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THE STRUCTURE OF THE FIXED POINT SET OF QUADRATIC OPERATORS ON THE SIMPLEX

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Abstract. We know that any linear operator associated with a positive square stochastic matrix has a unique fixed point in the simplex. However, in general, the similar result for a quadratic operator acting on the simplex does not hold true. Namely, there is a quadratic operator associated with a positive cubic stochastic matrix which has more than one fixed point in the simplex. The first attempt to give an example for such kind of quadratic operators was done by A.A. Krapivin and Yu.I. Lyubich. However, we showed that their examples are wrong. Therefore, in this paper, we decided to give a correct example for a quadratic operator with positive coefficients having three fixed points in the simplex. Moreover, we also describe the number of fixed points of the quadratic operator associated with a positive cubic stochastic matrix.

Key Words and Phrases: Cubic stochastic matrix, quadratic stochastic operator, fixed point.
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1. Introduction

In many fields, an equilibrium is the fundamental concept that can be described in term of a fixed point of some mapping. In general, a problem of showing an existence or/and uniqueness of fixed points of a mapping is a tedious task. However, there are some results which speak about an existence and uniqueness of fixed points of a continuous mapping in suitable spaces. For instance, the Perron–Frobenius theorem states that a linear operator associated with a positive square stochastic matrix has a unique fixed point in the simplex. However, the similar result in the nonlinear case does not hold true. This is the mainstream of the paper.

By being the simplest nonlinear mapping, a quadratic operator has an incredible application in population genetics [1, 3, 4, 7, 10], game theory [2], control systems [6, 17, 18]. In population genetics, the quadratic operator describes a distribution of the next generation of the system if the current distribution is given [10, 20]. In this sense, the quadratic operator is a primary source for investigations of evolution of population genetics. The detailed exposure of the theory of quadratic operators is presented in [5, 11, 12, 13, 14, 15, 16]. Meanwhile, a fixed point of the quadratic operator is an equilibrium for the system. In contrast to the linear case, the fixed point set of the quadratic operator is sophisticated. For instance, regardless of positivity of an associated cubic stochastic matrix, the quadratic operator may have more than one fixed point. The first attempt to give an example for such kind of quadratic operators was done by A. A. Krapivin in [8] and by Yu. I. Lyubich in [10]. However, it turns out that their examples are wrong. In fact, we shall show that Krapivin's example as well as Lyubich's example has a unique fixed point in the simplex. At the same time, we shall also provide an example for the quadratic operator associated with the positive cubic stochastic matrix which has three fixed points in the simplex. Moreover, we also describe the number of fixed points of the quadratic operator associated with a positive cubic stochastic matrix.

2. Preliminary

Let $\|\mathbf{x}\|_1 = \sum\limits_{k=1}^{m} |x_k|$ be a norm of a vector $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$. We say that $\mathbf{x} \geq 0$ (resp. $\mathbf{x} > 0$) if $x_k \geq 0$ (resp. $x_k > 0$) for all $k = \overline{1, m}$. Let

$$\mathbb{S}^{m-1} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_1 = 1,\ \mathbf{x} \geq 0\}$$

be the $(m-1)-$dimensional standard simplex. An element of the simplex $\mathbb{S}^{m-1}$ is called *a stochastic vector*.

A square matrix $\mathbb{P} = (p_{ij})_{i,j=1}^{m}$ is called *stochastic* if

$$\sum_{j=1}^{m} p_{ij} = 1,\ p_{ij} \geq 0,\ \forall i, j = \overline{1, m}.$$

Every square stochastic matrix is associated with a linear operator $\mathcal{L} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ as follows $\mathcal{L}(\mathbf{x}) = \mathbf{x}\mathbb{P}$, i.e.,

$$(\mathcal{L}(\mathbf{x}))_j = \sum_{i=1}^{m} x_i p_{ij}, \quad \forall\ j = \overline{1, m.} \tag{2.1}$$

A square stochastic matrix $\mathbb{P} = (p_{ij})_{i,j=1}^{m}$ is said to be *positive* (written $\mathbb{P} > 0$) if $p_{ij} > 0,\ \forall\ i, j = \overline{1, m}$. A linear operator associated with a positive square stochastic matrix is called *positive*. Let $\mathbf{Fix}(\mathcal{L}) = \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathcal{L}(\mathbf{x}) = \mathbf{x}\}$ be a fixed point set. If $\mathcal{L} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ is the positive linear operator then due to the Perron-Frobenius theorem, one has that $|\mathbf{Fix}(\mathcal{L})| = 1$.

A cubic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is called *stochastic* if

$$\sum_{k=1}^m p_{ijk} = 1,\ p_{ijk} \geq 0,\ \forall i,j,k = \overline{1,m}.$$

Every cubic stochastic matrix is associated with a quadratic operator $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ as follows

$$(\mathcal{Q}(\mathbf{x}))_k = \sum_{i,j=1}^m x_i x_j p_{ijk}, \quad \forall\ k = \overline{1,m}. \tag{2.2}$$

A cubic stochastic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is said to be *positive* (written $\mathcal{P} > 0$) if $p_{ijk} > 0,\ \forall i,j,k = \overline{1,m}$. A quadratic operator associated with a positive cubic stochastic matrix is called *positive*. Let $\mathbf{Fix}(\mathcal{Q}) = \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathcal{Q}(\mathbf{x}) = \mathbf{x}\}$ be a fixed point set. Due to Brouwer's fixed point theorem, $\mathbf{Fix}(\mathcal{Q}) \neq \emptyset$. In general, if $\mathcal{P} > 0$ then it is not necessary to be true that $|\mathbf{Fix}(\mathcal{Q})| = 1$. The first attempt to provide an example for the positive quadratic operator having more than one fixed point was done by A.A.Krapivin [8]. Later, Y.I. Lyubich also provided an example for the positive quadratic operator having three fixed points in his book [10] by slightly modifying Krapivin's example. It turns out that Krapivin's example as well as Lyubich's example has a unique fixed point. In the next section, we shall discuss those examples. Moreover, we shall also provide an example for positive quadratic operators having three fixed points in the simplex.

3. Krapivin's example

A. A. Krapivin has considered the following quadratic operator $V_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$, $V_\varepsilon(\mathbf{x}) = \mathbf{x}' = (x_1', x_2', x_3')$ in his paper [8]

$$V_\varepsilon : \begin{cases} x_1' = (1-4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1x_2 + (1+4\varepsilon)x_1x_3 + 8\varepsilon x_2x_3 \\ x_2' = 2\varepsilon x_1^2 + (1-3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2}+2\varepsilon)x_1x_2 + 2\varepsilon x_1x_3 + (1+8\varepsilon)x_2x_3 \\ x_3' = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1-11\varepsilon)x_3^2 + (\frac{3}{2}-6\varepsilon)x_1x_2 + (1-6\varepsilon)x_1x_3 + (1-16\varepsilon)x_2x_3 \end{cases}$$

A.A. Krapivin claimed [8] that the quadratic operator $V_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ has two fixed points on the line segment $L = \left\{\left(\frac{t}{2}, \frac{1-t}{2}, \frac{1}{2}\right)\right\}_{0\leq t\leq 1}$ where $0 < \varepsilon < \frac{1}{100}$. However, this claim is wrong.

Proposition 3.1. *The quadratic operator V_ε does not have any fixed point on L.*

Proof. We search for a fixed point $\mathbf{x}_0 = (\frac{t}{2}, \frac{1-t}{2}, \frac{1}{2})$ on the line segment L. We should have that $V_\varepsilon(\mathbf{x}_0) = \mathbf{x}_0$. After some algebraic calculations, we obtain the following system of equations

$$\begin{cases} (1-6\varepsilon)t^2 - (1+4\varepsilon)t + 20\varepsilon = 0 \\ (1-6\varepsilon)t^2 - (1-4\varepsilon)t + 12\varepsilon = 0 \\ (1-6\varepsilon)t^2 - \left(1+\frac{4}{3}\varepsilon\right)t + \frac{52}{3}\varepsilon = 0 \end{cases} \tag{3.1}$$

If we take a difference of the first two equations of the system (3.1) we then get that $8\varepsilon t = 8\varepsilon$ or $t = 1$. However, if we substitute $t = 1$ into the third equation in the system (3.1) we then have that $10\varepsilon = 0$ which contradicts to the condition $0 < \varepsilon < \frac{1}{100}$. Therefore, the system (3.1) does not have any solution. This completes the proof.

Now, we are aiming to prove that $|\mathbf{Fix}(V_\varepsilon)| = 1$.

It is clear that $V_\varepsilon(\mathbb{S}^2) \subset \mathrm{int}\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{S}^2 : x_1x_2x_3 > 0\}$. Hence, $\mathbf{Fix}(V_\varepsilon) \subset \mathrm{int}\mathbb{S}^2$.

In order to find all fixed points, we have to solve the system of equations

$$\begin{cases} x_1 = (1-4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1x_2 + (1+4\varepsilon)x_1x_3 + 8\varepsilon x_2x_3 \\ x_2 = 2\varepsilon x_1^2 + (1-3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2}+2\varepsilon)x_1x_2 + 2\varepsilon x_1x_3 + (1+8\varepsilon)x_2x_3 \\ x_3 = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1-11\varepsilon)x_3^2 + (\frac{3}{2}-6\varepsilon)x_1x_2 + (1-6\varepsilon)x_1x_3 + (1-16\varepsilon)x_2x_3 \end{cases}$$

Proposition 3.2. *One has $\xi_1 \neq \eta_1$, $\xi_2 \neq \eta_2$, $\xi_3 \neq \eta_3$ for $\xi, \eta \in \boldsymbol{Fix}(V_\varepsilon)$, $\xi \neq \eta$.*

Proof. Let ξ, η be two distinct solutions of the system given above (if any). Since $x_3 = 1 - x_1 - x_2$, we can rewrite the system in terms of x_1 and x_2 as follows

$$\begin{cases} 2\varepsilon x_1^2 + 4\varepsilon x_2^2 + (12\varepsilon - 1)x_1x_2 - 16\varepsilon x_1 - 12\varepsilon x_2 + 10\varepsilon = 0 \\ \varepsilon x_1^2 - 10\varepsilon x_2^2 - (\frac{1}{2}+6\varepsilon)x_1x_2 + 6\varepsilon x_2 + \varepsilon = 0 \end{cases} \tag{3.2}$$

Case $\xi_1 \neq \eta_1$. Our aim is to show that $\xi_1 \neq \eta_1$. We suppose the contrary that is $\xi_1 = \eta_1$. Since $\xi \neq \eta$, we must have that $\xi_2 \neq \eta_2$. This means that for $x_1 = \xi_1 = \eta_1$, the following two quadratic equations (with respect to x_2)

$$4\varepsilon x_2^2 + ((12\varepsilon - 1)\xi_1 - 12\varepsilon)x_2 + (2\xi_1^2 - 16\xi_1 + 10)\varepsilon = 0,$$
$$-10\varepsilon x_2^2 + \left[-\left(\frac{1}{2}+6\varepsilon\right)\xi_1 + 6\varepsilon\right]x_2 + (\xi_1^2+1)\varepsilon = 0$$

must have two distinct common roots ξ_2 and η_2. Consequently, we should have that

$$\frac{4\varepsilon}{-10\varepsilon} = \frac{(12\varepsilon-1)\xi_1 - 12\varepsilon}{-\left(\frac{1}{2}+6\varepsilon\right)\xi_1 + 6\varepsilon} = \frac{2\xi_1^2 - 16\xi_1 + 10}{\xi_1^2+1}.$$

It follows from the first equality that $\xi_1 = \frac{8\varepsilon}{8\varepsilon - 1}$. Since $0 < \varepsilon < \frac{1}{100}$, we get that $\xi_1 < 0$ which is a contradiction.

Case $\xi_2 \neq \eta_2$. Our aim is to show that $\xi_2 \neq \eta_2$. We again suppose the contrary that is $\xi_2 = \eta_2$. Since $\xi \neq \eta$, we must have that $\xi_1 \neq \eta_1$. It follows from (3.2) that for $x_2 = \xi_2 = \eta_2$, the following two quadratic equations (with respect to x_1)

$$2\varepsilon x_1^2 + ((12\varepsilon - 1)\xi_2 - 16\varepsilon)x_1 + (4\xi_2^2 - 12\xi_2 + 10)\varepsilon = 0,$$
$$\varepsilon x_1^2 - \left(\frac{1}{2}+6\varepsilon\right)\xi_2 x_1 + (-10\xi_2^2 + 6\xi_2 + 1)\varepsilon = 0$$

must have two distinct common roots ξ_1 and η_1. Consequently, we should have that

$$\frac{2\varepsilon}{\varepsilon} = \frac{(12\varepsilon-1)\xi_2 - 16\varepsilon}{-\left(\frac{1}{2}+6\varepsilon\right)\xi_2} = \frac{4\xi_2^2 - 12\xi_2 + 10}{-10\xi_2^2 + 6\xi_2 + 1}.$$

It follows from the first equality that $\xi_2 = \frac{2}{3}$. By substituting $\xi_2 = \frac{2}{3}$ into the last fraction, we get that $\frac{4\xi_2^2-12\xi_2+10}{-10\xi_2^2+6\xi_2+1} = \frac{34}{5} \neq 2 = \frac{2\varepsilon}{\varepsilon}$. This is again contradiction.

Case $\xi_3 \neq \eta_3$**.** Our aim is to show that $\xi_3 \neq \eta_3$. We again suppose the contrary that is $\xi_3 = \eta_3$. Since $\xi \neq \eta$, we must have that $\xi_1 \neq \eta_1$. In this case, we can rewrite the system of equations in terms of x_1 and x_3 as follows

$$\begin{cases}(1-6\varepsilon)x_1^2 + 4\varepsilon x_3^2 + (1-4\varepsilon)x_1x_3 - x_1 + 4\varepsilon x_3 + 2\varepsilon = 0\\ (9\varepsilon - \frac{3}{2})x_1^2 + 6\varepsilon x_3^2 + (18\varepsilon - \frac{3}{2})x_1x_3 + (\frac{3}{2} - 8\varepsilon)x_1 - 18\varepsilon x_3 + \varepsilon = 0\end{cases}$$

This yields that for $x_3 = \xi_3 = \eta_3$, the following two quadratic equations (with respect to x_1)

$$(1-6\varepsilon)x_1^2 + ((1-4\varepsilon)\xi_3 - 1)x_1 + (4\xi_3^2 + 4\xi_3 + 2)\varepsilon = 0,$$

$$(9\varepsilon - \frac{3}{2})x_1^2 + \left[\left(18\varepsilon - \frac{3}{2}\right)\xi_3 + \left(\frac{3}{2} - 8\varepsilon\right)\right]x_1 + (6\xi_3^2 - 18\xi_3 + 1)\varepsilon = 0$$

must have two distinct common roots ξ_1 and η_1. Consequently, we should have that

$$\frac{1-6\varepsilon}{9\varepsilon - \frac{3}{2}} = \frac{(1-4\varepsilon)\xi_3 - 1}{(18\varepsilon - \frac{3}{2})\xi_3 + \frac{3}{2} - 8\varepsilon} = \frac{4\xi_3^2 + 4\xi_3 + 2}{6\xi_3^2 - 18\xi_3 + 1}.$$

It follows from the first equality that $\xi_3 = \frac{2}{3}$. By substituting $\xi_3 = \frac{2}{3}$ into the last fraction, we get that $\frac{4\xi_3^2+4\xi_3+2}{6\xi_3^2-18\xi_3+1} = -\frac{58}{75} \neq -\frac{2}{3} = \frac{1-6\varepsilon}{9\varepsilon-\frac{3}{2}}$. This is again contradiction. This completes the proof.

Theorem 3.3. *Let* $V_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ *be the quadratic operator given above. Then for sufficiently small* ε, *one has that* $\boldsymbol{Fix}(V_\varepsilon) = \left\{\left(\frac{3a_0^2-3a_0+1}{2-3a_0},\ a_0,\ \frac{1-2a_0}{2-3a_0}\right)\right\}$ *where* a_0 *is the unique positive root in* $\left(0, \frac{1}{2}\right)$ *of the following quartic equation*

$$(9-54\varepsilon)a^4 + (132\varepsilon - 15)a^3 + (9-68\varepsilon)a^2 - (12\varepsilon + 2)a + 10\varepsilon = 0.$$

Proof. Since $x_3 = 1 - x_1 - x_2$, it is enough to find all solutions (x_1, x_2) of the following system of equations

$$\begin{cases}2\varepsilon x_1^2 + 4\varepsilon x_2^2 + (12\varepsilon - 1)x_1x_2 - 16\varepsilon x_1 - 12\varepsilon x_2 + 10\varepsilon = 0\\ \varepsilon x_1^2 - 10\varepsilon x_2^2 - (\frac{1}{2} + 6\varepsilon)x_1x_2 + 6\varepsilon x_2 + \varepsilon = 0\end{cases} \tag{3.3}$$

which satisfy the conditions $0 < x_1, x_2 < 1$ and $0 < x_1 + x_2 < 1$.

Let $x_1 = x$ be a variable and $x_2 = a$ be a parameter. Then the system (3.3) takes the following form

$$x^2 + \frac{(12\varepsilon - 1)a - 16\varepsilon}{2\varepsilon}x + (2a^2 - 6a + 5) = 0, \tag{3.4}$$

$$x^2 + \frac{a(-1-12\varepsilon)}{2\varepsilon}x + (-10a^2 + 6a + 1) = 0. \tag{3.5}$$

Due to Proposition 3, these two quadratic equations cannot have two common roots. Hence, the system (3.3) has a solution (x_1, x_2) with $0 < x_1, x_2 < 1,\ 0 < x_1 + x_2 < 1$ if and only if two quadratic equations (3.4) and (3.5) must have a unique common root in $(0,1)$ for $a \in (0,1)$. We know (see [19]) that two quadratic equations (3.4) and (3.5) have a unique common root if and only if their resultant is equal to zero, i.e.,

$$(9-54\varepsilon)a^4 + (132\varepsilon - 15)a^3 + (9-68\varepsilon)a^2 + (-12\varepsilon - 2)a + 10\varepsilon = 0. \tag{3.6}$$

In this case, $x = \frac{3a^2-3a+1}{2-3a}$ is the unique common root of two quadratic equations (3.4) and (3.5). In order to have conditions $0 < x_1, x_2 < 1,\ 0 < x_1 + x_2 < 1$, we have to solve the following system of inequalities

$$\begin{cases} 0 < \frac{3a^2-3a+1}{2-3a} < 1 \\ 0 < a < 1 \\ 0 < a + \frac{3a^2-3a+1}{2-3a} < 1. \end{cases} \tag{3.7}$$

The solution of the system (3.7) is $a \in (0, \frac{1}{2})$. Therefore, the total number of solutions (x_1, x_2), $0 < x_1, x_2 < 1,\ 0 < x_1 + x_2 < 1$ of the system (3.3) is the same as the total number of roots of the quartic equation (3.6) in the interval $(0, \frac{1}{2})$. Moreover, there is one-to-one correspondence between a root $a_0 \in (0, \frac{1}{2})$ of the quartic equation (3.6) and a fixed point $\left(\frac{3a_0^2-3a_0+1}{2-3a_0},\ a_0,\ \frac{1-2a_0}{2-3a_0}\right)$ of the quadratic operator $V_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$.

Now, we want to show that the quartic equation (3.6) has a unique root in the interval $(0, \frac{1}{2})$ for sufficiently small ε. To do so, we have to apply the Sturm theorem for the quartic equation (3.6) in $(0, \frac{1}{2})$ (see [19]).

Let $p(a) = (9-54\varepsilon)a^4 + (132\varepsilon - 15)a^3 + (9-68\varepsilon)a^2 + (-12\varepsilon - 2)a + 10\varepsilon$ be a quartic polynomial. Let $\{p_0(a), p_1(a), p_2(a), p_3(a), p_4(a)\}$ be a Sturm sequence of the quartic polynomial $p(a)$. Let $\sigma(\xi)$ be the number of sign changes (ignoring zero terms) in the sequence $p_0(\xi), p_1(\xi), p_2(\xi), p_3(\xi), p_4(\xi)$. Then due to the Sturm theorem, the number of roots of the quartic polynomial $p(x)$ in the interval $(0, \frac{1}{2})$ is equal to $\sigma(0) - \sigma(\frac{1}{2})$. Simple calculations show that for sufficiently small ε, one has that

$$\begin{aligned}
&p_0(0) = 10\varepsilon > 0, && p_0\left(\tfrac{1}{2}\right) = -\tfrac{1}{16} + \tfrac{1}{8}\varepsilon < 0,\\
&p_1(0) = -(12\varepsilon + 2) < 0, && p_1\left(\tfrac{1}{2}\right) = \tfrac{1}{4} - 8\varepsilon > 0,\\
&p_2(0) \simeq -\tfrac{5}{24(6\varepsilon-1)} > 0, && p_2\left(\tfrac{1}{2}\right) \simeq -\tfrac{13}{6\varepsilon-1} > 0,\\
&p_3(0) \simeq \tfrac{9}{(3-344\varepsilon)^2} > 0, && p_3\left(\tfrac{1}{2}\right) \simeq \tfrac{9}{(3-344\varepsilon)^2} > 0,\\
&p_4(0) \simeq \tfrac{3}{6\varepsilon-1} < 0, && p_4\left(\tfrac{1}{2}\right) \simeq \tfrac{3}{6\varepsilon-1} < 0.
\end{aligned}$$

Therefore, we get that $\sigma(0) - \sigma(\frac{1}{2}) = 3 - 2 = 1$. Consequently, this means that the quartic equation (3.6) has a unique root a_0 in the interval $(0, \frac{1}{2})$, or equivalently, the quadratic operator $V_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ has a unique fixed point $\left(\frac{3a_0^2-3a_0+1}{2-3a_0},\ a_0,\ \frac{1-2a_0}{2-3a_0}\right)$ in the simplex $\mathbb{S}^2$. This completes the proof.

4. Lyubich's example

Yu. I. Lyubich has considered (see [10], page 296) the following quadratic operator $W_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$, $W_\varepsilon(\mathbf{x}) = \mathbf{x}' = (x'_1, x'_2, x'_3)$

$$W_\varepsilon : \begin{cases} x'_1 = (1-4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1x_2 + (1+4\varepsilon)x_1x_3 + 8\varepsilon x_2x_3 \\ x'_2 = 2\varepsilon x_1^2 + (1-3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2}+2\varepsilon)x_1x_2 + 2\varepsilon x_1x_3 + (1-12\varepsilon)x_2x_3 \\ x'_3 = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1-11\varepsilon)x_3^2 + (\frac{3}{2}-6\varepsilon)x_1x_2 + (1-6\varepsilon)x_1x_3 + (1+4\varepsilon)x_2x_3 \end{cases}$$

where $0 < \varepsilon < \frac{1}{12}$. In commentaries and references section, Yu.I. Lyubich wrote that the quadratic operator $W_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ was constructed by A.A. Krapivin in [8]. However, Krapivin's example $V_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ considered in the previous section is slightly different from the quadratic operator $W_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ given in Lyubich's book [10]. Yu. I. Lyubich claimed that if $0 < \varepsilon < \frac{9-5\sqrt{2}}{124}$ then the quadratic operator $W_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ has three fixed points in the simplex $\mathbb{S}^2$. However, this is wrong. Namely, the quadratic operator W_ε has a unique fixed point for any $0 < \varepsilon < \frac{1}{12}$. For the sake of argument, we shall present its proof by repeating the same method used in Krapivin's example.

It is clear that $W_\varepsilon(\mathbb{S}^2) \subset \mathrm{int}\mathbb{S}^2$. Hence, $\mathbf{Fix}(W_\varepsilon) \subset \mathrm{int}\mathbb{S}^2$.

In order to find all fixed points, we have to solve the system of equations

$$\begin{cases} x_1 = (1-4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1x_2 + (1+4\varepsilon)x_1x_3 + 8\varepsilon x_2x_3 \\ x_2 = 2\varepsilon x_1^2 + (1-3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2}+2\varepsilon)x_1x_2 + 2\varepsilon x_1x_3 + (1-12\varepsilon)x_2x_3 \\ x_3 = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1-11\varepsilon)x_3^2 + (\frac{3}{2}-6\varepsilon)x_1x_2 + (1-6\varepsilon)x_1x_3 + (1+4\varepsilon)x_2x_3 \end{cases}$$

Proposition 4.1. *One has $\xi_1 \neq \eta_1, \xi_2 \neq \eta_2, \xi_3 \neq \eta_3$ for $\xi, \eta \in \boldsymbol{Fix}(W_\varepsilon)$, $\xi \neq \eta$.*

Proof. Let ξ, η be two distinct solutions of the system given above (if any). Since $x_3 = 1 - x_1 - x_2$, we can rewrite the system of equations in terms of x_1 and x_2 as

$$\begin{cases} 2\varepsilon x_1^2 + 4\varepsilon x_2^2 + (12\varepsilon - 1)x_1x_2 - 16\varepsilon x_1 - 12\varepsilon x_2 + 10\varepsilon = 0 \\ \varepsilon x_1^2 + 10\varepsilon x_2^2 + (14\varepsilon - \frac{1}{2})x_1x_2 - 14\varepsilon x_2 + \varepsilon = 0 \end{cases} \tag{4.1}$$

Case $\xi_1 \neq \eta_1$. Our aim is to show that $\xi_1 \neq \eta_1$. We suppose the contrary that is $\xi_1 = \eta_1$. Since $\xi \neq \eta$, we must have that $\xi_2 \neq \eta_2$. This means that for $x_1 = \xi_1 = \eta_1$, the following two quadratic equations (with respect to x_2)

$$4\varepsilon x_2^2 + ((12\varepsilon - 1)\xi_1 - 12\varepsilon)x_2 + (2\xi_1^2 - 16\xi_1 + 10)\varepsilon = 0,$$
$$10\varepsilon x_2^2 + ((14\varepsilon - \frac{1}{2})\xi_1 - 14\varepsilon)x_2 + (\xi_1^2 + 1)\varepsilon = 0$$

must have two distinct common roots ξ_2 and η_2. Consequently, we should have that

$$\frac{4\varepsilon}{10\varepsilon} = \frac{(12\varepsilon - 1)\xi_1 - 12\varepsilon}{(14\varepsilon - \frac{1}{2})\xi_1 - 14\varepsilon} = \frac{2\xi_1^2 - 16\xi_1 + 10}{\xi_1^2 + 1}.$$

It follows from the first equality that $\xi_1 = \frac{8\varepsilon}{8\varepsilon - 1}$. Since $0 < \varepsilon < \frac{1}{12}$, we get that $\xi_1 < 0$ which is a contradiction.

Case $\xi_2 \neq \eta_2$. Our aim is to show that $\xi_2 \neq \eta_2$. We again suppose the contrary that is $\xi_2 = \eta_2$. Since $\xi \neq \eta$, we must have that $\xi_1 \neq \eta_1$. It follows from (4.1) that for

$x_2 = \xi_2 = \eta_2$, the following two quadratic equations (with respect to x_1)

$$2\varepsilon x_1^2 + ((12\varepsilon - 1)\xi_2 - 16\varepsilon)x_1 + (4\xi_2^2 - 12\xi_2 + 10)\varepsilon = 0,$$
$$\varepsilon x_1^2 + (14\varepsilon - \frac{1}{2})\xi_2 x_1 + (10\xi_2^2 - 14\xi_2 + 1)\varepsilon = 0$$

must have two distinct common roots ξ_1 and η_1. Consequently, we should have that

$$\frac{2\varepsilon}{\varepsilon} = \frac{(12\varepsilon - 1)\xi_2 - 16\varepsilon}{(14\varepsilon - \frac{1}{2})\xi_2} = \frac{4\xi_2^2 - 12\xi_2 + 10}{10\xi_2^2 - 14\xi_2 + 1}.$$

It follows from the first equality that $\xi_2 = -1$ which is a contradiction.
Case $\xi_3 \neq \eta_3$. Our aim is to show that $\xi_3 \neq \eta_3$. We again suppose the contrary that is $\xi_3 = \eta_3$. Since $\xi \neq \eta$, we must have that $\xi_1 \neq \eta_1$. In this case, we can rewrite the system of equations in terms of x_1 and x_3 as follows

$$\begin{cases} (1 - 6\varepsilon)x_1^2 + 4\varepsilon x_3^2 + (1 - 4\varepsilon)x_1x_3 - x_1 + 4\varepsilon x_3 + 2\varepsilon = 0 \\ (9\varepsilon - \frac{3}{2})x_1^2 - 14\varepsilon x_3^2 - (\frac{3}{2} + 2\varepsilon)x_1x_3 + (\frac{3}{2} - 8\varepsilon)x_1 + 2\varepsilon x_3 + \varepsilon = 0 \end{cases}$$

This yields that for $x_3 = \xi_3 = \eta_3$, the following two quadratic equations (with respect to x_1)

$$(1 - 6\varepsilon)x_1^2 + ((1 - 4\varepsilon)\xi_3 - 1)x_1 + (4\xi_3^2 + 4\xi_3 + 2)\varepsilon = 0,$$
$$(9\varepsilon - \frac{3}{2})x_1^2 + \left[-\left(\frac{3}{2} + 2\varepsilon\right)\xi_3 + \left(\frac{3}{2} - 8\varepsilon\right)\right] x_1 + [-14\xi_3^2 + 2\xi_3 + 1]\varepsilon = 0$$

must have two distinct common roots ξ_1 and η_1. Consequently, we should have that

$$\frac{1 - 6\varepsilon}{9\varepsilon - \frac{3}{2}} = \frac{(1 - 4\varepsilon)\xi_3 - 1}{-(\frac{3}{2} + 2\varepsilon)\xi_3 + \frac{3}{2} - 8\varepsilon} = \frac{4\xi_3^2 + 4\xi_3 + 2}{-14\xi_3^2 + 2\xi_3 + 1}.$$

It follows from the first equality that $\xi_3 = -1$ which is a contradiction. This completes the proof.

Theorem 4.2. *Let $W_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ be the quadratic operator given above. Then for any $0 < \varepsilon < \frac{1}{12}$, one has that $\boldsymbol{Fix}(W_\varepsilon) = \left\{\left(\frac{1+2a_0(1-a_0)}{2(1+a_0)},\ a_0,\ \frac{1-2a_0}{2(1+a_0)}\right)\right\}$ where a_0 is the unique positive root in $(0, \frac{1}{2})$ of the following quartic equation*

$$(2 - 12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon = 0.$$

Proof. Since $x_3 = 1 - x_1 - x_2$, it is enough to find all solutions (x_1, x_2) of the following system of equations

$$\begin{cases} 2\varepsilon x_1^2 + 4\varepsilon x_2^2 + (12\varepsilon - 1)x_1x_2 - 16\varepsilon x_1 - 12\varepsilon x_2 + 10\varepsilon = 0 \\ \varepsilon x_1^2 + 10\varepsilon x_2^2 + (14\varepsilon - \frac{1}{2})x_1x_2 - 14\varepsilon x_2 + \varepsilon = 0 \end{cases} \tag{4.2}$$

which satisfy the conditions $0 < x_1, x_2 < 1$ and $0 < x_1 + x_2 < 1$.

Let $x_1 = x$ be a variable and $x_2 = a$ be a parameter. Then the system (4.2) takes the following form

$$x^2 + \frac{(12\varepsilon - 1)a - 16\varepsilon}{2\varepsilon}x + (2a^2 - 6a + 5) = 0, \tag{4.3}$$

$$x^2 + \frac{a(14\varepsilon - \frac{1}{2})}{\varepsilon}x + (10a^2 - 14a + 1) = 0. \tag{4.4}$$

Due to Proposition 4, these two quadratic equations cannot have two common roots. Hence, the system (4.2) has a solution (x_1, x_2) with $0 < x_1, x_2 < 1,\ 0 < x_1 + x_2 < 1$ if and only if two quadratic equations (4.3) and (4.4) must have a unique common root in $(0, 1)$ for $a \in (0, 1)$. We know (see [19]) that two quadratic equations (4.3) and (4.4) have a unique common root if and only if their resultant is equal to zero, i.e.,

$$(2 - 12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon = 0. \tag{4.5}$$

In this case, $x = \frac{1+2a(1-a)}{2(1+a)}$ is the unique common root of two quadratic equations (4.3) and (4.4). In order to have conditions $0 < x_1, x_2 < 1,\ 0 < x_1 + x_2 < 1$, we have to solve the following system of inequalities

$$\begin{cases} 0 < \frac{1+2a(1-a)}{2(1+a)} < 1 \\ 0 < a < 1 \\ 0 < a + \frac{1+2a(1-a)}{2(1+a)} < 1. \end{cases} \tag{4.6}$$

The solution of the system (4.6) is $a \in (0, \frac{1}{2})$. Therefore, the total number of solutions (x_1, x_2), $0 < x_1, x_2 < 1,\ 0 < x_1 + x_2 < 1$ of the system (4.2) is the same as the total number of roots of the quartic equation (4.5) in the interval $(0, \frac{1}{2})$. Moreover, there is one-to-one correspondence between a root $a_0 \in (0, \frac{1}{2})$ of the quartic equation (4.5) and a fixed point $\left(\frac{1+2a_0(1-a_0)}{2(1+a_0)},\ a_0,\ \frac{1-2a_0}{2(1+a_0)}\right)$ of the quadratic operator $W_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$.

We are aiming to study the number of positive roots of the quartic equation (4.5) in the interval $(0, \frac{1}{2})$. Let $f(a) = (2 - 12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon$. Since $0 < \varepsilon < \frac{1}{12}$, it is easy to check that

$$f(0) = 5\varepsilon > 0,\ f\left(\frac{1}{2}\right) = \frac{18\varepsilon - 9}{8} < 0,\ f(2) = 18 - 27\varepsilon > 0.$$

This means that the quartic equation (4.5) has at least two positive roots. On the other hand, due to Descartes's theorem, the number of positive roots cannot be more than the number of sign changes between consecutive nonzero coefficients $2 - 12\varepsilon,\ 16\varepsilon,\ 16\varepsilon - 3,\ -(16\varepsilon + 1),\ 5\varepsilon$ of the quartic equation (4.5) which is two.

Therefore, the quartic equation (4.5) has exactly two positive roots in which one of them belongs to $(0, \frac{1}{2})$ and another one belongs to $(\frac{1}{2}, 2)$. Hence, for any $0 < \varepsilon < \frac{1}{12}$, there exists a unique positive root a_0 of the quartic equation (4.5) in the interval $(0, \frac{1}{2})$. Consequently, for any $0 < \varepsilon < \frac{1}{12}$, the quadratic operator W_ε has a unique fixed point $\left(\frac{1+2a_0(1-a_0)}{2(1+a_0)},\ a_0,\ \frac{1-2a_0}{2(1+a_0)}\right)$. This completes the proof

5. Positive quadratic operator having three fixed points

In this section, we provide an example for a quadratic operator with positive coefficients having three fixed points in the simplex $\mathbb{S}^2$.

Let $\mathbf{A}(0.1, 0.2, 0.7)$, $\mathbf{B}(0.4, 0.3, 0.3)$ and $\mathbf{C}(0.59, 0.31, 0.1)$ be points in the simplex. We define a positive quadratic operator $\mathcal{Q}_0 : \mathbb{S}^2 \to \mathbb{S}^2$, $\mathcal{Q}_0(\mathbf{x}) = \mathbf{x}' = (x_1', x_2', x_3')$ as follows

$$\mathcal{Q}_0 : \begin{cases} x_1' = \frac{232873}{319300}x_1^2 + \frac{4717}{10300}x_2^2 + \frac{207}{63860}x_3^2 + \frac{7}{5}x_1x_2 + \frac{3}{5}x_1x_3 + \frac{1}{50}x_2x_3 \\ x_2' = \frac{27}{100}x_1^2 + \frac{1}{2}x_2^2 + \frac{3}{20}x_3^2 + \frac{470171}{814300}x_1x_2 + \frac{378421}{407150}x_1x_3 + \frac{158157}{814300}x_2x_3 \\ x_3' = \frac{54}{79825}x_1^2 + \frac{433}{10300}x_2^2 + \frac{27037}{31930}x_3^2 + \frac{18409}{814300}x_1x_2 + \frac{191589}{407150}x_1x_3 + \frac{1454157}{814300}x_2x_3 \end{cases}$$

The straightforward calculation shows that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are fixed points of the quadratic operator $\mathcal{Q}_0 : \mathbb{S}^2 \to \mathbb{S}^2$.

We can define another positive quadratic operator $\mathcal{Q}_1 : \mathbb{S}^2 \to \mathbb{S}^2$, $\mathcal{Q}_1(\mathbf{x}) = \mathbf{x}' = (x_1', x_2', x_3')$ as follows

$$\mathcal{Q}_1 : \begin{cases} x_1' = \frac{17322871}{22351000}x_1^2 + \frac{990257}{2163000}x_2^2 + \frac{1559}{13410600}x_3^2 + \frac{13}{10}x_1x_2 + \frac{16}{25}x_1x_3 + \frac{11}{500}x_2x_3 \\ x_2' = \frac{224}{1000}x_1^2 + \frac{488}{1000}x_2^2 + \frac{125}{1000}x_3^2 + \frac{703327}{1017875}x_1x_2 + \frac{19461451}{24429000}x_1x_3 + \frac{8271787}{24429000}x_2x_3 \\ x_3' = \frac{4301}{4470200}x_1^2 + \frac{117199}{2163000}x_2^2 + \frac{2933179}{3352650}x_3^2 + \frac{18371}{2035750}x_1x_2 + \frac{13761989}{24429000}x_1x_3 + \frac{1601951}{977160}x_2x_3 \end{cases}$$

The straightforward calculation shows that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are also fixed points of the quadratic operator $\mathcal{Q}_1 : \mathbb{S}^2 \to \mathbb{S}^2$.

Now, we can define a family of positive quadratic operators $\mathcal{Q}_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ as $\mathcal{Q}_\varepsilon(\mathbf{x}) = (1-\varepsilon)\mathcal{Q}_0(\mathbf{x}) + \varepsilon\mathcal{Q}_1(\mathbf{x})$ for any $x \in \mathbb{S}^2$ and $0 \le \varepsilon \le 1$. It is clear that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are also fixed points of the family of positive quadratic operators $\mathcal{Q}_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$.

In the paper [9], it was conjectured that *if the set of stationary vectors of a second-order Markov chain contains $k-$interior points of the $(k-1)-$dimensional face of the simplex then every vector in the $(k-1)-$dimensional face is a stationary vector.*

However, this conjecture is wrong. The family of quadratic stochastic operators $\mathcal{Q}_\varepsilon : \mathbb{S}^2 \to \mathbb{S}^2$ defined above are counterexamples to this conjecture.

6. The structure of the fixed point set of quadratic operator

Let $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$, $\mathcal{Q}(\mathbf{x}) = \mathbf{x}' = (x_1', x_2', x_3')$ be a positive quadratic operator

$$\mathcal{Q} : \begin{cases} x_1' = p_{11}x_1^2 + p_{22}x_2^2 + p_{33}x_3^2 + 2p_{12}x_1x_2 + 2p_{13}x_1x_3 + 2p_{23}x_2x_3 \\ x_2' = q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + 2q_{12}x_1x_2 + 2q_{13}x_1x_3 + 2q_{23}x_2x_3 \\ x_3' = r_{11}x_1^2 + r_{22}x_2^2 + r_{33}x_3^2 + 2r_{12}x_1x_2 + 2r_{13}x_1x_3 + 2r_{23}x_2x_3 \end{cases}$$

where $p_{ij}, q_{ij}, r_{ij} > 0,\ p_{ij} + q_{ij} + r_{ij} = 1,\ p_{ij} = p_{ji},\ q_{ij} = q_{ji},\ r_{ij} = r_{ji},\ \forall i, j = 1, 2, 3.$

The possible number of fixed points of the positive quadratic operator acting on 2D simplex was described in [8, 10].

Proposition 5.1. ([8, 10]) *One has that* $|\boldsymbol{Fix}(\mathcal{Q})| = 1$ or 3.

We shall describe the number of fixed points of positive quadratic operator in terms of its coefficients.

For that purpose, we introduce the following constants.

$$\alpha_{11} = p_{11} + p_{33} - 2p_{13},\ \alpha_{22} = p_{22} + p_{33} - 2p_{23},\ \alpha_{12} = p_{33} + p_{12} - p_{13} - p_{23},$$
$$\alpha_1 = p_{13} - p_{33},\ \alpha_2 = p_{23} - p_{33},\ \alpha_0 = p_{33},$$
$$\beta_{11} = q_{11} + q_{33} - 2q_{13},\ \beta_{22} = q_{22} + q_{33} - 2q_{23},\ \beta_{12} = q_{33} + q_{12} - q_{13} - q_{23},$$
$$\beta_1 = q_{13} - q_{33},\ \beta_2 = q_{23} - q_{33},\ \beta_0 = q_{33},$$
$$\gamma_1 = (2\beta_2 - 1)\alpha_{11} - 2\alpha_2\beta_{11},\ \gamma_2 = \alpha_{11}\beta_{22} - \alpha_{22}\beta_{11},\ \gamma_0 = \beta_0\alpha_{11} - \alpha_0\beta_{11},$$
$$\delta_1 = \alpha_{12}\beta_{11} - \beta_{12}\alpha_{11},\ \delta_0 = (2\alpha_1 - 1)\beta_{11} - 2\beta_1\alpha_{11},$$
$$\Delta_1 = \gamma_2\delta_0^2 - 2\gamma_1\delta_0\delta_1 + 4\gamma_0\delta_1^2,$$
$$\mu_1 = 2\beta_1\alpha_{22} - (2\alpha_1 - 1)\beta_{22},\ \mu_2 = \alpha_{22}\beta_{11} - \alpha_{11}\beta_{22},\ \mu_0 = \alpha_{22}\beta_0 - \alpha_0\beta_{22},$$
$$\nu_1 = \alpha_{12}\beta_{22} - \alpha_{22}\beta_{12},\ \nu_0 = 2\alpha_2\beta_{22} - (2\beta_2 - 1)\alpha_{22},$$
$$\Delta_2 = \mu_2\nu_0^2 - 2\mu_1\nu_0\nu_1 + 4\mu_0\nu_1^2,$$
$$\lambda_4 = \alpha_{11}\gamma_2^2 + 4\alpha_{12}\gamma_2\delta_1 + 4\alpha_{22}\delta_1^2,$$
$$\lambda_3 = 2\alpha_{11}\gamma_2\gamma_1 + 2\alpha_{12}\gamma_2\delta_0 + 4\alpha_{12}\gamma_1\delta_1 + 4\alpha_1\gamma_2\delta_1 - 2\gamma_2\delta_1 + 4\alpha_{22}\delta_1\delta_0 + 8\alpha_2\delta_1^2,$$
$$\lambda_2 = 2\alpha_{11}\gamma_2\gamma_0 + \alpha_{11}\gamma_1^2 + 2\alpha_{12}\gamma_1\delta_0 + 4\alpha_{12}\gamma_0\delta_1 + 2\alpha_1\gamma_2\delta_0 + 4\alpha_1\gamma_1\delta_1 - \gamma_2\delta_0 - 2\gamma_1\delta_1$$
$$+\alpha_{22}\delta_0^2 + 8\alpha_2\delta_1\delta_0 + 4\alpha_0\delta_1^2,$$
$$\lambda_1 = 2\alpha_{11}\gamma_1\gamma_0 + 2\alpha_{12}\gamma_0\delta_0 + 2\alpha_1\gamma_1\delta_0 + 4\alpha_1\gamma_0\delta_1 - \gamma_1\delta_0 - 2\gamma_0\delta_1 + 2\alpha_2\delta_0^2 + 4\alpha_0\delta_1\delta_0,$$
$$\lambda_0 = \alpha_{11}\gamma_0^2 + (2\alpha_1 - 1)\gamma_0\delta_0 + \alpha_0\delta_0^2.$$

Let us consider the following system of inequalities in the interval $t \in (0, 1)$

$$\begin{cases} 0 < \dfrac{\gamma_2 t^2 + \gamma_1 t + \gamma_0}{2\delta_1 t + \delta_0} < 1 \\ 0 < \dfrac{(\gamma_2 + 2\delta_1)t^2 + (\gamma_1 + \delta_0)t + \gamma_0}{2\delta_1 t + \delta_0} < 1 \end{cases} \tag{6.1}$$

It is clear that the set Ω of solutions of the system (6.1) is a union of finite number of disjoint open intervals, i.e., $\Omega = \bigcup_{i=1}^{n} \left(\omega_i^{(1)}, \omega_i^{(2)}\right)$.

Let $p(x) = \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ be a quartic polynomial. Let us construct the Sturm sequence $\{p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)\}$ of the polynomial $p(x)$ (see [19]). Let $\sigma(\xi)$ be the number of sign changes (ignoring zero terms) in the sequence $p_0(\xi), p_1(\xi), p_2(\xi), p_3(\xi), p_4(\xi)$. Then due to the Sturm theorem [19], the number of roots of the quartic polynomial $p(x)$ in the interval $\left(\omega_i^{(1)}, \omega_i^{(2)}\right)$ is equal to $\sigma\left(\omega_i^{(1)}\right) - \sigma\left(\omega_i^{(2)}\right)$. Consequently, the total number of roots of the quartic polynomial $p(x)$ in $\Omega = \bigcup_{i=1}^{n} \left(\omega_i^{(1)}, \omega_i^{(2)}\right)$ is

$$\#(p; \Omega) = \sum_{i=1}^{n} \left(\sigma\left(\omega_i^{(1)}\right) - \sigma\left(\omega_i^{(2)}\right)\right).$$

Theorem 5.2. *Let $p(x) = \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ be a quartic polynomial and $\alpha_{11}\beta_{11}\alpha_{22}\beta_{22}\Delta_1\Delta_2 \neq 0$. Let Ω be a solution set of the system* (6.1). *Then one has that $|\boldsymbol{Fix}(\mathcal{Q})| = \#(p; \Omega)$. Moreover, $(A_0, a_0, 1 - B_0)$ is the fixed point of the quadratic* *operator $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ corresponding to each root $a_0 \in \Omega$ of the quartic equation $p(x) = 0$ where*

$$A_0 = \frac{\gamma_2 a_0^2 + \gamma_1 a_0 + \gamma_0}{2\delta_1 a_0 + \delta_0}, \quad B_0 = \frac{(\gamma_2 + 2\delta_1)a_0^2 + (\gamma_1 + \delta_0)a_0 + \gamma_0}{2\delta_1 a_0 + \delta_0}.$$

Proof. In order to find all fixed points of the positive quadratic operator $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$, we have to solve the following system of equations

$$\begin{cases} x_1 = p_{11}x_1^2 + p_{22}x_2^2 + p_{33}x_3^2 + 2p_{12}x_1x_2 + 2p_{13}x_1x_3 + 2p_{23}x_2x_3 \\ x_2 = q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + 2q_{12}x_1x_2 + 2q_{13}x_1x_3 + 2q_{23}x_2x_3 \\ x_3 = r_{11}x_1^2 + r_{22}x_2^2 + r_{33}x_3^2 + 2r_{12}x_1x_2 + 2r_{13}x_1x_3 + 2r_{23}x_2x_3 \end{cases} \tag{6.2}$$

Since $x_3 = 1 - x_1 - x_2$, it is enough to find all solutions (x_1, x_2) of the first and second equations of the system (6.2) which satisfy the conditions $x_1, x_2 > 0$ and $0 < x_1 + x_2 < 1$. By plugging $x_3 = 1 - x_1 - x_2$ into the first and second equations of the system (6.2), we may get the following system of equations

$$\begin{cases} \alpha_{11}x_1^2 + \alpha_{22}x_2^2 + 2\alpha_{12}x_1x_2 + (2\alpha_1 - 1)x_1 + 2\alpha_2 x_2 + \alpha_0 = 0 \\ \beta_{11}x_1^2 + \beta_{22}x_2^2 + 2\beta_{12}x_1x_2 + 2\beta_1 x_1 + (2\beta_2 - 1)x_2 + \beta_0 = 0 \end{cases}$$

Let $x_1 = x$ be a variable and $x_2 = a$ be a parameter. Since $\alpha_{11}\beta_{11} \neq 0$, the last system of equations takes the following form

$$x^2 + \frac{2\alpha_{12}a + 2\alpha_1 - 1}{\alpha_{11}}x + \frac{\alpha_{22}a^2 + 2\alpha_2 a + \alpha_0}{\alpha_{11}} = 0,$$
$$x^2 + \frac{2\beta_{12}a + 2\beta_1}{\beta_{11}}x + \frac{\beta_{22}a^2 + (2\beta_2 - 1)a + \beta_0}{\beta_{11}} = 0.$$

Let

$$A_1 = \frac{2\alpha_{12}a + 2\alpha_1 - 1}{\alpha_{11}}, \; B_1 = \frac{\alpha_{22}a^2 + 2\alpha_2 a + \alpha_0}{\alpha_{11}},$$
$$A_2 = \frac{2\beta_{12}a + 2\beta_1}{\beta_{11}}, \; B_2 = \frac{\beta_{22}a^2 + (2\beta_2 - 1)a + \beta_0}{\beta_{11}}.$$

We then have the following two quadratic equations

$$x^2 + A_1 x + B_1 = 0, \quad x^2 + A_2 x + B_2 = 0.$$

Since $\Delta_1 \neq 0$, we have that $A_1 \neq A_2$ and $B_1 \neq B_2$. This means that for any $\mathbf{x}, \mathbf{y} \in \mathbf{Fix}(\mathcal{Q})$ one has that $x_2 \neq y_2$. Similarly, since $\alpha_{22}\beta_{22}\Delta_2 \neq 0$, one can show that $x_1 \neq y_1$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{Fix}(\mathcal{Q})$.

Therefore, the system of equations (6.2) has a solution (x_1, x_2) with $0 < x_1, x_2 < 1$, $0 < x_1 + x_2 < 1$ if and only if the last two quadratic equations must have a unique common root in $(0, 1)$ for $a \in (0, 1)$. We know [19] that the last two quadratic equations have a unique common root if and only if their resultant is equal to zero, i.e., $(B_2 - B_1)^2 + A_1(B_2 - B_1)(A_1 - A_2) + B_1(A_1 - A_2)^2 = 0$. In this case, $x = \frac{B_2 - B_1}{A_1 - A_2}$ is the unique common root. It is clear that

$$B_2 - B_1 = \frac{1}{\alpha_{11}\beta_{11}}(\gamma_2 a^2 + \gamma_1 a + \gamma_0), \quad A_1 - A_2 = \frac{1}{\alpha_{11}\beta_{11}}(2\delta_1 a + \delta_0).$$

After simple algebra, we get the following quartic equation

$$\lambda_4 a^4 + \lambda_3 a^3 + \lambda_2 a^2 + \lambda_1 a + \lambda_0 = 0. \tag{6.3}$$

and $x = \frac{B_2 - B_1}{A_1 - A_2} = \frac{\gamma_2 a^2 + \gamma_1 a + \gamma_0}{2\delta_1 a + \delta_0}$.

In order to have conditions $0 < x_1, x_2 < 1,\ 0 < x_1 + x_2 < 1$, we have to solve the following system of inequalities

$$\begin{cases} 0 < \dfrac{\gamma_2 a^2 + \gamma_1 a + \gamma_0}{2\delta_1 a + \delta_0} < 1 \\ 0 < a < 1 \\ 0 < \dfrac{\gamma_2 a^2 + \gamma_1 a + \gamma_0}{2\delta_1 a + \delta_0} + a = \dfrac{(\gamma_2 + 2\delta_1)a^2 + (\gamma_1 + \delta_0)a + \gamma_0}{2\delta_1 a + \delta_0} < 1 \end{cases}$$

Let Ω be a set of solutions of the system given above. Hence, the number of solutions (x_1, x_2), $0 < x_1, x_2 < 1$, $0 < x_1 + x_2 < 1$ of the system (6.2) is the same as the number of roots of the quartic equation (6.3) over the set Ω, i.e., $|\mathbf{Fix}(\mathcal{Q})| = \#(p; \Omega)$. Moreover, $(A_0, a_0, 1 - B_0)$ is the fixed point of the quadratic operator corresponding to each roots $a_0 \in \Omega$ of the quartic equation (6.3) where $A_0 = \frac{\gamma_2 a_0^2 + \gamma_1 a_0 + \gamma_0}{2\delta_1 a_0 + \delta_0}$ and $B_0 = A_0 + a_0 = \frac{(\gamma_2 + 2\delta_1)a_0^2 + (\gamma_1 + \delta_0)a_0 + \gamma_0}{2\delta_1 a_0 + \delta_0}$. This completes the proof.

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References

[1] S. Bernstein, *Solution of a mathematical problem connected with the theory of heredity,* Annals of Math. Statistics, **13**(1942), 53–61.

[2] N. Ganikhodjaev, R. Ganikhodjaev, U. Jamilov, *Quadratic stochastic operators and zero-sum game dynamics,* Ergodic Th. and Dynamical Systems, **35**(2015), no. 5, 1443–1473.

[3] N. Ganikhodjaev, M. Saburov, U. Jamilov, *Mendelian and Non-Mendelian quadratic operators,* App. Math. Info. Sci., **7**(2013), no. 5, 1721–1729.

[4] N. Ganikhodjaev, M. Saburov, A.M. Nawi, *Mutation and chaos in nonlinear models of heredity,* The Scientific World J., **2014**(2014), 1–11.

[5] R. Ganikhodzhaev, F. Mukhamedov, U. Rozikov, *Quadratic stochastic operators and processes: Results and Open Problems,* Inf. Dim. Anal. Quan. Prob. Rel. Top., **14**(2011), no. 2, 279–335.

[6] R. Ganikhodjaev, M. Saburov, Kh. Saburov, *Schur monotone decreasing sequences,* AIP Conference Proceedings, **1557**(2013), 108–111.

[7] H. Kesten, *Quadratic transformations: A model for population growth I,* Adv. App. Prob., **2**(1970), 1–82.

[8] A.A. Krapivin, *Fixed points of quadratic operators with positive coefficients,* Theory of Functions, Functional Anal. Appl., **2**5(1975), 62–67.

[9] C.-K. Li, S. Zhang, *Stationary probability vectors of higher-order Markov chains*, Linear Algebra App., **473**(2015), 114–125.

[10] Yu.I. Lyubich, *Mathematical Structures in Population Genetics,* Springer–Verlag, 1992.

[11] F. Mukhamedov, M. Saburov, *On homotopy of volterrian quadratic stochastic operator,* App. Math. Info. Sci., **4**(2010), 47–62.

[12] F. Mukhamedov, M. Saburov, *On dynamics of Lotka–Volterra type operators*, Bull. Malay. Math. Sci. Soc, **37**(2014), 59–64.

[13] F. Mukhamedov, M. Saburov, I. Qaralleh, *On ξ^s-quadratic stochastic operators on two-dimensional simplex and their behavior*, Abstract Applied Anal., **2013**(2013), 942038.

[14] M. Saburov, *Some strange properties of quadratic stochastic Volterra operators,* World Applied Science J., **21**(2013), 94–97.
[15] M. Saburov, *A class of nonergodic Lotka–Volterra operators,* Math. Notes, **97**(2015), no. 5, 759–763.
[16] M. Saburov, *On divergence of any order Cesaro mean of Lotka–Volterra operators,* Ann. Funct. Anal., **6**(2015), no. 4, 247–254.
[17] M. Saburov, Kh. Saburov, *Mathematical models of nonlinear uniform consensus*, Science Asia, **40**(2014), no. 4, 306–312.
[18] M. Saburov, Kh. Saburov, *Reaching a nonlinear consensus: Polynomial stochastic operators*, Inter. J. Cont. Auto. Systems, **12**(2014), no. 6, 1276–1282.
[19] B. Sturmfels, *Solving Systems of Polynomial Equations*, Texas, 2002.
[20] S. Ulam, *A Collection of Mathematical Problems,* New-York, London 1960.

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