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FIXED POINTS OF DISCONTINUOUS MAPPINGS IN UNIFORMLY CONVEX METRIC SPACES

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Abstract. Some fixed point theorems for discontinuous mappings in Banach spaces by Berinde and Păcurar [Fixed point theorems for non-self single-valued almost contractions, Fixed Point Theory 14 (2013), 301-311] and Kirk [Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type, Israel J. Math. 17 (1974), 339-346] are extended to uniformly convex metric spaces.
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1. Introduction

The celebrated Banach contraction principle asserts that a contraction on a complete metric space has a unique fixed point. A contraction (nonexpansive mapping) is continuous and hence this principle has a drawback that it is not applicable to discontinuous functions. Fixed point theorems for discontinuous mappings in Banach space setting have been established in [4, 21].

Recently in [4], the authors established some fixed point theorems for single-valued non-self almost contractions in Banach spaces. Since almost contractions form a large class of contractive type mappings that includes, amongst others, the Banach contraction mappings, therefore the results in [4] are significant generalization of some important metric fixed point theorems for single-valued self and non-self mappings; see for example [1, 10, 27].

For other very recent related results; see [28, 30, 29].

We state the main result in [4]:

Theorem 1.1. *Let E be a Banach space, K a nonempty closed subset of E and $T: K \to E$ a non-self almost contraction, that is, a mapping for which there exist two constants $\delta \in [0,1)$ and $L \geq 0$ such that*

$$||Tx - Ty|| \leq \delta \cdot ||x - y|| + L||y - Tx|| \,, \quad \text{for all } x, y \in K.$$

If T has property (M) (see Definition 3.1) and satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \text{ where } \partial K \text{ stands for the boundary of } K,$$

then T has a fixed point in K.

Note that here T may be discontinuous but T is continuous at the fixed point.

On the other hand, Goebel and Kirk [14] proved that if E is uniformly convex Banach space, K is a bounded, closed and convex subset of E and $T: K \to K$ is asymptotically nonexpansive on K, that is, if there exists a sequence $\{k_n\}$ of numbers such that $k_n \to 1$ as $n \to \infty$ and

$$||T^n x - T^n y|| \leq k_n ||x - y||, \; x, y \in K, \; n > N_0.$$

Then T has a fixed point. This generalizes fixed point theorem of Browder [6], Göhde [17] and Kirk [20] for nonexpansive mapping.

Later on, Kirk [21] substantially weakened the assumption of asymptotic nonexpansiveness of T as:

$$\limsup \{\sup_{y \in k} [||T^n x - T^n y|| - ||x - y||]\} \leq 0, \text{ for each } x \in K,$$

which may hold even if none of the iterates of T is Lipschitzian. Although, it is assumed that at least one of its iterates is continuous, the mapping itself need not be so. If, in addition, T is uniformly continuous, then it is said to be asymptotically nonexpansive in the intermediate sense [7].

Note that the Banach space is far away from being the most general setting in which Theorem 1.1 can be established. Moreover, property (M), a fundamental concept used in the proof, could also be naturally adapted in uniformly convex metric spaces (see Example 3.3). Reflexivity of uniformly convex Banach spaces used in the proof of the main results in [21], has its analogue in uniformly convex metric spaces (see Theorem 2.6).

With the help of these facts, in this paper, we obtain fixed point theorems for discontinuous mappings; non-self almost contractions and non-Lipschitzian mappings of asymptotically nonexpansive type in uniformly convex metric spaces.

2. Uniform convexity in metric spaces

Throughout this paper, (X, d) will stand for a metric space. Suppose that there exists a family $\mathcal{F}$ of metric segments such that any two points x, y in X are endpoints of a unique metric segment $[x, y] \in \mathcal{F}$ ($[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$). We shall denote by $(1-\beta)x \oplus \beta y$ the unique point z of $[x, y]$ which satisfies

$$d(x, z) = \beta d(x, y), \text{ and } d(z, y) = (1-\beta) d(x, y).$$

Such metric spaces are usually called *convex metric spaces* [25]. Moreover, if we have

$$d\left(\frac{1}{2}p \oplus \frac{1}{2}x, \frac{1}{2}p \oplus \frac{1}{2}y\right) \leq \frac{1}{2}d(x,y),$$

for all p, x, y in X, then X is said to be a *hyperbolic space* (see [26]).

Obviously, normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds [9], the Hilbert open unit ball equipped with the hyperbolic metric [15], and the $CAT(0)$ spaces [22]. We will say that a subset C of a hyperbolic metric space X is convex if $[x, y] \subset C$ whenever x, y are in C.

Definition 2.1. [19] Let (X, d) be a hyperbolic space. We say that X is uniformly convex if for any $a \in X$, for every $r > 0$, and for each $\varepsilon > 0$

$$\delta(r,\varepsilon) = \inf\Big\{1 - \frac{1}{r}\ d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon\Big\} > 0.$$

From now onwards we assume that X is a hyperbolic space.

Remark 2.2. (i) $\delta(r, 0) = 0$, and $\delta(r,\varepsilon)$ is an increasing function of ε for every fixed r (see [19], Remark 2.1).
(ii) $\delta(r, 2) = 1$ for every fixed r. To show this, choose $a = \frac{1}{2}x \oplus \frac{1}{2}y$ in Definition 2.1.

Remark 2.3. [19] If (X, d) is uniformly convex, then it is strictly convex. i.e., whenever

$$d\Big(\alpha x \oplus (1-\alpha)y, a\Big) = d(x,a) = d(y,a)$$

for $\alpha \in (0, 1)$ and any $x, y, a \in X$, then we must have $x = y$.

A subset C of a metric space X is Chebyshev if for every $x \in X$, there exists $c_0 \in C$ such that $d\,(c_0, x) < d\,(c, x)$ for all $c \in C$, $c \neq c_0$. In other words, for each point of the space, there is a well-defined nearest point of C . We can then define the nearest point projection $P : X \to C$ by sending x to c_0. We have the following result.

Lemma 2.4. [19] *Let (X, d) be a complete uniformly convex. Let C be nonempty, convex and closed subset of X. Let $x \in X$ be such that $d(x, C) < \infty$. Then there exists a unique best approximant of x in C, i.e., there exists a unique $c_0 \in C$ such that*

$$d(x, c_0) = d(x, C) = \inf\{d(x, c); c \in C\},$$

i.e., C is Chebyshev.

Lemma 2.5. [19] *Assume that (X, d) is uniformly convex. Let $\{C_n\} \subset X$ be a sequence of nonempty, nonincreasing, convex, bounded and closed sets. Let $x \in X$ be such that*

$$0 < d = \lim_{n\to\infty} d(x, C_n) < \infty.$$

Let $x_n \in C_n$ be such that $d(x, x_n) \to d$. Then $\{x_n\}$ is a Cauchy sequence.

Recall that a hyperbolic metric space (X, d) is said to have the property (R) if any non-increasing sequence of nonempty, convex, bounded and closed sets, has a nonempty intersection.

The following result is an analogue of the well known fact that a uniformly convex Banach space is reflexive. For a reference the reader may consult Theorem 2.1 in [15].

Theorem 2.6. [19] *If (X, d) is complete and uniformly convex, then (X, d) has the property (R).*

3. Fixed point results

3.1. **Non-self almost contraction.** Let X be a uniformly convex metric space, C a nonempty closed subset of X and $T : C \to X$ a non-self mapping. If $x \in C$ is such that $Tx \notin C$, then we suppose throughout this paper that there exists $y \in \partial C$ with $y = (1-\lambda)x \oplus \lambda Tx\, (0 < \lambda < 1)$, such that

$$d(x, Tx) = d(x, y) + d(y, Tx),\ y \in \partial C. \tag{3.1}$$

Definition 3.1. Let X be a uniformly convex metric space, C a nonempty closed subset of X and $T : C \to X$ a non-self mapping. Let $x \in C$ with $Tx \notin C$ and let $y \in \partial C$ be the element given by (3.1). If, for any such element x, we have

$$d(y, Ty) \le d(x, Tx), \tag{3.2}$$

then we say that T has property (M).

Remark 3.2. Note that a condition similar to (3.2) has been used in [13].

The non-self mapping T in the following example has property (M).

Example 3.3. Let C be a nonempty convex and closed subset of a complete and uniformly convex metric space X. For a fixed $x \in X \setminus C$, set $c_o = P(x)$ where P is the nearest point projection from X onto C. Let $B = B(c_o, \frac{d(c_0,x)}{2})$ be the closed ball centered at c_o with radius $\frac{d(c_0,x)}{2}$.

Define $T : B \to X$ by $Tb = \frac{1}{2}b \oplus \frac{1}{2}c_0$, the midpoint of $[b, c_0]$, if $b \ne x$ and $Tb = x$, if $b = c_0$. Then T has property (M).

Indeed, the only $b \in B$ with $Tb \notin B$ is $b = c_0$; let $y \in \partial B$ be the element as in (3.1). The equation

$$d(y, Ty) = d(y, \frac{1}{2}y \oplus \frac{1}{2}c_0) = \frac{1}{2}d(y, c_0) = \frac{1}{4}d(c_0, Tc_0)$$

shows that (3.2) holds.

Theorem 3.4. *Let X be a complete and uniformly convex metric space, C a nonempty closed subset of X and $T : C \to X$ a non-self almost contraction, that is, a mapping for which there exist two constants $\delta \in [0, 1)$ and $L \ge 0$ such that*

$$d(Tx, Ty) \le \delta \cdot d(x, y) + Ld(y, Tx)\,, \quad \text{for all } x, y \in C. \tag{3.3}$$

If T has property (M) and satisfies Rothe's boundary condition

$$T(\partial C) \subset C, \tag{3.4}$$

then T has a fixed point in C.

Proof. Let $x_0 \in \partial C$. By (3.4), we know that $Tx_0 \in C$. Denote $x_1 = Tx_0$. Now, if $Tx_1 \in C$, set $x_2 = Tx_1$. If $Tx_1 \notin C$, then there exists unique x_2 on the segment $[x_1, Tx_1]$ which also belongs to ∂C, that is,

$$x_2 = (1-\lambda)x_1 \oplus \lambda Tx_1\, (0 < \lambda < 1).$$

Continuing in this way, we obtain a sequence $\{x_n\}$ whose terms satisfy one of the following properties:

i) $x_n = Tx_{n-1}$, if $Tx_{n-1} \in C$;

ii) $x_n = (1-\lambda)x_{n-1} \oplus \lambda Tx_{n-1} \in \partial C\,(0 < \lambda < 1)$, if $Tx_{n-1} \notin C$.

To simplify the argument in the proof, let us denote

$$P = \{x_k \in \{x_n\} : \, x_k = Tx_{k-1}\}$$

and

$$Q = \{x_k \in \{x_n\} : \, x_k \neq Tx_{k-1}\}.$$

Note that $\{x_n\} \subset C$ and that, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P. Moreover, by virtue of (3.4), we cannot have two consecutive terms of $\{x_n\}$ in the set Q (but we can have two consecutive terms of $\{x_n\}$ in the set P) .

We claim that $\{x_n\}$ is a Cauchy sequence. To prove this, we must discuss following three different cases:

Case I. $x_n,\, x_{n+1} \in P$.

In this case, we have $x_n = Tx_{n-1}$, $x_{n+1} = Tx_n$ and so by (3.3), we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \delta d(x_n, x_{n-1}) + Ld(x_n, Tx_{n-1}).$$

As $x_n = Tx_{n-1}$, so we have

$$d(x_{n+1}, x_n) \leq \delta d(x_n, x_{n-1}). \tag{3.5}$$

Case II. $x_n \in P,\ x_{n+1} \in Q$.

In this case, we have $x_n = Tx_{n-1}$, $x_{n+1} \neq Tx_n$ and

$$d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n).$$

Hence

$$d(x_n, x_{n+1}) \leq d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n)$$

and so by (3.3), we get

$$d(x_n, x_{n+1}) \leq \delta d(x_n, x_{n-1}) + Ld(x_n, Tx_{n-1}) = \delta d(x_n, x_{n-1}),$$

which again yields inequality (3.5).

Case III. $x_n \in Q,\ x_{n+1} \in P$.

In this situation, we have $x_{n-1} \in P$. By property (M), we have

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq d(x_{n-1}, Tx_{n-1})).$$

From $x_{n-1} \in P$, we have $x_{n-1} = Tx_{n-2}$ and so by (3.3), we get

$$d(Tx_{n-2}, Tx_{n-1}) \leq \delta d(x_{n-2}, x_{n-1}) + Ld(x_{n-1}, Tx_{n-2}) = \delta d(x_{n-2}, x_{n-1}).$$

which shows that

$$d(x_n, x_{n+1}) \leq \delta d(x_{n-2}, x_{n-1}). \tag{3.6}$$

Therefore, summarizing all the three cases and using (3.5) and (3.6), it follows that the sequence $\{x_n\}$ satisfies the inequality

$$d(x_n, x_{n+1}) \leq \delta \max\{d(x_{n-2}, x_{n-1}),\, d(x_{n-1}, x_n)\}, \tag{3.7}$$

for all $n \geq 2$. Now, by induction for $n \geq 2$, from (3.7) one obtains

$$d(x_n, x_{n+1}) \leq \delta^{[n/2]} \max\{d(x_0, x_1),\, d(x_1, x_2)\},$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Further, for $m > n > N$,

$$d(x_n, x_m) \leq \sum_{i=N}^{\infty} d(x_i, x_{i-1}) \leq 2\frac{\delta^{[N/2]}}{1-\delta} \max\{d(x_0, x_1),\, d(x_1, x_2)\},$$

which shows that $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset C$ and C is closed, $\{x_n\}$ converges to some point of C.

Denote

$$x^* = \lim_{n\to\infty} x_n\,, \tag{3.8}$$

and let $\{x_{n_k}\} \subset P$ be an infinite subsequence of $\{x_n\}$ (such a subsequence always exists) that we denote for simplicity by $\{x_n\}$ too.

Then

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) = d(x_{n+1}, x^*) + d(Tx_n, Tx^*)\,.$$

By (3.3), we have

$$d(Tx_n, Tx^*) \leq \delta\, d(x_n, x^*) + L\, d(x^*, Tx_n)$$

and hence

$$d(x^*, Tx^*) \leq (1+L)d(x^*, x_{n+1}) + \delta \cdot d(x_n, x^*), \text{ for all } n \geq 0. \tag{3.9}$$

Letting $n \to \infty$ in (3.9), we obtain

$$d(x^*, Tx^*) = 0,$$

which shows that x^* is a fixed point of T. □

Berinde [2] has shown that it is possible to obtain uniqueness of the fixed point of an almost contraction, by imposing an additional contractive condition, quite similar to (3.3).

The uniqueness of fixed point of an almost contraction on a nonlinear domain is given below; its proof is simple and so omitted.

Theorem 3.5. *Let X be a complete and uniformly convex metric space, C a nonempty closed subset of X and $T : C \to X$ a non-self almost contraction for which there exist $\theta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx)\,, \quad \textit{for all } x, y \in C\,.$$

If T has property (M) and satisfies Rothe's boundary condition

$$T(\partial C) \subset C,$$

then T has a unique fixed point in C.

3.2. **Non-Lipschitzian mappings.** In this section, we prove fixed points results for the class of non-Lipschitzian mappings of asymptotically nonexpansive type which contains the class of asymptotically nonexpansive mappings [14].

Theorem 3.6. *Let (X, d) be a complete and uniformly convex metric space, let $C \subset X$ be nonempty, bounded, and closed convex. Suppose that $T : C \to C$ has the property "T^N is continuous for some positive integer N", and T satisfies:*

$$\limsup\{\sup_{y\in k}[d(T^n x, T^n y) - d(x, y)]\} \le 0, \text{ for each } x \in C. \tag{3.10}$$

Then T has a fixed point in C.

Proof. For each $y \in C$ and $r > 0$, let $S(y, r)$ denote the ball centered at y with radius r. Let $x \in C$ be fixed, and let the set R_x consists of those numbers ρ for which there exists an integer k such that

$$C \cap \Big(\bigcap_{n=k}^{\infty} S(T^n x, \rho)\Big) \neq \phi.$$

If D is the diameter of C, then $D \in R_x$, so $R_x \neq \phi$. Let ρ_0 =g.l.b. R_x, and for each $\varepsilon > 0$, define $K_\varepsilon = \bigcup_{k=1}^{\infty} \big(\bigcap_{n=k}^{\infty} S(T^n x, \rho_0 + \varepsilon)\big)$. Thus for each $\varepsilon > 0$, the sets $K_\varepsilon \cap C$ are nonempty and convex and so the property (R) of (X, d) implies that

$$K = \bigcap_{\varepsilon>0} \big(\overline{K}_\varepsilon \cap C\big) \neq \phi.$$

Now let $z \in K$, and let

$$\tau(z) = \limsup_{n\to\infty} d(z, T^n z).$$

Suppose $\rho_0 = 0$, which implies that $\{T^i x\}$ is a Cauchy sequence. Hence $T^i x \to z$ as $i \to \infty$. Let $\eta > 0$. By (3.10), $n > M$

$$\sup_{y\in C}[d(T^n z, T^n y) - d(z, y)] \le \frac{1}{3}\eta, \text{ where } n > M.$$

As $T^i x \to z$ so there exists $m > n$ such that $d(T^m x, z) \le \frac{1}{3}\eta$ and $d(T^{m-n} x, z) \le \frac{1}{3}\eta$. Thus if $n \ge M$, then we get

$$\begin{aligned} d(z, T^n z) &\le d(z, T^m x) + d(T^m x, T^n z) \\ &\le d(z, T^m x) + d(T^n z, T^n(T^{m-n} x)) - d(z, T^{m-n} x) + d(z, T^{m-n} x) \\ &\le \frac{1}{3}\eta + \sup_{y\in C}[d(T^n z, T^n y) - d(z, y)] + \frac{1}{3}\eta \\ &= \eta. \end{aligned}$$

This proves that $T^i z \to z$ as $i \to \infty$, that is, $\rho(z) = 0$. But $\rho(z) = 0$ implies $T^{N_n} z \to z$ as $n \to \infty$ and the continuity of T^N yields $T^N z = z$. Thus

$$Tz = T(T^{N_n})z = T^{N_{n+1}} z \to z \text{ as } n \to \infty, \tag{3.11}$$

and $Tz = z$. Therefore, we may assume $\rho_0 > 0$ and $\tau(z) = 0$ (In fact, we may assume this for any $x, z \in C$.)

Now let $\varepsilon > 0,\ \varepsilon \leq \tau(z)$. By the definition of ρ_0, there exists an integer N^* such that for $n \geq N^*$m we have

$$d(z, T^n x) \leq \rho_0 + \varepsilon.$$

By (3.10), there exists N^{**} such that for $n \geq N^{**}$, we have

$$\sup_{y \in C}[d(T^n z, T^n y) - d(z, y)] \leq \varepsilon.$$

Select j so that $j \geq N^{**}$ and hence

$$d(z, T^j z) \geq \tau(z) - \varepsilon.$$

Thus if $n - j \geq N^*$, then we have

$$\begin{aligned} d(T^j z, T^n x) &= \{d(T^j z, T^j(T^{n-j}x)) - d(z, T^{n-j}x)\} + d(z, T^{n-j}x) \\ &= \varepsilon + (\rho_0 + \varepsilon) \\ &= \rho_0 + 2\varepsilon. \end{aligned}$$

For $m = \frac{1}{2}z \oplus \frac{1}{2}T^j z$, we have by uniform convexity of (X, d),

$$d(m, T^n x) \leq \big(1 - \delta\big(\rho_0 + 2\varepsilon, \frac{\tau(z) - \varepsilon}{\rho_0 + 2\varepsilon}\big)\big)(\rho_0 + 2\varepsilon),\ n \geq N^* + j.$$

By the minimality of ρ_0, this implies

$$\rho_0 \leq \big(1 - \delta\big(\rho_0 + 2\varepsilon, \frac{\tau(z) - \varepsilon}{\rho_0 + 2\varepsilon}\big)\big)(\rho_0 + 2\varepsilon);$$

letting $\varepsilon \to 0$,

$$\rho_0 \leq \big(1 - \delta\big(\rho_0, \frac{\tau(z)}{\rho_0}\big)\big)\rho_0.$$

So $\big(1 - \delta\big(\rho_0, \frac{\tau(z)}{\rho_0}\big)\big) \geq 1$ and hence $\delta\big(\rho_0, \frac{\tau(z)}{\rho_0}\big) = 0$; this implies that $\tau(z) = 0$. Hence as shown before in (3.11), $Tz = z$. □

Corollary 3.7. *Let (X, d) be a complete and uniformly convex metric space, and let $C \subset X$ be nonempty, bounded, closed and convex. Suppose $T : C \to C$ is asymptotically nonexpansive. Then T has a fixed point in C.*

Theorem 3.6 shows that $Fix(T)$, set of fixed points of T is not empty. The next theorem illustrates structure of the set $Fix(T)$.

Theorem 3.8. *Under the assumptions of Theorem 3.6, $Fix(T)$ is closed and convex.*

Proof. For the closeness of $Fix(T)$, let $\{x_n\} \subset Fix(T)$ be such that $x_n \to x$. Then $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^N x_n = T^N \lim_{n\to\infty} x_n = T^N x$. Hence as shown before in (3.11), $Tx = x$.

To show convexity, it is sufficient to prove that $z = \frac{1}{2}x \oplus \frac{1}{2}y \in Fix(T)$ for all $x, y \in Fix(T)$. We have

$$\limsup_{i\to\infty} d(T^i z, x) = \limsup_{i\to\infty} d(T^i z, T^i x) \leq d(z, x) = \frac{1}{2}d(x, y),$$

$$\limsup_{i\to\infty} d(T^i z, y) = \limsup_{i\to\infty} d(T^i z, T^i y) \leq d(z, y) = \frac{1}{2}d(x, y).$$

Thus

$$\limsup_{i\to\infty} d(T^i z, z) \le \frac{1}{2}\big(1 - \delta\big(\frac{1}{2}d(x,y), 2\big)\big)d(x,y)$$

and hence

$$z = \lim_{i\to\infty} T^i z = \lim_{i\to\infty} T^{i+N} z = T^N \lim_{i\to\infty} T^i z = T^N z.$$

Once again, as in (3.11), $Tz = z$. □

Remark 3.9. (1) Theorems 3.4-3.5 extend ([4], Theorem 3.3 and Theorem 3.6) on a nonlinear domain and which provides a positive answer to the Open Problem posed by Berinde [4].

(2) Corollary 3.7 sets analogue of the fixed point property for asymptotically nonexpansive mappings due to Kirk [23] and Khamsi [18] in $CAT(0)$ spaces and hyperconvex metric spaces, respectively.

(3) Theorem 3.6 is a natural generalization of fundamental fixed point theorem for nonexpansive mappings by Takahashi [31] and for asymptotically nonexpansive mappings by Kohlenbach and Leustean [24].

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