is bounded.

By $H(G)$ and $H^\infty(G)$ we denote the set of analytic functions and bounded analytic functions on a plane domain G, respectively.

Lemma 1.1. *Each point of* $\Omega=\{z:|z|<r\}$ *where* $r=\liminf_n \beta(n)^{\frac{1}{n}}$ *is a bounded point evaluation and* Ω *is the largest open disc such that* $H^p(\beta)\subset H(\Omega)$.

Proof. Let $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ be in $H^p(\beta)$. By Holder inequality we have:

$$\sum_{n=0}^{\infty}|\hat{f}(n)|\,|z|^n\leq\left(\sum_{n=0}^{\infty}|\hat{f}(n)|^p(\beta(n))^p\right)^{1/p}\left(\sum_{n=0}^{\infty}\frac{|z|^{nq}}{\beta(n)^q}\right)^{1/q}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Now by the Root test the radius of convergence of the series $\sum_n\frac{|z|^{nq}}{\beta(n)^q}$ is equal to r. So by the above inequality each point of Ω is a bounded point evaluation. Noting that $\sum_n\frac{z^n}{(n+1)^{2/p}\beta(n)}$ is in $H^p(\beta)$ and has r as its exact radius of convergence. So Ω is the largest open disc such that $H^p(\beta)\subset H(\Omega)$. □

Assume that $\liminf_n \beta(n)^{\frac{1}{n}}=1$, thus by Lemma 1.1, $H^p(\beta)$ consists of functions analytic on the open unit disk U. In this case, the function φ in $H^p(\beta)$ that maps the unit disc U into itself induces a composition operator C_φ on $H^p(\beta)$ defined by $C_\varphi f=f\circ\varphi$. Also, if $\psi\in H^p_\infty(\beta)$, the weighted composition operator $C_{\psi,\varphi}$ is defined on $H^p(\beta)$ by

$$C_{\psi,\varphi}f=M_\psi C_\varphi f=\psi.f\circ\varphi.$$

To avoid $C_{\psi,\varphi}$ being merely a multiplication operator, the composition map φ is taken to be different from identity.

We say an analytic self-map φ of U has an angular derivative at $w \in \partial U$, if for some $\eta \in \partial U$ the non-tangential limit of $\frac{\varphi(z)-\eta}{z-w}$ when $z \to w$, exists and is finite. We call this limit the angular derivative of φ at w and denoted it by $\varphi'(w)$.

Recall that $\psi : U \to \mathbb{C}$ is bounded away from zero toward the boundary of U if and only if $\liminf_{z\to\xi} |\psi(z)| > 0$ for each $\xi \in \partial U$.

For more information on weighted Hardy spaces and other discussed topics, we refer to [1–28].

2. Main results

Weighted composition operators has been the focus of attention for several decades and many properties of weighted composition operators on various spaces has been studied. In [7], M.D. Contreras and A.G. Hernandez-Dias characterized the boundedness and compactness of weighted composition operators between special weighted Banach spaces of analytic functions. They estimated the essential norm of a weighted composition operator and computed it for some Banach spaces which are isomorphic to c_0. They also showed that, when such an operator is not compact, it is an isomorphism on a subspace isomorphic to c_0 or l^∞. Finally, they applied these results to study composition operators between Bloch type spaces and little Blch type spaces. Also, M.D. Contreras and A.G. Hernandez-Dias dealed with the boundedness, compactness, weak compactness, and completely continuity of weighted composition operators on Hardy spaces H^p $(1 \le p < \infty)$ ([8]). In particular they proved that such an operator is compact on H^1 if and only if it is weakly compact on this space. This result depends on a technique which passes the weak compactness from an operator T to operators dominated in norm by T. V. Matache summarized the basic properties of bounded and compact weighted composition operators on the Hilbert Hardy space on the open unit disc and used them to study the composition operators on Hardy-Smirnov spaces ([17]). In [2], P.S. Bourdon and S.K. Narayan characterized those weighted composition operators on H^2 that are unitary, showing that in contrast to the unweighted case, every automorphism of the open unit disc U induces a unitary weighted composition operator. A conjugation argument, using these unitary operators, allows us to describe all normal weighted composition operators on H^2 for which the composition map fixes a point in U. This description shows both ψ and φ must be linear fractional in order for the weighted composition operator $C_{\psi,\varphi}$ to be normal (assuming φ fixes a point in U). In general, they showed that if $C_{\psi,\varphi}$ is normal on H^2 and ψ is not zero function, then φ must be either univalent on U or constant. Descriptions of spectra are provided for the operator $C_{\psi,\varphi}$ when it is unitary or when it is normal and φ fixes a point in U. The inevertibility of weighted composition operators $C_{\psi,\varphi}$ are studied by G. Gunatillake in [14]. The two maps ψ and φ are characterized when $C_{\psi,\varphi}$ acts on the Hardy-Hilbert space of the unit disc, H^2. Depending upon the nature of the fixed points of φ, spectra are then investigated. Numerical ranges of some classes of weighted composition operators on H^2 has been studied by G. Gunatillake, M. Jovovich, and W. Smith in [15]. They considered the case when a composition map is a rotation of the unit disc and then identified a class of convexoid operators. In the case of isometric weighted composition operators, they gave a complete classification of their numerical ranges, and the inclusion of zero in the interior of the numerical range has been considered. Also, G. Gunatillake studied that how the compactness of $C_{\psi,\varphi}$ acting on H^2, depends on the iteration between the two maps ψ and φ ([13]). The spectrum of compact weighted composition operators on the weighted Hardy spaces in the unit ball has been studied by Z.H. Zhou and C. Yuan in [28]. The characterization of the boundedness and compactness of the weighted composition operators on the Zygmund space and the little Zygmund space has been given by S. Ye and G. Hu in [20]. In [6], F. Colonna and S. Li provided several characterizations of the bounded and the compact weighted composition operators from the Bloch space $\mathcal{B}$ and the analytic Besov spaces B_p (with $1 < p < \infty$) into the Zygmund space $\mathcal{Z}$. As a special case, they showed that the bounded (resp., compact) composition operators from $\mathcal{B}$, B_p, and H^∞ to $\mathcal{Z}$ coincide. In addition, the boundedness and compactness of the composition operator can be characterized in terms of the boundedness (resp., convergence to 0, under the boundedness assumption of the operator) of the Zygmund norm of the powers of the symbol. Hypercyclic property of weighted composition operators on function spaces has been investigated by B. Yousefi and H. Rezaei in [26]. In [27], B. Yousefi and J. Izadi have studied the supercyclicity criterion for weighted composition operators. Furthermore, Z. Kamali and B. Yousefi provided several characterizations of the disjoint hypercyclicity of weighted composition operators in [16]. In [11], M. Fatehi and M. H. Shaabani characterized the essentially normal weighted composition operators $C_{\psi,\varphi}$ on the weighted Bergman spaces when φ is a linear fractional of the open unit disc that is not an automorphism and $\psi \in H^\infty$ is continuous at a point which φ has a finite angular derivative. Also, they consider the case whenever φ is an automorphism of the disc and ψ belongs to the disc algebra. Hermitian weighted composition operators on weighted Hardy spaces of the unit disc was studied by C.C. Cowen, G. Gunatillake, and E. Ko ([10]). In particular, necessary conditions have been provided for a weighted composition operator to be Hermitian on such spaces. On weighted Hardy spaces with special kernel functions, the Hermitian weighted composition operators are explicitly identified and their spectra and spectral decompositions have been described. They showed that some of these Hermitian operators are part of a family of closely related normal weighted composition operators. In addition, as a consequence of the properties of weighted composition operators, they computed the extremal functions for the subspaces associated with the usual atomic inner functions for weighted Bergman spaces and they also get explicit formulas for the projections of the kernel functions on these subspaces.

The main theorem of this section extends the same result which has been stated in [5, 12] for weighted composition operators acting only on the Hilbert spaces H^2 (Hardy space), A^2_α (Bergman space), and $H^2(\beta)$ (Hilbert weighted Hardy space). The result regarding the case of $H^2(\beta)$ is due to Gunatillake ([6, Theorem 2]) that is proved by using the assumption

$$\sum_{n=1}^{\infty} \frac{1}{\beta(n)^2} = +\infty.$$

In the main theorem, we will substituted this assumption by the general case:

$$\sum_{n=1}^{\infty} \frac{n^{qj}}{\beta(n)^q} = +\infty,$$

and also we will consider the Banach spaces $H^p(\beta)$ where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (note that the Hilbert spaces H^2, A^2_α and $H^2(\beta)$ are special cases of Banach spaces $H^p(\beta)$ and in the general case the situation is more complicated). The manuscript [12] contains some other valuable and interesting results, for example Gunatillake in [6] characterizes the spectrum of $C_{\psi,\varphi}$ when it is compact on the Hilbert space $H^2(\beta)$ and when the composition map φ has a fixed point inside the open unit disk. Similarly, one can see that the same results hold for Banach spaces $H^p(\beta)$, $1 \le p < \infty$.

Recall that the functional of evaluation of the j-th derivative at λ is denoted by $e_\lambda^{(j)}$, i.e., for all $f \in H^p(\beta)$, $< f, e_\lambda^{(j)} >= f^{(j)}(\lambda)$ where $f^{(j)}$ is the j-th derivative of f at λ. Note that the continuity of e_λ $(\lambda \in U)$ implies that the point evaluations of derivatives of all orders are continuous. This is a consequence of an easy automatic continuity result: If a finite codimensional linear subspace Y of a Banach space X is the range of a continuous linear mapping on X, then it must be closed ([19, Lemma 3.3]). Apply this to $X = kere_\lambda$, $Y = (M_z - \lambda)X = ker(e_\lambda^{(1)}|_X)$ where $e_\lambda^{(1)}(f) = f'(\lambda)$, to get the continuity of $e_\lambda^{(1)}$ (first on X and then on the whole space).

Also, note that by the Julia-Caratheodory Theorem ([18]), φ has an angular derivative at $w \in \partial U$ if and only if φ' has non-tangential limit at w, and φ has non-tangential limit of modulus one at w. Consider the open Euclidean disc, Julia disc,

$$J(\xi, a) = \{z \in U; |\xi - z|^2 < a(1 - |z|^2)\}$$

of radius $\frac{a}{1+a}$ and center at $\frac{\xi}{1+a}$, whose boundary is tangent to ∂U at ξ. By the Julia's Lemma ([1, 18]), if $\xi \in \partial U$ and φ is an analytic function satisfying

$$d(\xi) = \liminf_{z \to \xi} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty,$$

then

$$\varphi(J(\xi, a)) \subseteq J(\varphi(\xi), ad(\xi)).$$

Note that the Julia-Caratheodory Theorem also implies that $|\varphi'(\xi)| = d(\xi) > 0$.

If $\frac{1}{p} + \frac{1}{q} = 1$, then $H^p(\beta)^* = H^q(\beta^{p/q})$ and indeed by this equality we mean that $H^q(\beta^{p/q})$ is isometrically isomorphic to the dual space of $H^p(\beta)$. In the following lemma, we show that $H^q(\beta^{\frac{p}{q}})$ and $H^q(\beta^{-1})$, where $\beta^{-1} = \{\beta^{-1}(n)\}_{n=0}^\infty$, are isometrically isomorphic from which we conclude that $H^p(\beta)^* = H^q(\beta^{-1})$.

Lemma 2.1. *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $H^p(\beta)^* = H^q(\beta^{-1})$.

Proof. Define $L : H^q(\beta^{\frac{p}{q}}) \to H^q(\beta^{-1})$ by $L(f) = F$ where

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \text{ and } F(z) = \sum_{n=0}^{\infty} \hat{f}(n) \beta^p(n) z^n.$$

Then

$$\begin{aligned}\|F\|^q_{H^q(\beta^{-1})} &= \sum_{n=0}^{\infty} |\hat{f}(n)|^q \big(\beta^{pq}(n)/\beta^q(n)\big) \\ &= \sum_{n=0}^{\infty} |\hat{f}(n)|^q \beta^p(n) \\ &= \|f\|^q_{H^p(\beta^{\frac{p}{q}})}.\end{aligned}$$

Thus L is an isometry. It is also surjective, because if $F(z) = \sum_{n=0}^{\infty} \hat{F}(n) z^n \in H^q(\beta^{-1})$, then $L(f) = F$, where $f(z) = \sum_{n=0}^{\infty} \big(\hat{F}(n)/\beta^p(n)\big) z^n$. Hence $H^q(\beta^{\frac{p}{q}})$ and $H^q(\beta^{-1})$ are isometrically isomorphic. Since $H^p(\beta)^* = H^q(\beta^{\frac{p}{q}})$, the proof is complete. □

It is convenient and helpful to introduce the notation $< f, g >$ to stand for $g(f)$ where $f \in H^p(\beta)$ and $g \in H^q(\beta^{-1})$ where $\frac{1}{p} + \frac{1}{q} = 1$. Note that in this case we have

$$< f, g >= \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$

In the following theorem we use the fact that

$$e_w(z) = \sum_{n=0}^{\infty} \bar{w}^n z^n \in H^p(\beta)^* = H^q(\beta^{-1})$$

and

$$\|e_w\|^q = \sum \frac{|w|^{nq}}{\beta(n)^q} < \infty$$

for all w in $\{z : |z| < r\}$ where $r = \liminf_n \beta(n)^{\frac{1}{n}}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Thus w is a bounded point evaluation on $H^p(\beta)$ if and only if $\{w^n/\beta(n)\} \in l^q$.

In the rest of the paper we suppose that $\liminf_n \beta(n)^{\frac{1}{n}} = 1$, so $H^p(\beta) \subset H(U)$. Also we suppose that the multiplication operator M_ψ and the composition operator C_φ are bounded on $H^p(\beta)$.

Theorem 2.2. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\sum_{n=1}^{\infty} \frac{n^{qj}}{\beta(n)^q} = +\infty$ *for some nonnegative integer* j. *Let* $\varphi : U \to U$ *be holomorphic and* $\psi \in H^p_\infty(\beta)$ *be such that for all nonnegative integers* $i \le j$, $\psi^{(i)} : U \to \mathbb{C}$ *is holomorphic and bounded away from zero toward* ∂U. *If* $C_{\psi,\varphi}$ *is compact on* $H^p(\beta)$ *and for some* ξ *in* ∂U, $\varphi'(\xi)$ *exists and is nonzero, then it should be* $|\varphi'(\xi)| > 1$ *and so the composition map* φ *has a unique fixed point in* U.

Proof. Let $|\varphi'(\xi)| \le 1$ for some $\xi \in \partial U$ and $\{z_k\}_k$ be any sequence in U with $z_k \longrightarrow \xi$. Also let j be the least non-negative integer such that the sum $\sum_{n=1}^{\infty} \frac{n^{qj}}{\beta(n)^q} = +\infty$. Set

$e_k = \frac{e^{(j)}_{z_k}}{\|e^{(j)}_{z_k}\|}$, then $\|e_k\| = 1$. If $j = 0$, we have:

$$\lim_k \|e_{z_k}\|^q = \lim_k \sum_{n\geq 0} \frac{|z_k|^{nq}}{\beta(n)^q} = \sum_{n\geq 0} \frac{1}{\beta(n)^q} = \infty.$$

So if p is a polynomial, then

$$\lim_k < p, e_k >= \lim_k \frac{p(z_k)}{\|e_{z_k}\|} = 0.$$

But polynomials are dense in $H^p(\beta)$, thus $e_k \longrightarrow 0$ weakly as $k \longrightarrow \infty$. If $j > 0$, then since $|z_k| \longrightarrow 1$ and $\sum\limits_n \frac{n^{qj}}{\beta(n)^q} = \infty$, we have

$$\begin{aligned}
\lim_k \|e^{(j)}_{z_k}\|^q &= \lim_k \|\sum_{n=0}^{\infty} n(n-1)(n-2)\ldots(n-j+1)(\bar{z}_k)^{n-j} z^n\|^q \\
&= \lim_k \sum [n(n-1)\ldots(n-j+1)]^q \frac{|z_k|^{(n-j)q}}{\beta(n)^q} = \infty.
\end{aligned}$$

Since polynomials are dense in $H^p(\beta)$, by the same manner as in the previous case, we can see that $e_k \longrightarrow 0$ weakly as $k \longrightarrow \infty$. Thus

$$\varphi^{(j)}(z_k)/||e^{(j)}_{z_k}|| =< \varphi, e_k >\to 0.$$

(Note that C_φ is bounded and $z \in H^p(\beta)$, thus $C_\varphi z = \varphi \in H^p(\beta)$). Also since $C^*_{\psi,\varphi}$ is a compact operator, then it is completely continuous and since $e_k \longrightarrow 0$ weakly, it should be $\|C^*_{\psi,\varphi} e_k\| \longrightarrow 0$. Now we show that

$$\lim_k \frac{\|e^{(j)}_{\varphi(z_k)}\|}{\|e^{(j)}_{z_k}\|} = 0.$$

To prove this relation, first note that a straightforward computation gives the following equalities:

$$\begin{aligned}
C^*_{\psi,\varphi} e_{z_k} &= \psi(z_k) e_{\varphi(z_k)} \\
C^*_{\psi,\varphi} e^{(1)}_{z_k} &= \psi'(z_k) e_{\varphi(z_k)} + \psi(z_k)\varphi'(z_k) e^{(1)}_{\varphi(z_k)} \\
C^*_{\psi,\varphi} e^{(2)}_{z_k} &= \psi^{(2)}(z_k)^2 e_{\varphi(z_k)} + 2\psi'(z_k)\varphi'(z_k) e^{(1)}_{\varphi(z_k)} + \psi(z_k)[\varphi'(z_k)^2 e^{(2)}_{\varphi(z_k)} \\
&+ \varphi^{(2)}(z_k) e^{(1)}_{\varphi(z_k)}] \\
C^*_{\psi,\varphi} e^{(3)}_{z_k} &= \psi'(z_k)^3 e_{\varphi(z_k)} + 3\psi^{(2)}(z_k)\varphi'(z_k) e^{(1)}_{\varphi(z_k)} + 3\psi'(z_k)[\varphi'(z_k)^2 e^{(2)}_{\varphi(z_k)} \\
&+ \varphi^{(2)}(z_k) e^{(1)}_{\varphi(z_k)}] + \psi(z_k)[\varphi'(z_k)^3 e^{(3)}_{\varphi(z)} + \varphi^{(3)}(z_k) e^{(1)}_{\varphi(z_k)} \\
&+ 2\varphi^{(2)}(z_k) e^{(2)}_{\varphi(z_k)} + \varphi'(z_k)\varphi^{(2)}(z_k) e^{(2)}_{\varphi(z_k)}] \\
&\vdots \\
C^*_{\psi,\varphi} e^{(j)}_{z_k} &= \psi(z_k)\varphi'(z_k)^j e^{(j)}_{\varphi(z_k)} + \psi(z_k)\varphi^{(j)}(z_k) e^{(1)}_{\varphi(z_k)} + L_{j,k}
\end{aligned}$$

where $L_{j,k}$ involves functionals of evaluation of derivatives of order less than j at $\varphi(z_k)$ with coefficients involving products of derivatives of φ at z_k of order less than j, and derivatives of ψ at z_k of order less than or equal to j.

Note that $\psi^{(i)} : U \to \mathbb{C}$ is holomorphic and bounded away from zero toward ∂U for all $0 \le i \le j$. Thus there exists $\mu > 0$ and $\delta > 0$ such that whenever $w \in U$ and $d(w, \xi) < \delta$, then $|\psi^{(i)}(w)| > \mu$ for all $0 \le i \le j$.

If $j = 0$, then $e_k = \frac{e_{z_k}}{\|e_{z_k}\|}$. Since $C^*_{\psi,\varphi}$ is completely continuous, as we saw earlier

$$C^*_{\psi,\varphi} e_k = \psi(z_k) \frac{e_{\varphi(z_k)}}{\|e_{z_k}\|}$$

tends to zero. But for k large enough, $|\psi(z_k)| > \mu > 0$. Hence $\frac{e_{\varphi(z_k)}}{\|e_{z_k}\|}$ tends to zero.

If $j = 1$, then $e_k = \frac{e^{(1)}_{z_k}}{\|e^{(1)}_{z_k}\|}$. Note that

$$C^*_{\psi,\varphi} e_k = \psi'(z_k) \frac{e_{\varphi(z_k)}}{\|e^{(1)}_{z_k}\|} + \psi(z_k)\varphi'(z_k) \frac{e^{(1)}_{\varphi(z_k)}}{\|e^{(1)}_{z_k}\|}$$

tends to zero, since $C^*_{\psi,\varphi}$ is completely continuous. Note that $\frac{e_{\varphi(z_k)}}{\|e^{(1)}_{z_k}\|}$ tends to zero, and by the assumption the non-tangential limit of $\varphi'(\xi)$ exists and is nonzero. Also, for k large enough, we have

$$|\psi(z_k)| > \mu > 0, \;\; |\psi^{'}(z_k)| > \mu > 0.$$

Thus the relation $\lim_k \|C^*_{\psi,\varphi} e_k\| = 0$ implies that

$$\lim_k \|e^{(1)}_{\varphi(z_k)}\| / \|e^{(1)}_{z_k}\| = 0.$$

If $j > 1$, then $e_k = e^{(j)}_{z_k} / \|e^{(j)}_{z_k}\|$. Note that

$$C^*_{\psi,\varphi} e_k = \frac{1}{\|e^{(j)}_{z_k}\|} (\psi(z_k)\varphi'(z_k)^j e^{(j)}_{\varphi(z_k)} + \psi(z_k)\varphi^{(j)}(z_k) e^{(1)}_{\varphi(z_k)} + L_{j,k})$$

where $L_{j,k}$ is defined as before. Remark that since j is the least non-negative integer such that $\sum_{n=0}^{\infty} \frac{n^{jq}}{\beta(n)^q} = +\infty$, then we have

$$\begin{aligned} \|e^{(i)}_{\varphi(z_k)}\|^q &= \sum_{n=i}^{\infty} [n(n-1)\dots(n-i+1)]^q \frac{|\varphi(z_k)|^{n-i}}{\beta(n)^q} \\ &\le \sum_{n=i}^{\infty} \frac{n^{iq}}{\beta(n)^q} \\ &\le \sum_{n=1}^{\infty} \frac{n^{(i-1)q}}{\beta(n)^q} < \infty \end{aligned}$$

for all $i < j$. So $\lim_{k\to\infty} \|e^{(i)}_{\varphi(z_k)}\|$ remains bounded for all i less than j. Furthermore, for k large enough, we have $|\psi^{(i)}(z_k)| > \mu > 0$ for all $0 \le i \le j$. Now since $\|e^{(j)}_{z_k}\| \longrightarrow \infty$

as $k \to 0$, and also since

$$\begin{aligned} |\varphi^{(i)}(z_k)| &= |<\varphi, e_{z_k}^{(i)}>| \\ &\leq \|\varphi\| \, \|e_{z_k}^{(i)}\| \\ &\leq \|\varphi\| \sum_{n=1}^{\infty} \frac{n^{(i-1)q}}{\beta(n)^q} < \infty \end{aligned}$$

for all $i < j$ and all $k \in \mathbb{N}$, thus indeed $\lim_k \frac{\|L_{j,k}\|}{\|e_{z_k}^{(j)}\|} = 0$. By a similar method used in the case of $j = 1$, we can see that

$$\lim_k \frac{\|\psi(z_k)\varphi^{(j)}(z_k)e_{\varphi(z_k)}^{(1)}\|}{\|e_{z_k}^{(j)}\|} = 0.$$

Therefore

$$\lim_k \|C_{\psi,\varphi}^* e_k\| = \lim_k |\psi(z_k)| \, |\varphi'(z_k)|^j \frac{\|e_{\varphi(z_k)}^{(j)}\|}{\|e_{z_k}^{(j)}\|} = 0.$$

Since $\varphi'(\xi) \neq 0$ and $|\psi(z_k)| > \mu$ for large k, we get

$$\lim_k \frac{\|e_{\varphi(z_k)}^{(j)}\|}{\|e_{z_k}^{(j)}\|} = 0. \qquad (*)$$

On the other hand note that by the Julia's Lemma ([1, 18]), φ maps the disks

$$\{z \in U : |z - \xi|^2 \leq c(1 - |z|^2)\}$$

into themselves, for each $c > 0$. So it follows that $|\varphi(r\xi)| \geq |r\xi| = r$ for all $r \in (0, 1)$ ([5, page 49]). Now since

$$\|e_z^{(j)}\|^q = \sum_{n=j}^{\infty} \frac{n!}{(n-j)!} \frac{|z|^{n-j}}{\beta(n)^q},$$

the norm $\|e_z^{(j)}\|$ increases with $|z|$. Thus for all k we have

$$\|e_{\varphi(z_k)}^{(j)}\| / \|e_{z_k}^{(j)}\| \geq 1.$$

This contradicts with relation $(*)$, and so it should be $|\varphi'(\xi)| > 1$ for all $\xi \in \partial U$ such that $\varphi'(\xi)$ exists. Now we show that φ has a unique fixed point in U. If φ is a disc automorphism, then it is elliptic, hyperbolic or parabolic. Since every elliptic automorphisms have a unique fixed point in U, so we only consider that φ is hyperbolic or parabolic. In these cases, we know that by the Wolff's Theorem ([18, page 81]) there is a unique fixed point $\xi \in \partial U$ with $\varphi'(\xi) \leq 1$. This is a contradiction by the above discussion. So if φ is a disc automorphism, then it should be elliptic. Now suppose that φ is not a disc automorphism and also assume that φ has no fixed point in U. Let ξ denotes its Denjoy-Wolff point in $\overline{U}$. So the angular derivative at ξ, $\varphi'(\xi)$, exists and is in $(0, 1]$. Now since $\xi \in \partial U$, thus it is a contradiction. So φ has a fixed point ξ in U. Obviously this fixed point is unique, because the composition map φ is not identity. □

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