Fixed Point Theory, 19(2018), No. 1, 335-358
DOI 10.24193/fpt-ro.2018.1.26
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

CONVERGENCE ANALYSIS OF COMMON SOLUTION OF CERTAIN NONLINEAR PROBLEMS

F.U. OGBUISI* AND O.T. MEWOMO**

*School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
Durban, South Africa
E-mail: 215082189@stu.ukzn.ac.za fudochukwu@yahoo.com

**School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
Durban, South Africa
E-mail:mewomoo@ukzn.ac.za

Abstract. We introduce an iterative algorithm for approximating a common fixed point of an infinite family of left Bregman strongly nonexpansive mappings which is also a common solution of a finite system of generalized mixed equilibrium problems and a common zero of a finite family of maximal monotone operators in a reflexive real Banach space. A strong convergence theorem is also proved for finding an element in the intersection of the set of solution of a fixed point problem for infinite family of left Bregman strongly nonexpansive mappings, the set of solutions of a system of generalized mixed equilibrium problems and the set of zero points of a finite family of maximal monotone operators in a reflexive real Banach space. The result of this paper complement many related and important results in the literature.

Key Words and Phrases: Bregman distance, Bregman projection, maximal monotone operator, generalized mixed equilibrium problem, resolvent, Legendre function, reflexive real Banach space, zero point.
2010 Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25

1. Introduction

Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Throughout this paper, we shall denote the dual space of E by E^*. The norm and the duality pairing between E and E^* are respectively denoted by $\|.\|$ and $\langle ., . \rangle$. $\mathbb{R}$ stands for the set of real numbers.

Let $T : C \to C$ be a mapping, a point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$.

Let $g : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function and $B : C \to E^*$ be a nonlinear mapping. The *Generalized mixed equilibrium problem* is to find $u \in C$ such that

$$g(u, y) + \langle Bu, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C. \tag{1.1}$$

Denote the set of solutions of the problem (1.1) by $GMEP(g, \varphi, B)$. That is

$$GMEP(g, \varphi, B) = \{u \in C : g(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) \geq 0,\ \forall y \in C\}.$$

If $B = 0$, then the generalized mixed equilibrium problem (1.1) reduces to the following *mixed equilibrium problem*, find $u \in C$ such that

$$g(u, y) + \varphi(y) - \varphi(u) \geq 0,\ \forall y \in C.$$

If $\varphi = 0$, then the generalized mixed equilibrium problem (1.1) becomes the *generalized equilibrium problem*, find $u \in C$ such that

$$g(u, y) + \langle Bu, y - u\rangle \geq 0,\ \forall y \in C.$$

Again if $B = \varphi = 0$, then the generalized mixed equilibrium problem (1.1) becomes the *equilibrium problem*, find $u \in C$ such that

$$g(u, y) \geq 0,\ \forall y \in C.$$

Equilibrium problems and their generalizations are well known to have been important tools for solving problems arising in the fields of linear or nonlinear programming, variational inequalities, complementary problems, optimization problems, fixed point problems and have been widely applied to physics, structural analysis, management sciences and economics, etc (see, for example [9, 26, 40, 41]).

In solving equilibrium problem (1.1), the bifunction g is said to satisfy the following conditions:
(A1) $g(x, x) = 0$ for all $x \in C$;
(A2) g is monotone, i.e., $g(x, y) + g(y, x) \geq 0$ for all $x, y \in C$;
(A3) for each $x, y \in C$, $\lim_{t\to 0} g(tz + (1 - t)x; y) \leq g(x; y)$;
(A4) for each $x \in C$; $y \mapsto g(x, y)$ is convex and lower semicontinuous.

Cholamjiak and Suantai [25] proposed a hybrid iterative scheme for finding a common element in the solution set of system of equilibrium problems and the common fixed points set of an infinitely countable family of quasi-nonexpansive mappings and prove the following strong convergence theorem.

Theorem 1.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed and convex subset of E. Let $\{f_j\}_{j=1}^{M}$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions $(A1) - (A4)$, and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself. Assume that $F := (\cap_{i=1}^{\infty} F(T_i)) \cap (\cap_{j=1}^{M} EP(f_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_0} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n), \\ u_{n,i} = T^{f_M}_{r_{M,n}} T^{f_{M-1}}_{r_{M-1,n}} \ldots T^{f_1}_{r_{1,n}} y_{n,i}, \\ C_{n+1} = \{z \in C_n : \sup_{i\geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ n \geq 0. \end{cases} \tag{1.2}$$

Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, ..., M$ are sequences which satisfy the following conditions:
(B1) $\limsup_{n\to\infty} \alpha_n < 1$,
(B2) $\liminf_{n\to\infty} r_{j,n} > 0$.
Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Many authors have contributed in developing efficient and implementable algorithms for solving equilibrium problems and some of their generalizations, (see, for example,[1, 26, 33] and the references therein).

Let $A : E \to 2^{E^*}$ be a set valued mapping. The domain of A is the set $domA = \{x \in E : Ax \neq \emptyset\}$ and the graph of A is the set $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. A set-valued mapping A is said to be monotone if $\langle x^* - y^*, x - y\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in G(A)$. If in addition that the graph of A is not contained in the graph of any other monotone operator, then A is said to be a maximal monotone operator on E. It is well known that if A is maximal monotone , then the set $A^{-1}(0^*) = \{z \in E : 0^* \in Az\}$ is closed and convex.
The problem of finding the zeroes of a maximal monotone operator is very vital in optimization, because it can be reduced to a convex minimization problem and variational inequality problem.

Rockafeller [49], motivated by the work of Martinet [35], introduced in a Hilbert space H the following proximal point iterative algorithm:

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = J_{\lambda_n} x_n, \ \forall n \geq 1, \end{cases} \tag{1.3}$$

where $\{\lambda_n\} \subset (0, \infty)$ and J_{λ_n} is the resolvent of A defined by $J_\lambda = (I + \lambda A)^{-1}$ for all $\lambda > 0$, and A is a maximal monotone operator on H. He proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to an element in $A^{-1}(0)$ provided $\liminf_{n\to\infty} \lambda_n > 0$.
A weak convergence result was also obtained by Kamimura and Takahashi [30] in a real Hilbert space with the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n} x_n, \ \forall n \geq 1,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ under some suitable conditions on $\{\lambda_n\} \subset (0, \infty)$.
Inspired by the result of Kamimura and Takahashi [30], Kohsaka and Takahashi [32], in reflexive Banach space introduced the following iterative algorithm:

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(J_{\lambda_n} x_n)), \ \forall n \geq 1,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$, $f : E \to \mathbb{R}$ is a Bregman function and $J_\lambda = (\nabla f + \lambda A)^{-1}\nabla f$ for all $\lambda > 0$. They obtained a weak convergence result with the proposed algorithm.
For some other existing results for finding zero points of maximal monotone operators see for example [13, 14, 24, 29, 38, 44, 50] and some of the references therein.

In 1967, Bregman [12] introduced a nice and effective method for using the so called Bregman distance function D_f (see Definition 2.1 in Section 2) in the process of designing and analysing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms for solving equilibria, for approximating equilibria, and for computing fixed points of nonlinear mappings (see, e.g.,[2, 3, 4, 5, 8, 12, 16, 17, 19, 20, 21, 27, 31, 42, 43, 44, 45, 46, 47, 48, 51, 23] and the references therein).

Inspired and motivated by the researches going on in this direction, we propose an iterative algorithm for approximating a fixed point of an infinite family of left Bregman strongly nonexpansive mappings, which is a common solution to a finite system of equilibrium problems and also a zero point of a finite family of maximal monotone operators in a reflexive real Banach space and prove its strong convergence in this work.

2. Preliminaries

In this section, we present the basic notions and facts that are needed in the sequel. The pairing $\langle \xi, x\rangle$ is defined by the action of $\xi \in E^*$ at $x \in E$, that is, $\langle \xi, x\rangle := \xi(x)$. The domain of a convex function $f : E \to \mathbb{R}$ is defined to be

$$domf := \{x \in E : f(x) < +\infty\}.$$

When $domf \neq \emptyset$, we say that f is proper. The *Fenchel conjugate* function of f is the convex function $f^* : E^* \to \mathbb{R}$ defined by

$$f^*(\xi) = sup\{\langle \xi, x\rangle - f(x) : x \in E\}.$$

It is not difficult to check that whenever f is proper and lower semicontinuous, so is f^*. The function f is said to be *cofinite* if $domf^* = E^*$.
Let $x \in int(domf)$, for any $y \in E$, we define the directional derivative of f at x by

$$f^o(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

If the limit as $t \to 0^+$ in (2.1) exists for each y, then the function f is said to be *Gâteaux differentiable* at x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y\rangle := f^o(x, y)$ for all $y \in E$ (see [22]). The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at each $x \in int(domf)$. When the limit as $t \to 0$ in (2.1) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is *Fréchet* differentiable at x. Throughout this paper, $f : E \to \mathbb{R}$ is always an admissible function, that is, a proper, lower semicontinuous, convex and Gâteaux differentiable function. Under these conditions we know that f is continuous in $int(domf)$ (see [6]).
The function f is said to be *Legendre* if it satisfies the following two conditions.
(L1) $int(domf) \neq \emptyset$, and the subdifferential ∂f is single-valued on its domain.
(L2) $int(domf^*) \neq \emptyset$, and ∂f^* is single-valued on its domain.
The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [6]. Their definition is equivalent to conditions (L1) and (L2) because the space E is assumed to be reflexive (see [6], Theorems 5.4 and 5.6, page 634). It is well known that in reflexive Banach spaces $\nabla f = (\nabla f^*)^{-1}$ (see [10], page 83). When this fact is combined with conditions (L1) and (L2), we obtain

$$ran\nabla f = dom\nabla f^* = int(domf)^* \text{and } ran\nabla f^* = dom\nabla f = int(domf).$$

It also follows that f is Legendre if and only if f^* is Legendre (see [6], Corollary 5.5, page 634) and that the functions f and f^* are Gateaux differentiable and strictly convex in the interior of their respective domains.

When the Banach space E is smooth and strictly convex, in particular, a Hilbert space, the function $\left(\frac{1}{p}\right)\|.\|^p$ with $p \in (1, \infty)$ is Legendre (cf. [6], Lemma 6.2, page 639). For examples and more information regarding Legendre functions, see, for instance, [5, 6].

Definition 2.1. The bifunction $D_f : domf \times int(domf) \to [0; +\infty)$ defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{2.2}$$

is called the *Bregman distance* (cf. [12, 22]).

The Bregman distance does not satisfy the well-known properties of a metric, but it does have the following important property, which is called the three point identity: for any $x \in domf$ and $y, z \in int(domf)$

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{2.3}$$

The modulus of total convexity of f is the bifunction $v_f : int(domf) \times [0, +\infty) \to [0, +\infty]$ which is defined by

$$v_f(x,t) := inf\{D_f(y,x) : y \in domf,\ \|y - x\| = t\rangle.$$

The function f is said to be *totally convex at a point* $x \in int(domf)$ if $v_f(x;t) > 0$ whenever $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point $x \in int(domf)$. This property is less stringent than uniform convexity (see [16], Section 2.3, page 92). Examples of totally convex functions can be found, for instance, in [11, 16, 19].

We remark that f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [19], Theorem 2.10, page 9).

The Bregman projection (cf. [12]) with respect to f of $x \in int(domf)$ onto a nonempty, closed and convex set $C \subset int(domf)$ is defined as the necessarily unique vector $Proj^f_C(x) \in C$ which satisfies

$$D_f(Proj^f_C(x), x) = inf\{D_f(y,x) : y \in C\}. \tag{2.4}$$

Let C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. It is known from [19] that $z = Proj^f_C x$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \le 0$ for all $y \in C$. We also have

$$D_f(y, Proj^f_C(x)) + D_f(Proj^f_C(x), x) \le D_f(y,x),\ \forall x \in E, y \in C. \tag{2.5}$$

Similar to the metric projection in Hilbert spaces, the Bregman projection with respect to totally convex and Gâteaux differentiable functions has a variational characterization (cf. [19], Corollary 4.4, page 23).

Proposition 2.2. (see [43]) (Characterization of Bregman Projections) *Suppose that $f : E \to \mathbb{R}$ is totally convex and Gâteaux differentiable in $int(domf)$. Let $x \in int(domf)$ and let $C \subset int(domf)$ be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent.*

(i) The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f.

(ii) The vector $\hat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \ge 0,\ \forall y \in C.$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x) \ \forall y \in C.$$

Recall that the function f is said to be sequentially consistent [7] *if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first is bounded,*

$$\lim_{n\to\infty} D_f(x_n,y_n) = 0 \Rightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.6}$$

The resolvent of a bifunction $g : C \times C \to \mathbb{R}$ (see, [45]*) is the operator $Res_g^f : E \to C$ defined by*

$$Res_g^f(x) = \{z \in C : g(z,y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \ \forall y \in C\}. \tag{2.7}$$

For any $x \in E$, there exists $z \in C$ such that $z = Res_g^f(x)$, see [45].
Let C be a convex subset of $int(domf)$ and let T be a self-mapping of C. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Recall that the Bregman distance is not symmetric, we define the following operators.

Definition 2.3. A mapping T with a nonempty asymptotic fixed point set is said to be:
(i) *left Bregman strongly nonexpansive* (see [37]) with respect to a nonempty $\hat{F}(T)$ if

$$D_f(p,Tx) \leq D_f(p,x) \ \forall x \in C, p \in \hat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to\infty} (D_f(p,x_n) - D_f(p,Tx_n)) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(Tx_n,x_n) = 0.$$

According to Martín-Márquez et al. [36], a left Bregman strongly nonexpansive mapping T with respect to a nonempty $\hat{F}(T)$ is called *strictly left Bregman strongly nonexpansive mapping.*
(ii) An operator $T : C \to int(domf)$ is said to be *left Bregman firmly nonexpansive* (L-BFNE) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle$$

for any $x, y \in C$, or equivalently,

$$D_f(Tx,Ty) + D_f(Ty,Tx) + D_f(Tx,x) + D_f(Ty,y) \leq D_f(Tx,y) + D_f(Ty,x).$$

See [7, 11, 46], for more information and examples of L-BFNE operators (operators in this class are also called D_f -firm and BFNE). For two recent studies of the existence and approximation of fixed points of left Bregman firmly nonexpansive operators, see [37, 46]. It is also known that if T is left Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then $F(T) = \hat{F}(T)$ and $F(T)$ is closed and convex (see [46]). It also follows that every left Bregman firmly nonexpansive mapping is left Bregman strongly nonexpansive with respect to $F(T) = \hat{F}(T)$.

Let $V_f : E \times E^* \to [0, \infty)$ associated with f be defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \forall x \in E, x^* \in E^*.$$

Observe that V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \forall x \in E, x^* \in E^*$.

Let $f : E \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. In addition, if $f : E \to (-\infty; +\infty]$ is a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper *weak** lower semi-continuous and convex function (see [39]). Hence V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i)) \leq \sum_{i=1}^{N} t_i D_f(z, x_i) \tag{2.8}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Let A be a maximal monotone operator, the resolvent of A denoted by $Res_A^f : E \to 2^E$, is defined as follows [7]:

$$Res_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

It is known that $F(Res_A^f) = A^{-1}(0^*)$, and Res_A^f is single valued (see [7]). If f is Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E, then $\hat{F}(Res_A^f) = F(Res_A^f)$ (see [7]). The Yosida approximation $A_\lambda : E \to E,\ \lambda > 0$ is also defined by

$$A_\lambda(x) = \frac{1}{\lambda}(\nabla f(x) - \nabla f(Res_{\lambda A}^f(x)))$$

for all $x \in E$. From Proposition 2.7 [44], it is known that $(Res_{\lambda A}^f(x)), A_\lambda(x)) \in G(A)$, and $0^* \in Ax$ if and only if $0^* \in A_\lambda x$ for all $x \in E$ and $\lambda > 0$.

The following lemmas are very useful in establishing our main results.

Lemma 2.4. (Reich and Sabach [43]) *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.5. (Butnariu and Iusem [16]) *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 2.6. (Reich and Sabach [44]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^\infty$ is bounded, then the sequence $\{x_n\}_{n=1}^\infty$ is also bounded.*

Lemma 2.7. (Reich and Sabach [45]) *Let $f : E \to (-\infty, +\infty)$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of E. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), then,*

1. *Res_g^f is single-valued;*
2. *Res_g^f is a Bregman firmly nonexpansive mapping;*
3. *$F(Res_g^f) = EP(g)$;*
4. *$EP(g)$ is a closed and convex subset of C;*

5. for all $x \in E$ and $q \in F(Res_g^f)$,

$$D_f(q, Res_g^f(x)) + D_f(Res_g^f(x), x) \leq D_f(q, x).$$

Lemma 2.8. (Xu [53]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relations:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \geq 0,$$

where,
(i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$;
(ii) $\limsup \sigma_n \leq 0$;
(iii) $\gamma_n \geq 0 (n \geq 0), \sum \gamma_n < \infty$.
Then $\{a_n\} \to 0$, as $n \to \infty$.
Lemma 2.9. (Mainge [34]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.
Lemma 2.10. (Suantai et al.[52]) *Let E be a reflexive real Banach space. Let C be a nonempty, closed and convex function of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Suppose T is a left Bregman strongly nonexpansive mappings of C into E such that $F(T) = \hat{F}(T) \neq 0$. If $\{x_n\}_{n=0}^{\infty}$ is bounded sequence such that $x_n - Tx_n \to 0$ and $z := \overleftarrow{Proj}_{\Omega}^f u$, then*

$$\limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

Lemma 2.11. ([44]) *Let $A : E \to 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0^*) = \emptyset$. Then*

$$D_f(p, Res_{\lambda A}^f(x) + D_f(Res_{\lambda A}^f(x), x) \leq D_f(p, x)$$

for all $\lambda > 0, p \in A^{-1}(0^*)$ and $x \in E$.

3. Main results

Theorem 3.1. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^{\infty}$ be an infinite family of left Bregman strongly nonexpasive mappings from C into itself and $F(T_j) = \hat{F}(T_j),\ \forall j \geq 1$. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be bifunctions satisfying conditions $(A1) - (A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ be proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ and $A_i : E \to 2^{E^*} (i = 1, 2, ..., N)$ be maximal monotone operators, such* *that* $\Omega := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \cap (\cap_{i=1}^{N} A_i^{-1}(0)) \neq \emptyset$. *Then the sequence* $\{x_n\}$ *generated for arbitrary* $u,\ x_1 \in E$ *by*

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ \ldots \circ Res^f_{G_2} \circ Res^f_{G_1} y_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)),\ \forall n \geq 1, \end{cases} \tag{3.1}$$

converges strongly to a point $p = \overleftarrow{P}roj^f_\Omega u \in \Omega$, *where*

$$G(x,y) := g(x,y) + \langle Bx, y-x \rangle + \varphi(y) - \varphi(x)$$

and the sequences $\alpha_n, \beta_n, \gamma_{nj}$ *and* λ_n *satisfy the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\alpha_n + \beta_n + \displaystyle\sum_{j=1}^{\infty} \gamma_{nj} = 1$;

(iv) $0 < a < \beta_n,\ \displaystyle\sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;

(v) $\liminf\limits_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

A prototype example of the control sequences are:

$$\alpha_n = \frac{1}{n+6},\ \beta_n = \frac{n^2+6n+9}{(n+6)(n+3)} \text{ and } \gamma_{nj} = \frac{1}{2^j(n+3)}.$$

Proof. It is known (see [54]) that the function

$$G(x,y) := g(x,y) + \langle Bx, y-x \rangle + \varphi(y) - \varphi(x)$$

satisfies $(A1) - (A4)$ and $GMEP(g, \varphi, B)$ is closed and convex.
For any $x^* \in \Omega$, then from (3.1), we have that

$$\begin{aligned} D_f(x^*, y_n) &= D_f(x^*, Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n) \\ &\leq D_f(x^*, Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n) \\ &\vdots \\ &\leq D_f(x^*, x_n). \end{aligned} \tag{3.2}$$

Also from (3.1), we have

$$\begin{aligned} D_f(x^*, u_n) &= D_f(x^*, Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ \ldots \circ Res^f_{G_2} \circ Res^f_{G_1} y_n) \\ &\leq D_f(x^*, Res^f_{G_{N-1}} \circ \ldots \circ Res^f_{G_2} \circ Res^f_{G_1} y_n) \\ &\vdots \\ &\leq D_f(x^*, y_n). \end{aligned} \tag{3.3}$$

Again from (2.8), (3.1), (3.2) and (3.3), we have

$$\begin{aligned}
D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n))) \\
&\leq \alpha_n D_f(x^*, u) + \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, T_j(u_n)) \\
&\leq \alpha_n D_f(x^*, u) + \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, u_n) \\
&= \alpha_n D_f(x^*, u) + (\beta_n + \sum_{j=1}^{\infty} \gamma_{nj}) D_f(x^*, u_n) \\
&\leq \alpha_n D_f(x^*, u) + (\beta_n + \sum_{j=1}^{\infty} \gamma_{nj}) D_f(x^*, y_n) \\
&\leq \alpha_n D_f(x^*, u) + (\beta_n + \sum_{j=1}^{\infty} \gamma_{nj}) D_f(x^*, x_n) \\
&= \alpha_n D_f(x^*, u) + (1 - \alpha_n) D_f(x^*, x_n) \\
&\leq \max\{D_f(x^*, u), D_f(x^*, x_n)\} \\
&\vdots \\
&\leq \max\{D_f(x^*, u), D_f(x^*, x_1)\}. \qquad (3.4)
\end{aligned}$$

Therefore $\{D_f(x^*, x_n)\}$ is bounded and so also are $\{D_f(x^*, u_n)\}$ and $\{D_f(x^*, y_n)\}$, and consequently, we have that the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are bounded. Moreover,

$$\begin{aligned}
D_f(x^*, u_{n+1}) &\leq D_f(x^*, x_{n+1}) \\
&= V_f(x^*, \alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)) \\
&\leq V_f(x^*, \alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n) \\
&\quad - \alpha_n(\nabla f(u) - \nabla f(x^*))) - \langle \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) \\
&\quad + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)) - x^*, -\alpha_n(\nabla f(u) - \nabla f(x^*))\rangle \\
&= V_f(x^*, \alpha_n \nabla f(x^*) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)) \\
&\quad + \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= D_f(x^*, \nabla f^*(\alpha_n \nabla f(x^*) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)) \\
&+ \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*) \rangle \\
&\leq \alpha_n D_f(x^*, x^*) + \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, T_j u_n) \\
&+ \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*) \rangle \\
&= \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, T_j u_n) \\
&+ \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*) \rangle \\
&\leq \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, u_n) \\
&+ \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*) \rangle \\
&= (1 - \alpha_n) D_f(x^*, u_n) + \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*) \rangle. \qquad (3.5)
\end{aligned}
$$

We now consider two cases to obtain strong convergence.
Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(x^*, u_n)\}_{n=1}^{\infty}$ is monotonically nonincreasing. Then $\{D_f(x^*, u_n)\}_{n=1}^{\infty}$ converges and

$$
D_f(x^*, u_{n+1}) - D_f(x^*, u_n) \to 0, n \to \infty.
$$

Let $s_n := \nabla f^*\big(\frac{\beta_n}{1-\alpha_n} \nabla f(u_n) + \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} \nabla f(T_j u_n)\big)$. Then,

$$
\begin{aligned}
D_f(x^*, s_n) &= D_f(x^*, \nabla f^*\big(\frac{\beta_n}{1-\alpha_n} \nabla f(u_n) + \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} \nabla f(T_j u_n)\big) \\
&\leq \frac{\beta_n}{1-\alpha_n} D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} D_f(x^*, T_j u_n) \\
&\leq \frac{\beta_n}{1-\alpha_n} D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} D_f(x^*, u_n) \\
&\leq \frac{\beta_n + \sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} D_f(x^*, u_n). \qquad (3.6)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0 &\leq D_f(x^*, u_n) - D_f(x^*, s_n) \\
&= D_f(x^*, u_n) - D_f(x^*, u_{n+1}) + D_f(x^*, u_{n+1}) - D_f(x^*, s_n) \\
&\leq D_f(x^*, u_n) - D_f(x^*, u_{n+1}) + D_f(x^*, x_{n+1}) - D_f(x^*, s_n) \\
&\leq D_f(x^*, u_n) - D_f(x^*, u_{n+1}) + \alpha_n D_f(x^*, u) \\
&\quad + (1-\alpha_n) D_f(x^*, s_n) - D_f(x^*, s_n) \\
&= D_f(x^*, u_n) - D_f(x^*, u_{n+1}) \\
&\quad + \alpha_n (D_f(x^*, u) - D_f(x^*, s_n)) \to 0, n \to \infty. \qquad (3.7)
\end{aligned}
$$

Furthermore,

$$\begin{aligned} D_f(x^*, s_n) &\leq \frac{\beta_n}{1-\alpha_n} D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} D_f(x^*, T_j u_n) \\ &= (1 - \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n}) D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} D_f(x^*, T_j u_n) \\ &= D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} (D_f(x^*, T_j u_n) - D_f(x^*, u_n)). \end{aligned} \tag{3.8}$$

Therefore from (3.8), we have

$$\frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} (D_f(x^*, u_n) - D_f(x^*, T_j u_n)) \leq D_f(x^*, u_n) - D_f(x^*, s_n) \to 0, n \to \infty. \tag{3.9}$$

Since T_j is left Bregman strongly nonexpansive, we obtain that

$$\lim_{n\to\infty} D_f(u_n, T_j u_n) = 0,$$

which implies that

$$\lim_{n\to\infty} ||u_n - T_j u_n|| = 0. \tag{3.10}$$

Since $\{u_n\}$ is bounded and E is a reflexive Banach space, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ that converges weakly to $p \in C$. It then follows from (3.10) that $p \in \cap_{j=1}^{\infty} F(T_j)$, since $F(T_j) = \hat{F}(T_j)$.

We next show that $p \in \cap_{k=1}^{\infty} EP(G_k) = \cap_{k=1}^{\infty} GMEP(F_k, \varphi_k, B_k)$.

Denote $\Theta_k = Res^f_{G_k} \circ Res^f_{G_{k-1}} \circ, ..., \circ Res^f_{G_1}$ for $k = 1, 2, ...N$ and $\Theta_0 = I$. We note that $u_n = \Theta_N y_n$. Now, by using the fact that $Res^f_{G_k}, k = 1, 2, \cdots, N$ is properly left Bregman nonexpansive mapping, we have

$$\begin{aligned} D_f(x^*, u_n) &= D_f(x^*, \Theta_N y_n) \\ &= D_f(x^*, Res^f_{G_N} \Theta_{N-1} y_n) \\ &\leq D_f(x^*, \Theta_{N-1} y_n) \leq \cdots \leq D_f(x^*, y_n) \\ &\leq D_f(x^*, x_n). \end{aligned} \tag{3.11}$$

Since $x^* \in EP(G_N) = F(Res^f_{G_n})$, then from Lemma 2.7, (3.3) and (3.11), we have

$$\begin{aligned} D_f(u_n, Res^f_{G_N} \Theta_{N-1} y_n) &= D_f(Res^f_{G_N} \Theta_{N-1} y_n), \Theta_{N-1} y_n) \\ &\leq D_f(x^*, \Theta_{N-1} y_n) - D_f(x^*, u_n) \\ &\leq D_f(x^*, y_n) - D_f(x^*, u_n) \\ &\leq D_f(x^*, x_n) - D_f(x^*, u_n) \\ &\leq (1 - \alpha_{n-1}) D_f(x^*, u_{n-1}) \\ &\quad + \alpha_{n-1} \langle x_n - x^*, \nabla f(u) - \nabla f(x^*) \rangle - D_f(x^*, u_n) \\ &\leq \alpha_{n-1} M \\ &\quad + D_f(x^*, u_{n-1}) - D_f(x^*, u_n) \to 0, n \to \infty, \end{aligned} \tag{3.12}$$

where $M > 0$ is such that $D_f(x^*, u_{n-1}) + \langle x_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle \leq M$.
Therefore,

$$\lim_{n\to\infty} D_f(\Theta_N y_n, \Theta_{N-1} y_n) = \lim_{n\to\infty} D_f(u_n, \Theta_{N-1} y_n) = 0.$$

From Lemma 2.5, we have

$$\lim_{n\to\infty} ||\Theta^N y_n - \Theta^{N-1} y_n|| = \lim_{n\to\infty} ||u_n - \Theta^{N-1} y_n|| = 0. \tag{3.13}$$

Thus, we have from (3.13) that

$$\lim_{n\to\infty} ||\nabla f(\Theta^N y_n) - \nabla f(\Theta^{N-1} y_n)|| = 0. \tag{3.14}$$

Again, since $x^* \in EP(G_{N-1}) = F(Res^f_{G_{N-1}})$, it follows from (3.11) and Lemma 2.7 that

$$\begin{aligned} D_f(\Theta^{N-1} y_n, \Theta^{N-2} y_n) &= D_f(Res^f_{G_{N-1}} \Theta^{N-2} y_n, \Theta^{N-2} y_n) \\ &\leq D_f(x^*, \Theta^{N-2} y_n) - D_f(x^*, \Theta^{N-1} y_n) \\ &\leq D_f(x^*, y_n) - D_f(x^*, u_n) \\ &\leq D_f(x^*, x_n) - D_f(x^*, u_n) \\ &\leq \alpha_{n-1} M \\ &\quad + D_f(x^*, u_{n-1}) - D_f(x^*, u_n) \to 0, n \to \infty. \end{aligned} \tag{3.15}$$

That is,

$$\lim_{n\to\infty} D_f(\Theta^{N-1} y_n, \Theta^{N-2} y_n) = 0.$$

Hence from Lemma 2.5, we have

$$\lim_{n\to\infty} ||\Theta^{N-1} y_n - \Theta^{N-2} y_n|| = 0, \tag{3.16}$$

and consequently we have

$$\lim_{n\to\infty} ||\nabla f(\Theta^{N-1} y_n) - \nabla f(\Theta^{N-2} y_n)|| = 0. \tag{3.17}$$

In a similar way, we can verify that

$$\lim_{n\to\infty} ||\Theta^{N-2} y_n - \Theta^{N-3} y_n|| = \cdots = \lim_{n\to\infty} ||\Theta^1 y_n - y_n|| = 0. \tag{3.18}$$

It is now easily seen from (3.13),(3.16) and (3.18), that

$$\lim_{n\to\infty} ||\Theta^k y_n - \Theta^{k-1} y_n|| = 0, k = 1, 2, \cdots, N. \tag{3.19}$$

and

$$\lim_{n\to\infty} ||u_n - y_n|| = 0.$$

Now since $u_{n_j} \rightharpoonup p$ and $\lim_{n\to\infty} ||u_n - y_n|| = 0$, we have that $y_{n_j} \rightharpoonup p$. Also from (3.13), (3.16), (3.18) and $y_{n_j} \rightharpoonup p$, we have that $\Theta^k y_{n_j} \rightharpoonup p, j \to \infty$, for each $k = 1, 2, \cdots, N$. Again using (3.19), we get that

$$\lim_{n\to\infty} ||\nabla f(\Theta^k y_n) - \nabla f(\Theta^{k-1} y_n)|| = 0, k = 1, 2, \cdots, N. \tag{3.20}$$

Therefore by (2.7), we have that for each $k = 1, 2, \cdots, N$,

$$G_k(\Theta^k y_{n_j}, y) + \langle y - \Theta^k y_{n_j}, \nabla f(\Theta^k y_{n_j}) - \nabla f(\Theta^{k-1} y_{n_j})\rangle \geq 0, \ \forall y \in C.$$

Again using (A2), we obtain

$$\langle y-\Theta^k y_{n_j}, \nabla f(\Theta^k y_{n_j})-\nabla f(\Theta^{k-1}y_{n_j})\rangle \geq G_k(y,\Theta^k y_{n_j}). \tag{3.21}$$

Thus, a combination of (A4),(3.20),(3.21) and $\Theta^k y_{n_j} \rightharpoonup p, j\to\infty$, gives us that for each $k=1,2,\cdots,N$,

$$G_k(y,p)\leq 0,\ \forall y\in C.$$

Then for fixed $y\in C$, let $z_{t,y}:=ty+(1-t)p$ for all $t\in(0,1]$. This implies that $z_{t,y}\in C$ and further yields that $G_k(z_{t,y},p)\leq 0$. It then follows from (A1) and (A4) that

$$\begin{aligned} 0 &= G_k(z_{t,y},z_{t,y}) \\ &\leq tG_k(z_{t,y},y)+(1-t)G_k(z_{t,y},p) \\ &\leq tG_k(z_{t,y},y). \end{aligned}$$

Hence, from condition (A3), we obtain $G_k(p,y)\geq 0,\ \forall y\in C$, which implies that

$$p\in \cap_{k=1}^N EP(G_k).$$

Next, we show that $p\in\cap_{i=1}^N A_i^{-1}(0)=\cap_{i=1}^N F(Res^f_{\lambda_n^i A_i})$.
Set $\Phi^i = Res^f_{\lambda_n^i A_i}\circ Res^f_{\lambda_n^{i-1}A_{i-1}}\circ\cdots\circ Res^f_{\lambda_n^2 A_2}\circ Res^f_{\lambda_n^1 A_1}$, for each $i=1,2,\cdots,N$, and $\Phi^0=I$. We note that $y_n=\Phi^i x_n$.
Since $x^*\in A_N^{-1}(0)$, by Lemma 2.11, we have

$$\begin{aligned} D_f(y_n,\Phi^{N-1}(x_n)) &\leq D_f(x^*,\Phi^{N-1}(x_n))-D_f(x^*,y_n) \\ &\leq D_f(x^*,x_n)-D_f(x^*,y_n) \\ &= (1-\alpha_{n-1})D_f(x^*,u_{n-1}) \\ &\quad +\alpha_{n-1}\langle z_{n-1}-x^*,\nabla f(u)-\nabla f(x^*)\rangle - D_f(x^*,y_n) \\ &\leq \alpha_{n-1}M_1+D_f(x^*,u_{n-1}-D_f(x^*,u_n)\to 0, n\to\infty \end{aligned} \tag{3.22}$$

where M_1 is such that $D_f(x^*,u_{n-1})+\alpha_{n-1}\langle z_{n-1}-x^*,\nabla f(u)-\nabla f(x^*)\rangle\leq M_1$. Since f is sequentially consistent, then we have from (3.22) that

$$\lim_{n\to\infty}||y_n-\Phi^{N-1}x_n||=0, \tag{3.23}$$

and hence

$$\lim_{n\to\infty}||\nabla f(x_n)-\nabla f(\Phi^{N-1}x_n)||=0. \tag{3.24}$$

Again, since $x^*\in A_{N-1}^{-1}(0)$, by Lemma 2.11, we have

$$\begin{aligned} D_f(\Phi^{N-1}(x_n),\Phi^{N-2}(x_n)) &\leq D_f(x^*,\Phi^{N-2}(x_n))-D_f(x^*,\Phi^{N-1}(x_n)) \\ &\leq D_f(x^*,x_n)-D_f(x^*,y_n) \\ &\leq \alpha_{n-1}M_1 \\ &\quad +D_f(x^*,u_{n-1})-D_f(x^*,u_n)\to 0, n\to\infty \end{aligned} \tag{3.25}$$

Thus since f is sequentially consistent, we have

$$\lim_{n\to\infty}||\Phi^{N-1}(x_n)-\Phi^{N-2}(x_n)||=0, \tag{3.26}$$

and hence

$$\lim_{n\to\infty} ||\nabla f(\Phi^{N-1}(x_n)) - \nabla f(\Phi^{N-2}(x_n))|| = 0.$$

Following the same procedure, we have that

$$\lim_{n\to\infty} ||\Phi^{N-2}(x_n) - \Phi^{N-3}(x_n)|| = \cdots = \lim_{n\to\infty} ||\Phi^1(x_n) - x_n|| = 0. \tag{3.27}$$

Therefore, from (3.23),(3.26) and (3.27), we conclude that

$$\lim_{n\to\infty} ||\Phi^i(x_n) - \Phi^{i-1}(x_n)|| = 0,\ i = 1, 2, \cdots, N$$

and

$$\lim_{n\to\infty} ||\nabla f(\Phi^i(x_n)) - \nabla f(\Phi^{i-1}(x_n))|| = 0. \tag{3.28}$$

Thus we have that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Since $y_{n_j} \rightharpoonup p$ and $\lim_{n\to\infty} ||y_n - x_n|| = 0$, we have that $x_{n_j} \rightharpoonup p$. For each $i = 1, 2, \cdots, N$, we note that $\Phi^i(x_n) = Res^f_{\lambda^i_n A_i} \Phi^{i-1}(x_n)$ and therefore

$$||A_{\lambda^i_n} \Phi^i x_n|| = \frac{1}{\lambda^i_n} ||\nabla f(\Phi^{i-1}(x_n)) - \nabla f(\Phi^i(x_n))||.$$

Hence from (3.28) and the condition $\lim_{n\to\infty} \lambda^i_n > 0$, we have

$$\lim_{n\to\infty} ||A_{\lambda^i_n} \Phi^i x_n|| = 0. \tag{3.29}$$

Now since $(\Phi^i x_n, A_{\lambda^i_n} \Phi^{i-1}(x_n)) \in G(A_i)$ for each $i = 1, 2, \cdots, N$. If $(w, w^*) \in G(A_i)$ for each $i = 1, 2, \cdots, N$, then it follows from the monotonicity of $A_i, i = 1, 2, \cdots, N$, that

$$\langle w^* - A_{\lambda^i_n} \Phi^{i-1}(x_n), w - \Phi^i(x_n)\rangle \geq 0.$$

Since $x_{n_j} \rightharpoonup p$, then $\Phi^i(x_{n_j}) \rightharpoonup p$ for each $i = 1, 2, \cdots, N$. Thus from (3.29), we have

$$\langle w^*, w - p\rangle \geq 0,$$

and since A_i is maximally monotone for each $i = 1, 2, \cdots, N$, we conclude that $p \in \cap_{i=1}^N A_i^{-1}(0)$.
Thus we have

$$p \in F(T) \cap (\cap_{k=1}^N EP(G_k)) \cap (\cap_{i=1}^N A_{i=1}^{-1}(0)),$$

that is

$$p \in F(T) \cap (\cap_{k=1}^N GMEP(g_k, \varphi_k, B_k)) \cap (\cap_{i=1}^N A_{i=1}^{-1}(0)).$$

We now show that $\{x_n\}$ converges strongly to $z := \overleftarrow{P}roj^f_\Omega u$.

$$\begin{aligned} D_f(u_n, x_{n+1}) &= D_f(u_n, \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^\infty \gamma_{nj} \nabla f(Tu_n))) \\ &\leq \alpha_n D_f(u_n, u) + \beta_n D_f(u_n, u_n) + \sum_{j=1}^\infty \gamma_{nj} D_f(u_n, Tu_n) \to 0, n \to \infty. \end{aligned}$$

Therefore, by Lemma 2.5, it follows that

$$||u_n - x_{n+1}|| \to 0, n \to \infty.$$

Now,

$$||x_n - x_{n+1}|| \leq ||x_n - y_n|| + ||y_n - u_n|| + ||u_n - x_{n+1}|| \to 0, n \to \infty.$$

Let $z := \overleftarrow{Proj}^f_\Omega u$, we now show that

$$\limsup_{n\to\infty}\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - z, \nabla f(u) - \nabla f(z)\rangle = \lim_{j\to\infty}\langle x_{n_j} - z, \nabla f(u) - \nabla f(z)\rangle.$$

Then, from $||x_n - x_{n+1}|| \to 0, n \to \infty$ and Lemma 2.10, we have

$$\limsup_{n\to\infty}\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \quad = \quad \limsup_{n\to\infty}\langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

Now, from (3.5),

$$\begin{aligned} D_f(z, x_{n+1}) &\leq (1-\alpha_n)D_f(z, u_n) + \alpha_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\leq (1-\alpha_n)D_f(z, y_n) + \alpha_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\leq (1-\alpha_n)D_f(z, x_n) + \alpha_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle. \end{aligned}$$

Hence by Lemma 2.8, we obtain $D_f(z, x_n) \to 0, n \to \infty$ and so

$$||x_n - z|| \to 0.$$

That is $\{x_n\}$ converges strongly to $z := \overleftarrow{Proj}^f_\Omega u$.

Case 2. Suppose there exists a subsequence $\{n_\iota\}$ of $\{x_n\}$ such that

$$D_f(x^*, x_{n_\iota}) \leq D_f(x^*, x_{n_\iota+1}) \ \forall \iota \in \mathbb{N}.$$

Then by Lemma 2.9, there exists a nondecreasing sequence $\{m_\tau\} \subset \mathbb{N}$ such that $m_\tau \to \infty, \tau \to \infty$,

$$D_f(x^*, x_{m_\tau}) \leq D_f(x^*, x_{m_\tau+1})$$

and

$$D_f(x^*, x_\tau) \leq D_f(x^*, x_{m_\tau+1}) \ \forall \tau \in \mathbb{N}.$$

Again, let

$$s_{n_\tau} := \nabla f^*(\frac{\beta_{n_\tau}}{1-\alpha_{n_\tau}}\nabla f(u_{n_\tau}) + \frac{\sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}}\nabla f(T_j u_{n_\tau})).$$

Then

$$\begin{aligned} D_f(x^*, s_{n_\tau}) \quad &+ \quad D_f(x^*, \nabla f^*(\frac{\beta_{n_\tau}}{1-\alpha_{n_\tau}}\nabla f(u_{n_\tau}) + \frac{\sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}}\nabla f(T_j u_{n_\tau}))) \\ &\leq (\frac{\beta_{n_\tau}}{1-\alpha_{n_\tau}})D_f(x^*, u_{n_\tau}) + \frac{\sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}})D_f(x^*, T_j u_{n_\tau}) \\ &\leq (\frac{\beta_{n_\tau} + \sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}})D_f(x^*, u_{n_\tau}) \\ &\leq D_f(x^*, u_{n_\tau}). \end{aligned}$$

Therefore,

$$\begin{array}{rcl}
0 & \leq & D_f(x^*, u_{n_\tau}) - D_f(x^*, s_{n_\tau}) \\
& = & D_f(x^*, u_{n_\tau}) - D_f(x^*, u_{n_\tau+1}) + D_f(x^*, u_{n_\tau+1}) - D_f(x^*, s_{n_\tau}) \\
& \leq & D_f(x^*, u_{n_\tau}) - D_f(x^*, u_{n_\tau+1}) + D_f(x^*, x_{n_\tau+1}) - D_f(x^*, s_{n_\tau}) \\
& \leq & D_f(x^*, u_{n_\tau}) - D_f(x^*, u_{n_\tau+1}) + \alpha_{n_\tau} D_f(x^*, u) \\
& & +(1-\alpha_{n_\tau}) D_f(x^*, s_{n_\tau}) - D_f(x^*, s_{n_\tau}) \\
& = & D_f(x^*, u_{n_\tau}) - D_f(x^*, u_{n_\tau+1}) + \alpha_{n_\tau}(D_f(x^*, u) - D_f(x^*, s_{n_\tau}) \to 0, n \to \infty.
\end{array}$$

Furthermore,

$$\begin{aligned}
D_f(x^*, s_{n_\tau}) &\leq \frac{\beta_{n_\tau}}{1-\alpha_{n_\tau}} D_f(x^*, u_{n_\tau}) + \frac{\sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} D_f(x^*, T_j u_{n_\tau}) \\
&= \big(1 - \frac{\sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}}\big) D_f(x^*, u_{n_\tau}) + \frac{\sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} D_f(x^*, T_j u_{n_\tau}) \\
&= D_f(x^*, u_{n_\tau}) + \frac{\sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} (D_f(x^*, T_j u_{n_\tau}) - D_f(x^*, u_{n_\tau})). \qquad (3.30)
\end{aligned}$$

Thus from (3.30), we have

$$\frac{\sum_{j=1}^\infty \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} (D_f(x^*, u_{n_\tau}) - D_f(x^*, T_j u_{n_\tau})) \leq D_f(x^*, u_{n_\tau}) - D_f(x^*, s_{n_\tau}) \to 0, n \to \infty. \qquad (3.31)$$

Since T is left Bregman strongly nonexpansive, we obtain that

$$\lim_{\tau\to\infty} D_f(u_{n_\tau}, T_j u_{n\tau}) = 0,$$

which implies that

$$\lim_{\tau\to\infty} ||u_{n_\tau} - T_j u_{n_\tau}|| = 0. \qquad (3.32)$$

By the same arguments as in Case 1, we obtain that

$$\limsup_{\tau\to\infty} \langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0, \qquad (3.33)$$

and

$$D_f(z, x_{n_\tau+1}) \quad \leq \quad (1-\alpha_{n_\tau}) D_f(z, x_{n_\tau}) + \alpha_{n_\tau} \langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle,$$

which since $D_f(z, x_{n_\tau}) \leq D_f(z, x_{n_\tau+1})$ implies

$$\begin{array}{rcl}
\alpha_{n_\tau} D_f(z, x_{n_\tau}) & \leq & D_f(z, x_{n_\tau}) - D_f(z, x_{n_\tau+1}) + \alpha_{n_\tau} \langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle \\
& \leq & \alpha_{n_\tau} \langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle.
\end{array}$$

Thus since $\alpha_{n_\tau} > 0$, we have

$$D_f(z, x_{n_\tau}) \quad \leq \quad \langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle.$$

Hence it follows from (3.33) that

$$\lim_{\tau\to\infty} D_f(z, x_{n_\tau}) = 0.$$

Since $D_f(z, x_\tau) \leq D_f(z, x_{m_\tau+1})$ for all $\tau \in \mathbb{N}$, we conclude that $x_\tau \to z, \tau \to \infty$. This implies that $x_n \to z, n \to \infty$, which completes the proof.

Corollary 3.2. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^{\infty}$ be an infinite family of left Bregman nonexpasive mappings from C into itself and $F(T_j) = \hat{F}(T_j)$, $\forall j \geq 1$. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ and $A_i : E \to 2^{E^*}(i = 1, 2, ..., N)$ be maximal monotone operators, such that $\Omega := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{i=1}^{N} A_i^{-1}(0)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j y_n)), \ \forall n \geq 1, \end{cases} \tag{3.34}$$

converges strongly to a point $p = \overleftarrow{Proj}^f_\Omega u \in \Omega$, where the sequences $\alpha_n, \beta_n, \gamma_{nj}$ and λ_n satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

Corollary 3.3. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^{\infty}$ be an infinite family of left Bregman nonexpasive mappings from C into itself and $F(T_j) = \hat{F}(T_j)$, $\forall j \geq 1$. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be bifunctions satisfying conditions $(A1) - (A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup +\infty, (k = 1, 2, ..., N)$ be proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$, such that $\Omega := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} x_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)), \ \forall n \geq 1, \end{cases} \tag{3.35}$$

converges strongly to a point $p = \overleftarrow{Proj}^f_\Omega u \in \Omega$, where $G(x, y) := g(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ and the sequences α_n, β_n, and γ_{nj} satisfies the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$.

4. Applications

4.1. **Convex feasibility problem.** Let $\{K_j\}_{j=1}^{\infty}$ be nonempty closed and convex subsets of E such that $\cap_{j=1}^{\infty} K_j \neq \emptyset$. The convex feasibility problem (CFP) is to find

$x \in \cap_{j=1}^{\infty} K_j$. Obviously $F(\overleftarrow{Proj}^f_{K_j}) = K_j$ for all $j \geq 1$. If the Legendre function is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the Bregman projection $\overleftarrow{Proj}^f_{K_j}$ is BFNE, hence BSNE and $F(\overleftarrow{Proj}^f_{K_j}) = \hat{F}(\overleftarrow{Proj}^f_{K_j})$ (see, [46] Lemma 1.2.3). Thus, if we take $T_j = \overleftarrow{Proj}^f_{K_j}$ in Theorem 3.1, we get a strong convergence theorem for approximating the solution of convex feasibility problems, a common solution to a finite system of generalized mixed equilibrium problems and a common element of the set of zeros of a finite family of maximal monotone operators.

Theorem 4.1. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $T_j = \overleftarrow{Proj}^f_{K_j}$, where $\{K_j\}_{j=1}^{\infty}$, are nonempty closed and convex subsets of C. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be a bifunction satisfying conditions $(A1)-(A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be a continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ be a proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ and $A_i : E \to 2^{E^*} (i = 1, 2, ..., N)$ be maximal monotone operators, such that $\Omega := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \cap (\cap_{i=1}^{N} A_i) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} y_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)), \ \forall n \geq 1, \end{cases} \tag{4.1}$$

converges strongly to a point $p = \overleftarrow{Proj}^f_{\Omega} u \in \Omega$, where $G(x, y) := g(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ and the sequences $\alpha_n, \beta_n, \gamma_{nj}$ and λ_n satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ for each $k = 1, 2, ..., N$.

4.2. **Zeroes of Bregman inversely strongly monotone operators.** Let the Legendre function f be such that

$$ran(\nabla f - A) \subseteq ran(\nabla f). \tag{4.2}$$

The operator $A : E \to 2^{E^*}$ is called *Bregman inversly strongly monotone* (BISM) if

$$(domA) \cap (int(domf) \neq \emptyset$$

and for any $x, y \in int(domf)$, and each $\xi \in Ax, \eta \in Ay$, we have

$$\langle \xi - \eta, \nabla f^*(\nabla f(x) - \xi) - \nabla f^*(\nabla f(y) - \eta)\rangle \geq 0.$$

This class of operators was introduced by Butnariu and Kassey (see [18]). For any operator $A : E \to 2^{E^*}$, the anti resolvent $A^f : E \to 2^E$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A).$$

Observe that $domA^f \subseteq (domA) \cap (int(domf)$ and $ranA^f \subseteq int(domf)$. The operator A [18] is BISM if and only if the anti-resolvent A^f is a single valued BFNE operator. Some examples of BISM operator can be seen in [18]. From the definition of anti-resolvent and ([18], Lemma 3.5), we obtain the following proposition.

Proposition 4.2. *Let $f : E \to (-\infty, +\infty]$ be a Legendre function and let $A : E \to 2^{E^*}$ be a BISM operator such that $A^{-1}(0)^* \neq \emptyset$. Then the following statements holds;*
(i) $A^{-1}(0)^* = F(A^f)$,
(ii) For any $u \in A^{-1}(0)^*$ and $x \in domA^f$, we have

$$D_f(u, A^f) + D_f(A^f x, x) \leq D_f(u, x).$$

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the resolvent of A^f of A is a single-valued BSNE operator which satisfies $F(A^f) = \hat{F}(A^f)$ ([46] Lemma 1.3.2).

In Theorem 3.1, if we let $T_i = A_i^f$ and let f be the Legendre function such that (4.2) is satisfied then we obtain the following result for approximating a common zeroes of infinite family Bregman Inversely Strongly Monotone Operators, a common solution to a finite system of generalized mixed equilibrium problems and a common element of the set of zeros of a finite family of maximal monotone operators.

Theorem 4.3. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^{\infty} = \{A_j^f\}_{j=1}^{\infty}$. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be bifunctions satisfying conditions $(A1) - (A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ be proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset$ int(domf) and $A_i : E \to 2^{E^*} (i = 1, 2, ..., N)$ be maximal monotone operators, such that $\Omega := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \cap (\cap_{i=1}^{N} A_i^{-1}(0)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} y_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)), \ \forall n \geq 1, \end{cases} \tag{4.3}$$

converges strongly to a point $p = \overleftarrow{Proj}^f_{\Omega} u \in \Omega$, where $G(x, y) := g(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ and the sequences $\alpha_n, \beta_n, \gamma_{nj}$ and λ_n satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

4.3. **Variational inequalities.** Let $A : E \to E^*$ be a BISM operator and let C be a nonempty, closed and convex subset of domA. The variational inequality problem corresponding to A is to find $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \ \forall y \in C. \tag{4.4}$$

The set of solutions of (4.4) is denoted by VI(A,C).

Proposition 4.4. ([45]Proposition 8) *Let* $f : E \to (-\infty, +\infty]$ *be a Legendre and totally convex function which satisfies the range condition* (4.2). *Let* $A : E \to E^*$ *be a BISM operator. If* C *is a nonempty, closed and convex subset of* $domA \cap int(domf)$, *then* $VI(A,C) = F(\overleftarrow{Proj}_C^f \circ A^f)$.

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, the anti-resolvent A^f is single-valued ([18], Lemma 3.5(d))and BSNE operator (see Section 2 and [18] Lemma 3.5(c)) which satisfy $F(A^f) = \hat{F}(A^f)$. Since the Bregman projection $\overleftarrow{Proj}_C^f$ is a BFNE operator, it is a BSNE which satisfy $F(\overleftarrow{Proj}_C^f) = \hat{F}(\overleftarrow{Proj}_C^f)$. It now follows (see [42] Lemma 2) that $\overleftarrow{Proj}_C^f \circ A^f$ is also a BSNE operator which satisfies $F(\overleftarrow{Proj}_C^f \circ A^f) = \hat{F}(\overleftarrow{Proj}_C^f \circ A^f)$. From Proposition 4.4, we know that $F(\overleftarrow{Proj}_C^f \circ A^f) = V(A,C)$. Therefore in Theorem 3.1, if we let $T_i = \overleftarrow{Proj}_C^f \circ A^f$, we get an algorithm for finding a common solution to the variational inequality problem corresponding to infinitely many BISM operators and system of equilibrium problem.

Theorem 4.5. *Let* E *be a reflexive real Banach space and* C *a nonempty, closed and convex subset of* E. *Let* $A_j : E \to E^*$, $j \geq 1$, *be an infinite family of BISM operators such that* $C \subset domA_j$ *and* $\{T_j\}_{n=1}^{\infty} = \{\overleftarrow{Proj}_C^f \circ A_j^f\}_{j=1}^{\infty}$. *Let* $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ *be bifunctions satisfying conditions* $(A1) - (A4)$. *Let* $B_k : E \to E^*, (k = 1, 2, ..., N)$ *be continuous and monotone mappings,* $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ *be proper lower semicontinuous and convex functions. Let* $f : E \to \mathbb{R}$ *be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* E *such that* $C \subset$ *int(domf) and* $A_i : E \to 2^{E^*} (i = 1, 2, ..., N)$ *be maximal monotone operators, such that* $\Omega := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \cap (\cap_{i=1}^{N} A_i^{-1}(0)) \neq \emptyset$. *Then the sequence* $\{x_n\}$ *generated for arbitrary* u, $x_1 \in E$ *by*

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} y_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)), \ \forall n \geq 1, \end{cases} \tag{4.5}$$

converges strongly to a point $p = \overleftarrow{Proj}_{\Omega}^f u \in \Omega$, *where* $G(x,y) := g(x,y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ *and the sequences* $\alpha_n, \beta_n, \gamma_{nj}$ *and* λ_n *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

Acknowledgement. The first author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Re- search Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Doctoral Bursary. Opinions ex- pressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

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Received: September 22, 2015; Accepted: July 28, 2016.