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HYERS-ULAM STABILITY OF THE LAPLACE OPERATOR

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Abstract. We investigate the Hyers-Ulam stability of the Laplace operator Δ and of a multiple of it, acting on suitable domains. Moreover, we obtain an explicit representation of the Hyers-Ulam constant.

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1. Introduction

Let A, B be normed spaces and $L : A \to B$ a linear operator. The following definition can be found in [11]; see also the references therein.

Definition 1.1. We say that L is Hyers-Ulam stable with constant $K > 0$ if for each $\varepsilon > 0$ and each $f \in A$ with $\|Lf\| \le \varepsilon$ there exists $g \in A$ such that $Lg = 0$ and $\|f - g\| \le K\varepsilon$.

Obviously, L is HU-stable with constant K if and only if for each $f \in A$ with $\|Lf\| \le 1$ there exists g in the null space $N(L)$ of L such that $\|f - g\| \le K$.

Hyers-Ulam stability is one of the main topics in functional equations theory. A functional equation is called HU-stable if for every approximate solution of the equation there exists an exact solution near it. In the last years in many papers the authors studied the HU-stability of ordinary differential equations and partial differential equations. Some of these papers deal with second order partial differential equations; for more details on stability of functional equations see [8].

In this paper we investigate the HU-stability of the Laplace operator Δ acting on a certain space of functions. The HU-stability of Δ on other spaces was studied with different methods in [4], [5], [7].

We investigate also the HU-stability of the operator $p(x)\Delta$, where

$$p(x) := \frac{1}{2n}(1 - x_1^2 - \cdots - x_n^2)$$

for all x in the unit ball of $\mathbb{R}^n$. This operator is related to the infinitesimal generator of a C_0-semigroup systematically studied in [2], [3], [10] and the references given there. Moreover, we give explicit forms of the involved HU-constants for Δ and $p(x)\Delta$.

2. The HU-stability of Δ

Let $G \subset \mathbb{R}^n$, $n \geq 1$, be an open and connected set bounded by a surface S of class C^1. Consider the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_n^2}$, acting on the space

$$D(\Delta) := \left\{u \in C^2(G) \cap C(\overline{G}) : \Delta u \in C^1(G) \cap C(\overline{G})\right\}.$$

On $C(\overline{G})$ and on its subspaces we consider the supremum norm denoted by $\|.\|$.

In our approach we need the following result; for more details see [9, p. 68].

Theorem 2.1. *For each $f \in C^1(G) \cap C(\overline{G})$ there exists a unique $u \in D(\Delta)$ such that $\Delta u = f$ and $u|_S = 0$.*

Consequently $\Delta : D(\Delta) \to R(\Delta)$ is surjective, where the range of Δ is

$$R(\Delta) = C^1(G) \cap C(\overline{G}).$$

Moreover, according to Theorem 2.1 there exists a unique $q \in D(\Delta)$ such that

$$\begin{cases} \Delta q = -1 \\ q|_S = 0. \end{cases}$$

Theorem 2.2. *The operator $\Delta : D(\Delta) \to R(\Delta)$ is HU-stable with constant $\|q\|$.*

Proof. Let $f \in D(\Delta)$, $\|\Delta f\| \leq 1$. Then $\Delta f \in C^1(G) \cap C(\overline{G})$, and Theorem 2.1 guarantees the existence of $u \in D(\Delta)$ such that $\Delta u = \Delta f$ and $u|_S = 0$.

Then $\|\Delta u\| \leq 1$, which entails $\Delta(u+q) = \Delta u - 1 \leq 0$. Since $(u+q)|_S = 0$, we get $u + q \geq 0$. Similarly, $\Delta(u-q) = \Delta u + 1 \geq 0$ and $(u-q)|_S = 0$ imply $u - q \leq 0$.

So we have $-q \leq u \leq q$, i.e., $\|u\| \leq \|q\|$.

Let $g := f - u$. Then $\Delta g = 0$ and $\|f - g\| = \|u\| \leq \|q\|$, which concludes the proof. □

In particular, if G is the unit ball of $\mathbb{R}^n$, then

$$q(x) = \frac{1}{2n}(1 - x_1^2 - \cdots - x_n^2) \quad \text{and} \quad \|q\| = \frac{1}{2n}.$$

Consequently we have

Corollary 2.3. *If G is the unit ball of $\mathbb{R}^n$, then $\Delta : D(\Delta) \to R(\Delta)$ is HU-stable with constant $\frac{1}{2n}$.*

3. The HU-stability of $p\Delta$

In this section G will be the unit ball of $\mathbb{R}^n$, $n \geq 1$. Consider the function

$$p(x) = \frac{1}{2n}(1 - x_1^2 - \cdots - x_n^2), \ x \in G,$$

and the operator $W := p\Delta$ with domain

$$D(W) := \left\{u \in C^2(G) \cap C(\overline{G}) : \Delta u \in C^1(G) \cap C(\overline{G})\right\}.$$

According to Theorem 2.1, the range of W is $R(W) = \left\{pv : v \in C^1(G) \cap C(\overline{G})\right\}$.

Consider the function

$$h_n(t) := \frac{2n}{t^{n-1}} \int_0^t \frac{s^{n-1}}{1 - s^2} ds, \ t \in [0, 1).$$

For $x \in G$ let $r(x) := (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and $\phi_n(x) := \int_{r(x)}^1 h_n(t)dt$.

Then $p(x)\Delta\phi_n(x) = -1$, and $\phi_n|_S = 0$.

Theorem 3.1. *The operator $W : D(W) \to R(W)$ is HU-stable with constant $\|\phi_n\|$.*

Proof. Let $f \in D(W)$, $\|Wf\| \le 1$. Then $p\Delta f = Wf \in R(W)$, so that $p\Delta f = pv$, with $v \in C^1(G) \cap C(\overline{G})$. This entails $\Delta f = v$, and Theorem 2.1 guarantees the existence of $u \in C^2(G) \cap C(\overline{G})$ with $\Delta u = v$ and $u|_S = 0$. Now $p\Delta f = p\Delta u$, i.e., $\|Wu\| = \|Wf\| \le 1$.

In particular, $p(x)\Delta u(x) \ge -1 = p(x)\Delta\phi_n(x)$, so that $\Delta(u - \phi_n) \ge 0$. Since $(u - \phi_n)|_S = 0$, we get $u - \phi_n \le 0$. Similarly we deduce $u + \phi_n \ge 0$, and finally, $-\phi_n \le u \le \phi_n$. This means that $\|u\| \le \|\phi_n\|$.

Now let $g := f - u$. Then $Wg = Wf - Wu = p\Delta f - p\Delta u = 0$, i.e., $Wg = 0$ and $\|f - g\| = \|u\| \le \|\phi_n\|$. So the proof is finished. □

Now let's evaluate the constant $\|\phi_n\|$. Since h_n is nonnegative, we have

$$\|\phi_n\| = \int_0^1 h_n(t)dt,$$

and it is easy to infer that

$$h_n(t) = 2n\sum_{k=0}^{\infty} \frac{t^{2k+1}}{n+2k}, n \ge 1.$$

Therefore,

$$\|\phi_n\| = n\sum_{k=0}^{\infty} \frac{1}{(k+1)(n+2k)}, n \ge 1.$$

It follows immediately that

$$\|\phi_1\| = \log 4, \quad \|\phi_2\| = \frac{\pi^2}{6}.$$

Let $n \ge 3$. By using [1, (6.3.16)] with $z = \frac{n}{2} - 1$ we get

$$\psi\left(\frac{n}{2}\right) = -\gamma + (n-2)\sum_{k=0}^{\infty} \frac{1}{(k+1)(n+2k)},$$

where ψ is the Digamma function and $\gamma = -\psi(1)$ is Euler's constant.
So we have

$$\|\phi_n\| = \frac{n}{n-2}(\psi(\frac{n}{2}) + \gamma).$$

According to [1, (6.3.4)] and [1, (6.3.6)],

$$\psi\left(n + \frac{1}{2}\right) = 2\sum_{k=1}^{n} \frac{1}{2k-1} - \log 4 - \gamma,$$

$$\psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma.$$

Therefore we get the following result.

Theorem 3.2. *A HU-constant of W is $\|\phi_n\|$, where:*

$$\|\phi_1\| = \log 4, \quad \|\phi_2\| = \frac{\pi^2}{6},$$

$$\|\phi_{2m}\| = \frac{m}{m-1}\sum_{k=1}^{m-1}\frac{1}{k}, \quad m \geq 2,$$

$$\|\phi_{2m+1}\| = \frac{2m+1}{2m-1}\left(2\sum_{k=1}^{m}\frac{1}{2k-1} - \log 4\right), \quad m \geq 1.$$

Remark 3.3. It is known (see, e.g., [6]) that the infimum of the set of HU-constants for an operator is not necessarily a HU-constant. It would be interesting to find, for Δ and $p(x)\Delta$, the corresponding infima and to see if they are HU-constants.

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