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RANDOM FIXED POINT THEOREMS FOR LOWER SEMICONTINUOUS CONDENSING RANDOM OPERATORS

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Abstract. In this paper, we study the existence of the random fixed points for lower semicontinuous condensing random operators defined on Banach spaces. Our results extend corresponding ones present in literature.

Key Words and Phrases: random fixed point theorem, lower semicontinuous operator, condensing operator.
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1. Introduction

Fixed point theory has been developed in the past decades as a very powerful tool used in the majority of mathematical applications. Some of its notable contributions have been extended and generalized to study a wide class of problems arising in mechanics, physics, engineering sciences, economics and equilibrium theory etc. New results concerning the existence of the deterministic or random fixed points were obtained, for instance, in [1-4], [6-8], [10], [12-31].

The main aim of this work is to establish random fixed point theorems for lower semicontinuous condensing random operators defined on Banach spaces. Our research enables us to improve some theorems obtained recently in [8].

The rest of the paper is organized as follows. In the following section, some notational and terminological conventions are given. We also present, for the reader's convenience, some results on continuity and measurability of the operators. The fixed point theorems for lower semicontinuous condensing random operators are stated in Section 3. Section 4 presents the conclusions of our research.

2. Preliminaries

Throughout this paper, we shall use the following notation:

2^D denotes the set of all non-empty subsets of the set D. If $D \subset Y$, where Y is a topological space, $\mathrm{cl}D$ denotes the closure of D.

For the reader's convenience, we review a few basic definitions and results from continuity and measurability of correspondences.

Let X and Y be non-empty sets. The graph of $T : X \to 2^Y$ is the set $\mathrm{Gr}(T) := \{(x, y) \in X \times Y : y \in T(x)\}$. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. T is said to be *lower semicontinuous* if, for each $z \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. The lower sections of T are defined by $T^{-1}(y) := \{x \in X : y \in T(x)\}$ for each $y \in Y$.

Let (X, d) be a metric space. We denote $B(x, r) = \{y \in E : d(y, x) < r\}$. If C is a subset of X, then, we will denote $B(C, r) = \{y \in E : d(y, C) < r\}$, where $d(y, C) = \inf_{x \in C} d(y, x)$.

Let now $(\Omega, \mathcal{F}, \mu)$ be a complete, finite measure space, and Y be a topological space. The correspondence $T : \Omega \to 2^Y$ is said to have a *measurable graph* if $\mathrm{Gr}(T) \in \mathcal{F} \otimes \alpha(Y)$, where $\alpha(Y)$ denotes the Borel σ-algebra on Y and $\otimes$ denotes the product σ-algebra. The correspondence $T : \Omega \to 2^Y$ is said to be *lower measurable* if, for every open subset V of Y, the set $T^{-1}(V) = \{\omega \in \Omega : T(\omega) \cap V \neq \emptyset\}$ is an element of $\mathcal{F}$. This notion of measurability is also called in literature *weak measurability* or just *measurability*, in comparison with strong measurability: the correspondence $T : \Omega \to 2^Y$ is said to be *strong measurable* if, for every closed subset V of Y, the set $\{\omega \in \Omega : T(\omega) \cap V \neq \emptyset\}$ is an element of $\mathcal{F}$. In the case when Y is separable, the strong measurability coincides with the lower measurability.

Recall (see Debreu [5], p. 359) that if $T : \Omega \to 2^Y$ has a measurable graph, then T is lower measurable. Furthermore, if T is closed valued and lower measurable, then $T : \Omega \to 2^Y$ has a measurable graph.

A mapping $T : \Omega \times X \to Y$ is called a *random operator* if, for each $x \in X$, the mapping $T(\cdot, x) : \Omega \to Y$ is measurable. Similarly, a correspondence $T : \Omega \times X \to 2^Y$ is also called a random operator if, for each $x \in X$, $T(\cdot, x) : \Omega \to 2^Y$ is measurable. A measurable mapping $\xi : \Omega \to Y$ is called a *measurable selection of the operator* $T : \Omega \to 2^Y$ if $\xi(\omega) \in T(\omega)$ for each $\omega \in \Omega$. A measurable mapping $\xi : \Omega \to Y$ is called a *random fixed point* of the random operator $T : \Omega \times X \to Y$ (or $T : \Omega \times X \to 2^Y$) if for every $\omega \in \Omega$, $\xi(\omega) = T(\omega, \xi(\omega))$ (or $\xi(\omega) \in T(\omega, \xi(\omega))$).

We will need the following measurable selection theorem in order to prove our results.

Proposition 2.1. (Kuratowski-Ryll-Nardzewski Selection Theorem [9]). *A weakly measurable correspondence with non-empty closed values from a measurable space into a Polish space admits a measurable selector.*

3. Main results

This section is meant to extend some results established in [8]. The main theorems obtained in this paper are Theorem 3.1 and Theorem 3.4, which consider lower semicontinuous condensing random operators defined on Banach spaces. New assumptions which induce the property of $\mathcal{C}$-almost hemicompactness are formulated and used in the statements of our theorems.

Firstly, we recall the definitions of condensing and $\mathcal{C}$-almost hemicompact correspondences.

Let (X, d) be a metric space and E be a non-empty subset of X.

The correspondence $T : E \to 2^X$ is said to be condensing (see [31]), if for each subset C of E such that $\gamma(C) > 0$, one has $\gamma(T(C)) < \gamma(C)$, where $T(C) = \cup_{x\in C}T(x)$ and γ is the Kuratowski measure of noncompactness, i.e., for each bounded subset A of E,

$\gamma(A)$ =inf$\{e > 0 : A$ is covered by a finite number of sets of diameter $\leq e\}$.

If A is not a bound subset of E, we assign $\gamma(A) = \infty$.

T is said to be countably condensing [24] if $T(E)$ is bounded and $\gamma(T(C)) < \gamma(C)$ for all countably bounded sets C of E with $\gamma(C) > 0$.

If Ω is any non-empty set, we say that the operator $T : \Omega \times E \to 2^X$ is condensing if, for each $\omega \in \Omega$, the correspondence $T(\omega, \cdot) : E \to 2^X$ is condensing.

The mapping $T : E \to X$ is said to satisfy condition (A) [23] if for any sequence $(x_n : n \in \mathbb{N})$ in E and $D \in C(E)$ such that $d(x_n; D) \to 0$ and $d(x_n; T(x_n)) \to 0$ as $n \to \infty$, there exists $x_0 \in D$ with $x_0 \in T(x_0)$. The map T is called hemicompact [23] if each sequence $(x_n : n \in \mathbb{N})$ in E has a convergent subsequence whenever $d(x_n; T(x_n)) \to 0$ as $n \to \infty$. We observe that every continuous hemicompact map satisfies condition (A). It is also known (see [24]) that if (X, d) is a Fréchet space, E a closed subset of X and $T : E \to X$ is a countably condensing map, then T is hemicompact.

Let E be a subset of X and $\mathcal{C}$ be a subfamily of 2^E. We say that τ_E is $\sigma-$generated by $\mathcal{C}$ (see [8]), if for each $x \in E$, $\{x\} \in \mathcal{C}$ and for each non-empty open subset A of E, there exists a sequence $(C_n; n \in \mathbb{N})$ in $\mathcal{C}$ such that $A = \cup_{n=0}^{\infty} C_n$. If $(\Omega, \mathcal{F})$ is a measurable space, a correspondence $F : \Omega \to 2^E$ is measurable if $F^{-1}(C) \in \mathcal{F}$ for each $C \in \mathcal{C}$. In particular, if E is separable, τ_E is $\sigma-$generated by all closed balls of E and if E is separable and locally compact, τ_E is $\sigma-$generated by the family of non-empty compact subsets of E.

The correspondence $T : E \to 2^X$ is said to be $\mathcal{C}$-almost hemicompact (see [8]), if τ_E is $\sigma-$generated by $\mathcal{C}$ and for each sequence $(x_n : n \in \mathbb{N})$ in E and $C \in \mathcal{C}$ such that $d(x_n, C) + h_T(x_n) \to 0$ as $n \to \infty$, there exists $x \in C$ such that $h_T(x) = 0$, where $h_T : E \to \mathbb{R}$ is the function defined by $h_T(x) = d(x, T(x))$ for each $x \in E$.

If Ω is any non-empty set, we say that the operator $T : \Omega \times E \to 2^X$ is $\mathcal{C}$-almost hemicompact if, for each $\omega \in \Omega$, the correspondence $T(\omega, \cdot) : E \to 2^X$ is $\mathcal{C}$-almost hemicompact.

The operator $T : \Omega \times E \to 2^X$ is lower semicontinuous if, for each $\omega \in \Omega$, the correspondence $T(\omega, \cdot) : E \to 2^X$ is lower semicontinuous.

We also present the following result in [8], concerning lower semicontinuous, $\mathcal{C}-$almost hemicompact operators, which will be further extended.

Lemma 3.1. (Corollary 3.4 in [8]) *Let* (X, d) be a metric space, E *be a complete and separable subset of* X, $T : \Omega \times E \to 2^X$ *be a random operator and* $\mathcal{C} \subseteq 2^E$. *If for each* $\omega \in \Omega$, $T(\omega, \cdot)$ *is lower semicontinuous,* $\mathcal{C}-$*almost hemicompact and there exists* $x_\omega \in E$ *such that* $x_\omega \in$cl$T(\omega, x_\omega)$, *then, clT has a random fixed point.*

The following lemmas are useful in order to prove Theorem 4.1.

Lemma 3.2. (Lemma 3.7 in [8]) *Let* (X, d) *be a metric space, and* $(x_n; n \in \mathbb{N})$ *and* $(y_n; n \in \mathbb{N})$ *be two sequences in* X *such that* $d(x_n, y_n) \to 0$ as $n \to \infty$. *Then,* $\gamma(A) = \gamma(B)$, *where* $A = \{x_n; n \in \mathbb{N}\}$ and $B = \{y_n; n \in \mathbb{N}\}$ and γ is the Kuratowski measure of noncompactness.

Lemma 3.3. *Let* (X, d) *be a metric space,* E *be a non-empty closed separable subset of* X, $\mathcal{C}$ *be the family of all closed subsets of* E *such that* τ_E *is* $\sigma-$*generated by* $\mathcal{C}$ *and* $T : E \to 2^X$ *be a condensing correspondence which satisfies the following condition:*

$x_0 \notin T(x_0)$ *implies the existence of a real* $r > 0$ *such that* $x_0 \notin B(T(x); r) \cap B(x, r)$ *for each* $x \in B(x_0, r)$.

Then, T *is* $\mathcal{C}-$*almost hemicompact.*

Proof. Let us consider $C \in \mathcal{C}$ and a sequence $(x_n : n \in \mathbb{N})$ in E, for which $d(x_n, C) + d(x_n, T(x_n)) \to 0$ as $n \to \infty$. Let A be $\{x_n : n \in \mathbb{N}\}$ and suppose $\gamma(A) > 0$. Firstly, let us denote for each $n \in \mathbb{N}$, $r_n = d(x_n, C)$. Since $d(x_n, T(x_n)) \to 0$ as $n \to \infty$, there exists a sequence $\{y_n : n \in \mathbb{N}\}$ in X such that for each $n \in \mathbb{N}$, $y_n \in T(x_n)$ and hence, $d(x_n, y_n) \to 0$ as $n \to \infty$. Lemma 3.2 implies $\gamma(A) = \gamma(B)$, where $B = \{y_n : n \in \mathbb{N}\}$. Further, we exploit the fact that T is condensing. Therefore, we obtain $\gamma(\cup_{n\in\mathbb{N}}T(x_n)) < \gamma(A) = \gamma(B)$. We notice that $B \subset \cup_{n\in\mathbb{N}}T(x_n)$ shows that the last assertion is a contradiction, and then, $\gamma(A) = 0$. Consequently, the sequence $(x_n : n \in \mathbb{N})$ has a convergent subsequence $(x_{n_k} : k \in \mathbb{N})$. Let $x_0 \in X$ be $x_0 = \lim_{n_k\to\infty} x_{n_k}$. Since $d(x_0, C) \leq d(x_0, x_{n_k}) + d(x_{n_k}, C)$, $d(x_0, C)$ must be 0 and the closedness of C implies that $x_0 \in C$.

Further we will prove that $x_0 \in T(x_0)$. Let us assume, by contrary, that $x_0 \notin T(x_0)$. Then, according to the hypotheses, there exists $r > 0$ such that $x_0 \notin B(T(x); r) \cap B(x, r)$ for each $x \in B(x_0, r)$. The convergence of $(x_{n_k} : k \in \mathbb{N}\}$ to x_0 implies the existence of a natural number $N(r) \in \mathbb{N}$ such that $x_{n_k} \in B(x_0, r)$ for each $n_k > N(r)$. Consequently, $x_0 \notin B(T(x_{n_k}); r) \cap B(x_{n_k}, r)$ for each $n_k > N(r)$. Since for each $n_k > N(r)$, $x_0 \in B(x_{n_k}, r)$, it follows that if $n_k > N(r)$, $x_0 \notin B(T(x_{n_k}); r)$, that is $d(x_0, T(x_{n_k})) > r$. This fact contradicts $d(x_0, T(x_{n_k})) \to 0$ when $n_k \to \infty$, which is true from the hypotheses and because $x_0 = \lim_{n_k\to\infty} x_{n_k}$. This means that our assumption is false, and it results that $x_0 \in T(x_0)$. We proved that T is $\mathcal{C}-$almost hemicompact. □

Lemma 3.4. *Let* (X, d) *be a metric space,* E *be a non-empty closed separable subset of* X, $\mathcal{C}$ *the family of all closed subsets of* E *such that* τ_E *is* $\sigma-$*generated by* $\mathcal{C}$ *and* $T : E \to 2^X$ *be a correspondence such that* T *and* T^{-1} *have closed values. Therefore, if* $x_0 \notin T(x_0)$, *there exists a real* $r > 0$ *such that* $x_0 \notin B(T(x); r) \cap B(x, r)$ *for each* $x \in B(x_0, r)$. *If, in addition,* T *is condensing, then,* T *is* $\mathcal{C}-$*almost hemicompact.*

Proof. Let us consider $x_0 \in E$ such that $x_0 \notin T(x_0)$. Since $\{x_0\} \cap T^{-1}(x_0) = \emptyset$ and X is a regular space, there exists $r_1 > 0$ such that $B(x_0, r_1) \cap B(T^{-1}(x_0); r_1)) = \emptyset$, and then, $B(x_0, r_1) \cap T^{-1}(x_0) = \emptyset$. Consequently, for each $x \in B(x_0, r_1)$, we have that $x \notin T^{-1}(x_0)$, which is equivalent with $x_0 \notin T(x)$ or $\{x_0\} \cap T(x) = \emptyset$. The closedeness of each $T(x)$ and the regularity of X imply the existence of a real number $r_2 > 0$ such that $B(x_0, r_2) \cap T(x) = \emptyset$ for each $x \in B(x_0, r_1)$, which implies $x_0 \notin B(T(x); r_2)$ for each $x \in B(x_0, r_1)$. Let $r = \min\{r_1, r_2\}$. Hence, $x_0 \notin B(T(x); r)$ for each $x \in B(x_0, r)$, and thus, the conclusion is fulfilled. In view of Lemma 3.3, the last assertion is true. □

The next result, due to Michael, is very important in the theory of continuous selections.

Lemma 3.5. (Michael [11]). *Let X be a T_1, paracompact space. If Y is a Banach space, then each lower semicontinuous convex closed valued correspondence $T : X \to 2^Y$ admits a continuous selection.*

Lemma 3.6. *Let X denote a nonempty closed convex subset of a Hausdorff locally convex topological vector space E. If $T : X \to 2^X$ is condensing, then there exists a nonempty compact convex subset K of X such that $T(x) \subset K$ for each $x \in K$.*

By using the above lemma, we obtain the main result of our paper, that is Theorem 3.1, which states the existence of the random fixed points for lower semicontinuous condensing random operators defined on Banach spaces.

Theorem 3.1. *Let $(\Omega, \mathcal{F})$ be a measurable space, E be a non-empty closed convex and separable subset of a Banach space X and let $T : \Omega \times E \to 2^X$ be a lower semicontinuous and condensing random operator with closed and convex values. Suppose that, for each $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} : X \to 2^E$ is closed valued.*

Then, T has a random fixed point.

Proof. Let $\mathcal{C}$ be a family of all closed subsets of E and fix $\omega \in \Omega$. The correspondence $T(\omega, \cdot) : E \to 2^X$ is condensing and closed valued and $(T(\omega, \cdot))^{-1} : X \to 2^E$ is also closed valued and therefore, by applying Lemma 3.4, we obtain that T is $\mathcal{C}-$almost hemicompact. In order to apply Lemma 3.1, we will prove that the set $F(\omega) := \{x \in E : x \in T(\omega, x)\} \neq \emptyset$.

According to Lemma 3.6, there exists a non-empty compact convex subset $K(\omega)$ of E such that $T(\omega, x) \subset K(\omega)$ for each $x \in K(\omega)$. Lemma 3.5 implies the existence of a continuous function $f_\omega : K(\omega) \to X$ such that $f_\omega(x) \in T(\omega, x)$ for each $x \in K(\omega)$. Since $f_\omega(K(\omega)) \subset K(\omega)$, we can apply the Brouwer-Schauder fixed point theorem and we conclude that there exists $x_\omega = f_\omega(x_\omega) \in T(\omega, x_\omega)$, or, equivalently, $F(\omega) \neq \emptyset$.

All the assumption of Lemma 3.1 are fulfilled and then, T has a random fixed point. □

Remark 3.1. Theorem 3.1 is a stronger result than Theorem 3.9 in [8].

The existence of the random fixed points remains valid if for each $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} : X \to 2^E$ is lower semicontinuous. In this case, we establish Theorem 3.2.

Theorem 3.2. *Let $(\Omega, \mathcal{F})$ be a measurable space, Let $(\Omega, \mathcal{F})$ be a measurable space, E be a non-empty closed convex and separable subset of a Banach space X and let $T : \Omega \times E \to 2^X$ be an operator with closed values.*

Suppose that, for each $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} = U(\omega, \cdot) : X \to 2^E$ is lower semicontinuous, condensing and closed convex valued, such that $U(\cdot, x)$ is measurable for each *$x \in X$.*

Then, T has a random fixed point.

Proof. According to Theorem 3.1, there exists a measurable mapping $\xi : \Omega \to E$ such that for each $\omega \in \Omega$¸ $\xi(\omega) \in (T(\omega,\cdot))^{-1}(\xi(\omega))$, that is, for each $\omega \in \Omega$¸ $\xi(\omega) \in T(\omega, \xi(\omega))$. Therefore, we obtained a random fixed point for T. □

Another main result of this section is Theorem 3.4, which involves the condensing random operators enjoying a property which will be introduced further.

We start with defining the following condition, which we call (*) and which is necessary to prove the existence of random fixed points. For this purpose, we denote $\mathrm{Fix}(T) = \{x \in E : x \in T(x)\}$.

Definition 3.1. Let (X, d) be a complete metric space, E be a non-empty closed separable subset of X, $\mathcal{C}$ be the family of all closed subsets of E and $Z = \{z_n\}$ be a countable dense subset of E. We say that the correspondence $T : E \to 2^X$ satisfies *condition (*)* if, for each $C \in \mathcal{C}$ with the property that $C \cap \mathrm{Fix}(T) \neq \emptyset$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $d(z_{n_k}, C) < 1/k$ and $d(z_{n_k}, T(z_{n_k})) < 1/k$ for each $k \in \mathbb{N}$.

Our work will consider a simpler assumption which implies condition (*). We refer to condition α defined in [20].

Let (X, d) be a metric space and E be a non-empty subset of X. We say that the correspondence $T : E \to 2^X$ *satisfies condition* α (see [20]) if $x_0 \in T(x_0)$ implies that for each $\varepsilon > 0$, there exists an open neighborhood $U_\varepsilon(x_0)$ of x_0 such that $x_0 \in B(T(x); \varepsilon)$ for each $x \in U_\varepsilon(x_0)$. If Ω is a non-empty set, we say that the operator $T : \Omega \times E \to 2^X$ *satisfies condition* α if, for each $\omega \in \Omega$, the correspondence $T(\omega, \cdot) : E \to 2^X$ satisfies condition α.

Next lemma shows that condition α is stronger than condition (*).

Lemma 3.7. *Let (X, d) be a complete metric space, E a non-empty closed separable subset of X and $\mathcal{C}$ be the family of all closed subsets of E. If the correspondence $T : E \to 2^X$ satisfies condition α, then, T satisfies (*).*

Proof. Let $Z = \{z_n\}$ be a countable dense subset of E. Let $C \in \mathcal{C}$ such that $C \cap \mathrm{Fix}(T) \neq \emptyset$. Then, there exists $x_0 \in C$ with the property that $x_0 \in T(x_0)$. According to condition α, for each $k \in \mathbb{N}$, there exists an open neighborhood $U_k(x_0)$ of x_0 such that $x_0 \in B(T(x), 1/k)$ for each $x \in U_k(x_0)$. Then, $x_0 \in B(T(x), 1/k)$ for each $x \in B(x_0, 1/k) \cap U_k(x_0)$ and thus, the intersection $B(x_0, 1/k) \cap U_k(x_0) \cap B(T(x), 1/k)$ is non-empty for each $x \in B(x_0, 1/k) \cap U_k(x_0)$. Since $B(x_0, 1/k) \cap U_k(x_0)$ and $B(T(x), 1/k)$ are open sets, $B(x_0, 1/k) \cap U_k(x_0) \cap B(T(x), 1/k) \neq \{x_0\}$. Therefore, for each $k \in \mathbb{N}$, we can choose $z_{n_k} \in B(x_0, 1/k) \cap E \cap Z$, $z_{n_k} \neq x_0$. Consequently, for each $k \in \mathbb{N}$, $z_{n_k} \in B(C, 1/k) \cap E \cap Z$ and $d(z_{n_k}, T(z_{n_k})) < 1/k$, that is T satisfies (*). □

We establish the following random fixed point theorem.

Theorem 3.3. *Let $(\Omega, \mathcal{F})$ be a measurable space, E be a non-empty closed separable subset of a complete metric space and let $T : \Omega \times E \to 2^X$ be a $\mathcal{C}-$almost hemicompact random operator which enjoys condition (*) and is closed valued. Suppose that, for each $\omega \in \Omega$, the set*

$F(\omega) := \{x \in E : x \in T(\omega, x)\} \neq \emptyset.$

Then, T has a random fixed point.

Proof. Let $\mathcal{C}$ be the family of all closed subsets of E and $Z = \{z_n\}$ be a countable dense subset of E. Let us define $F : \Omega \to 2^E$ by $F(\omega) = \{x \in E : x \in T(\omega, x)\}$. We notice that $F(\omega)$ is non-empty and it is also closed, since $T(\omega, \cdot)$ is $\mathcal{C}-$almost hemicompact.

Let us define $h_T : \Omega \times E \to \mathbb{R}$ by $h_T(\omega, x) = d(x, T(\omega, x))$. The measurability of $T(\cdot, x)$, for each $x \in E$ implies the measurability of $h_T(\cdot, x)$, for each $x \in E$.

We will prove the measurability of F. In order to do this, we consider $C \in \mathcal{C}$, and we denote $D_n = \{x \in E : d(x, C) < 1/n\} \cap Z = B(C, 1/n) \cap Z$ and

$$L(C) := \bigcap_{n=1}^{\infty} \bigcup_{x \in D_n} \{\omega \in \Omega : h_T(\omega, x) < 1/n\}.$$

$L(C)$ is measurable and we will prove further that $F^{-1}(C) = L(C)$.

Firstly, let us consider $\omega \in F^{-1}(C)$ and hence there exists $x_0 \in C$ such that $x_0 \in (T(\omega, \cdot))^{-1}(x_0)$.

Since T satisfies condition (*), for each $k \in \mathbb{N}$, there exists $z_{n_k} \in B(C, 1/k) \cap Z$ such that $d(z_{n_k}, T(\omega, z_{n_k})) < 1/k$. Therefore, $\omega \in L(C)$ and then, $F^{-1}(C) \subseteq L(C)$.

For the inverse inclusion, $L(C) \subseteq F^{-1}(C)$, let us consider $\omega \in L(C)$. Consequently, for each $n \geq 1$, there exists $x_n \in D_n$ such that $h_T(\omega, x_n) < 1/n$ and $d(x_n, C) < 1/n$. The property of C-almost hemicompactness of $T(\omega, \cdot)$ assures the existence of $x \in C$ such that $h_T(x) = 0$. Therefore, $x \in F(\omega) \cap C$ and $\omega \in F^{-1}(C)$.

We proved that for each $C \in \mathcal{C}$, $L(C) = F^{-1}(C)$. Therefore, F is measurable with non-empty closed values, and according to the Kuratowski and Ryll-Nardzewski Proposition 2.1, F has a measurable selection $\xi : \Omega \to E$ such that $\xi(\omega) \in T(\omega, (\xi, \omega))$ for each $\omega \in \Omega$. □

Based on Theorem 3.3 and Lemma 3.4, we obtain the next theorem concerning the condensing random operators which satisfy condition α.

Theorem 3.4. *Let $(\Omega, \mathcal{F})$ be a measurable space, E be a non-empty closed separable subset of a complete metric space and let $T : \Omega \times E \to 2^X$ be a condensing random operator which enjoys condition α and is closed valued, such that for each $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} : X \to 2^E$ is closed valued. Supposing that, for each $\omega \in \Omega$, the set $F(\omega) := \{x \in E : x \in T(\omega, x)\} \neq \emptyset$, then, T has a random fixed point.*

Proof. Since T is condensing and for each $\omega \in \Omega$, $T(\omega, \cdot) : E \to 2^X$ and $(T(\omega, \cdot))^{-1} : X \to 2^E$ are closed valued, then Lemma 3.4 implies that T is $\mathcal{C}$-almost hemicompact. In order to complete the proof, we apply Theorem 3.3. □

Remark 3.2. Random fixed point theorems for multivalued countably condensing random operators have been obtained, for instance, in [4].

4. Concluding remarks

We have proven the existence of random fixed points for condensing and lower semicontinuous random operators defined on Banach spaces. Our study has extended on some results which had already existed in literature. It is an open problem to prove the existence of random fixed points for new types of operators which satisfy weak continuity assumptions.

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