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GENERALIZED LÜDERS OPERATION FOR NORMAL STATES ON A VON NEUMANN ALGEBRA

KATARZYNA LUBNAUER*, ANDRZEJ ŁUCZAK** AND HANNA PODSĘDKOWSKA***

Faculty of Mathematics and Computer Sciences, University of Łódź
ul. S. Banacha 22, 90-238 Łódź, Poland
*E-mail: lubnauer@math.uni.lodz.pl
**E-mail: anluczak@math.uni.lodz.pl
***E-mail: hpodsedk@math.uni.lodz.pl

Abstract. A generalized Lüders operation on states of a von Neumann algebra is considered, and the fixed points of this operation are investigated. In particular, a description of the fixed points is obtained for arbitrary semifinite von Neumann algebras, generalizing thus the one obtained for the full algebra. There is some similarity between the results for the Lüders operation on states and the results for the analogous Lüders operation on the algebra, however, the analogy is not complete and the difference is already seen for the full algebra.
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Introduction

In 1951 G. Lüders [6] defined the following operation for bounded operators on a complex separable Hilbert space $\mathcal{H}$

$$x \mapsto \sum_i e_i x e_i, \quad x \in \mathbb{B}(\mathcal{H}),$$

where $\{e_i\}$ is a countable partition of the identity into pairwise orthogonal projections, and characterised the fixed points of this operation as those x's which commute with all the e_i. Since then various generalizations of the Lüders operation have been considered and their fixed points analysed (see [1, 2, 5, 7, 9]). The most general form of this generalized Lüders operation is

$$x \mapsto \sum_i a_i^* x a_i, \quad x \in \mathbb{B}(\mathcal{H}),$$

with the assumption

$$\sum_i a_i^* a_i = \sum_i a_i a_i^* = \mathbf{1}.$$

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It is immediately seen that if x commutes with all the a_i, then it is a fixed point of this operation; however, the natural conjecture that the converse is also true turned out false as shown in [1]. On the other hand, if the generalized Lüders operation is considered for normal states on $\mathbb{B}(\mathcal{H})$, given as is well known by density matrices (i.e. positive operators of trace one), then the fixed points coincide with the density matrices which commute with all the a_i (see [2]). This result raised the natural question about the fixed points of the generalized Lüders operation for normal states on an arbitrary von Neumann algebra, or slightly more general: the fixed points of the generalized Lüders operation on the predual of a von Neumann algebra. In the paper, we investigate this problem for semifinite algebras, and show that the fixed points again coincide with the normal functionals that commute with all the a_i. It turns out that the same also holds true for an arbitrary algebra, but under the additional assumption that the von Neumann algebra generated by the a_i is abelian.

1. Preliminaries and notation

Let $\mathcal{M}$ be an arbitrary von Neumann algebra with identity $\mathbf{1}$ acting on a Hilbert space $\mathcal{H}$. The predual $\mathcal{M}_*$ of $\mathcal{M}$ is a Banach space of all *normal*, i.e. continuous in the ultraweak topology, linear functionals on $\mathcal{M}$. In the sequel, three basic norms will be considered: the norm in $\mathcal{M}$, the norm in $\mathcal{M}_*$, and the norm in the space $L^1(\mathcal{M},\tau)$. They will be denoted respectively by $\|\cdot\|_\infty$ — norm for bounded operators in $\mathcal{M}$, $\|\cdot\|$ — norm for bounded functionals, and $\|\cdot\|_1$ — L^1-norm. The same notation will be employed also for maps on these spaces, thus e.g. for a map Φ, the symbol $\|\Phi\|_1$ will denote the norm of Φ regarded as a map on $L^1(\mathcal{M},\tau)$.

A *state* on $\mathcal{M}$ is a bounded positive linear functional $\rho\colon \mathcal{M}\to\mathbb{C}$ of norm one. For a normal state ρ its *support*, denoted by $\mathbf{s}(\rho)$, is defined as the smallest projection in $\mathcal{M}$ such that $\rho(\mathbf{s}(\rho))=\rho(\mathbf{1})$. In particular, we have

$$\rho(\mathbf{s}(\rho)x)=\rho(x\mathbf{s}(\rho))=\rho(x),\quad x\in\mathcal{M},$$

and if $\rho(\mathbf{s}(\rho)x\mathbf{s}(\rho))=0$ for $\mathbf{s}(\rho)x\mathbf{s}(\rho)\geqslant 0$, then $\mathbf{s}(\rho)x\mathbf{s}(\rho)=0$.

For any $a\in\mathcal{M}$ and $\rho\in\mathcal{M}_*$, we define functionals $a\rho,\rho a\in\mathcal{M}_*$ as

$$a\rho=\rho(\cdot\, a),\quad \rho a=\rho(a\,\cdot).$$

Let $\{a_i\}$ be a countable (finite or infinite) family of elements of $\mathcal{M}$ such that

$$\sum_i a_i^*a_i=\mathbf{1}\qquad\text{and}\qquad\sum_i a_ia_i^*=\mathbf{1}, \tag{1.1}$$

(for an infinite family, the series is assumed to converge in the σ-weak topology, i.e. the $\sigma(\mathcal{M},\mathcal{M}_*)$-topology). By $\mathcal{A}$ is denoted the von Neumann algebra generated by all the a_i, $\mathcal{A}=W^*(\{a_i\})$. Define a *generalized Lüders operation* $\Phi_*\colon\mathcal{M}_*\to\mathcal{M}_*$ by the formula

$$\Phi_*(\rho)=\sum_i a_i\rho a_i^*,\quad \rho\in\mathcal{M}_*.$$

Observe that if ρ is positive, then $a_i\rho a_i^*$ is also positive, and we have

$$\sum_i\|a_i\rho a_i^*\|=\sum_i(a_i\rho a_i^*)(\mathbf{1})=\sum_i\rho(a_i^*a_i)=\rho(\mathbf{1}),$$

thus the series is norm-convergent for positive ρ's, which yields its norm-convergence for all $\rho \in \mathcal{M}_*$.

The dual map $\Phi\colon \mathcal{M} \to \mathcal{M}$ is defined by the condition

$$\Phi_*(\rho) = \rho \circ \Phi, \quad \rho \in \mathcal{M}_*,$$

and it is easily seen that we have

$$\Phi(x) = \sum_i a_i^* x a_i, \quad x \in \mathcal{M},$$

where the series is convergent in the σ-weak topology.

Let us look closer at conditions (1.1). The first equality there yields for a state ρ

$$(\Phi_*(\rho))(\mathbf{1}) = \sum_i (a_i \rho a_i^*)(\mathbf{1}) = \sum_i \rho(a_i^* a_i) = \rho(\mathbf{1}),$$

thus Φ_* sends states to states. The second equality in (1.1) is called the *trace-preserving property* (of Φ). It is justified by the fact that for the full algebra $\mathbb{B}(\mathcal{H})$ and canonical trace "tr" we have, roughly speaking,

$$\operatorname{tr} \Phi(h) = \sum_i \operatorname{tr}(a_i^* h a_i) = \sum_i \operatorname{tr}(a_i a_i^* h) = \operatorname{tr} \big(\sum_i a_i a_i^*\big) h = \operatorname{tr} h,$$

for trace-class $h \in \mathbb{B}(\mathcal{H})$. The trace-preserving property has also one more important consequence. Namely, denote by $\operatorname{Fix} \Phi_*$ the fixed-point space of Φ_*, i.e.

$$\operatorname{Fix} \Phi_* = \{\rho \in \mathcal{M}_* : \Phi_*(\rho) = \rho\}.$$

By a small abuse of notation, let us agree to denote by $\mathcal{M}_* \cap \mathcal{A}'$ the set of all normal functionals which commute with all elements in $\mathcal{A}$, i.e.

$$\mathcal{M}_* \cap \mathcal{A}' = \{\rho \in \mathcal{M}_* : a\rho = \rho a \text{ for all } a \in \mathcal{A}\}.$$

Now, the trace-preserving property yields

$$\mathcal{M}_* \cap \mathcal{A}' \subset \operatorname{Fix} \Phi_*,$$

and for $\mathcal{M} = \mathbb{B}(\mathcal{H})$, a result in [2], mentioned in the Introduction, gives the equality

$$\operatorname{Fix} \Phi_* = \mathcal{M}_* \cap \mathcal{A}'. \tag{FP*}$$

We are interested in the situations when the above equality holds true. A similar problem, which however has attracted so far much more attention, is as follows. Denote the fixed-point space of Φ by $\operatorname{Fix} \Phi$, i.e.

$$\operatorname{Fix} \Phi = \{x \in \mathcal{M} : \Phi(x) = x\}.$$

Obviously, we have

$$\mathcal{M} \cap \mathcal{A}' \subset \operatorname{Fix} \Phi,$$

and the question is when there is equality above. In general, the inclusion is strict as shown in [1]. However, if there is a normal faithful Φ-invariant state φ, i.e. $\Phi_*(\varphi) = \varphi$, then ergodic theory for von Neumann algebras yields the following facts:

(i) $\operatorname{Fix} \Phi = \mathcal{M} \cap \mathcal{A}'$,

(ii) there exists a normal faithful conditional expectation $\mathbb{E}\colon \mathcal{M} \to \operatorname{Fix}\Phi$ such that $\varphi \circ \mathbb{E} = \varphi$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k(x) = \mathbb{E}x, \quad x \in \mathcal{M},$$

where the series is convergent in the σ-weak topology,

(iii) $\operatorname{Fix}\Phi_* = \{\rho \circ \mathbb{E} : \rho \in \mathcal{M}_*\}$.

In this case, we further obtain for each $\rho \in \mathcal{M}_*$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho\big(\Phi^k(x)\big) = \rho(\mathbb{E}x), \quad x \in \mathcal{M},$$

i.e.

$$\textit{weak} - \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi_*^k(\rho) = \rho \circ \mathbb{E}, \quad \rho \in \mathcal{M}_*.$$

On account of the Mean Ergodic Theorem in Banach spaces (cf. [4, Theorem 1.1]), we infer that for each $\rho \in \mathcal{M}_*$

$$\textit{norm} - \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi_*^k(\rho) = \rho \circ \mathbb{E}.$$

To finish our considerations concerning the fixed-points of Φ, let us note that if the algebra $\mathcal{A}$ is abelian, then we again have the equality $\operatorname{Fix}\Phi = \mathcal{M} \cap \mathcal{A}'$ as shown in [9].

In the course of our analysis, we shall be concerned with slightly weaker conditions on the a_i, namely,

$$\sum_i a_i^* a_i = \mathbf{1} \qquad \text{and} \qquad \sum_i a_i a_i^* = c \leqslant \mathbf{1}, \tag{1.2}$$

where c is some positive operator in $\mathcal{M}$. Consequently, in this situation, we shall aim at showing the inclusion

$$\operatorname{Fix}\Phi_* \subset \mathcal{M}_* \cap \mathcal{A}'.$$

Together with the operations Φ_* and Φ we shall consider *conjugate operations* $\widetilde{\Phi}_*$ and $\widetilde{\Phi}$ defined on $\mathcal{M}_*$ and $\mathcal{M}$, respectively, by the formulae

$$\widetilde{\Phi}_*(\rho) = \sum_i a_i^* \rho a_i, \quad \rho \in \mathcal{M}_*,$$

and

$$\widetilde{\Phi}(x) = \sum_i a_i x a_i^*, \quad x \in \mathcal{M}.$$

The maps Φ and $\widetilde{\Phi}$ are completely positive and

$$\|\Phi\|_\infty = 1, \qquad \|\widetilde{\Phi}\|_\infty = \|c\|_\infty;$$

in particular, the following Schwarz-Kadison inequalities hold for all $x \in \mathcal{M}$

$$\Phi(x^*)\Phi(x) \leqslant \Phi(x^*x), \quad \widetilde{\Phi}(x^*)\widetilde{\Phi}(x) \leqslant \|\widetilde{\Phi}\|_\infty \widetilde{\Phi}(x^*x) \leqslant \widetilde{\Phi}(x^*x). \tag{1.3}$$

(Later it will be shown that if $\operatorname{Fix}\Phi_* \neq \{0\}$, then $\|c\|_\infty = 1$.) Moreover, since $(\Phi_*)^* = \Phi$ and $(\widetilde{\Phi}_*)^* = \widetilde{\Phi}$, we have

$$\|\Phi_*\| = \|\Phi\|_\infty = 1 \qquad \text{and} \qquad \|\widetilde{\Phi}_*\| = \|\widetilde{\Phi}\|_\infty = \|c\|_\infty.$$

2. Fixed points

Lemma 2.1. *Let* $0 \leqslant \varphi \in \operatorname{Fix}\Phi_*$ *be faithful. Then the linear space* $\{a'\varphi : a' \in \operatorname{Fix}\Phi\}$ *is norm-dense in* $\operatorname{Fix}\Phi_*$.

Proof. The results in ergodic theory mentioned earlier, yield that $\operatorname{Fix}\Phi = \mathcal{M}\cap\mathcal{A}'$ and $\operatorname{Fix}\Phi = \mathbb{E}(\mathcal{M})$, where $\mathbb{E}$ is the conditional expectation described before. On account of [3, Lemma 7], $\operatorname{Fix}\Phi_*$ is isometrically isomorphic to the predual of $\operatorname{Fix}\Phi$, and this isomorphism is given by

$$\operatorname{Fix}\Phi_* \ni \rho \longleftrightarrow \rho|\operatorname{Fix}\Phi \in (\operatorname{Fix}\Phi)_*.$$

Since φ is faithful, the space

$$\{a'(\varphi|\operatorname{Fix}\Phi) : a' \in \operatorname{Fix}\Phi\}$$

is norm-dense in $(\operatorname{Fix}\Phi)_*$. Indeed, should it be not so, then there would be an $0 \neq x_0 \in ((\operatorname{Fix}\Phi)_*)^* = \operatorname{Fix}\Phi$ such that

$$(a'\varphi)(x_0) = 0$$

for all $a' \in \operatorname{Fix}\Phi$. Taking $a' = x_0^*$, we would obtain

$$\varphi(x_0x_0^*) = 0,$$

so $x_0 = 0$, giving a contradiction.

For every $a' \in \operatorname{Fix}\Phi$ and every $x \in \mathcal{M}$, we have

$$(a'\varphi)(\mathbb{E}x) = \varphi((\mathbb{E}x)a') = \varphi(\mathbb{E}(xa')) = \varphi(xa') = (a'\varphi)(x),$$

showing that

$$(a'\varphi)\circ\mathbb{E} = a'\varphi.$$

Now take arbitrary $\rho \in \operatorname{Fix}\Phi_*$. There is a net $\{a'_\gamma\varphi\}$, with $a'_\gamma \in \operatorname{Fix}\Phi$, such that

$$norm - \lim_\gamma (a'_\gamma\varphi|\operatorname{Fix}\Phi) = \rho|\operatorname{Fix}\Phi.$$

Since ρ is $\mathbb{E}$-invariant, we obtain

$$\|\rho - a'_\gamma\varphi\| = \|(\rho - a'_\gamma\varphi)\circ\mathbb{E}\| = \|(\rho - a'_\gamma\varphi)|\operatorname{Fix}\Phi\| \underset{\gamma}{\to} 0$$

which proves the claim. □

As a starting point in our investigation of the fixed points, let us note the following result.

Theorem 2.2. *Let* $\mathcal{M}$ *be an arbitrary von Neumann algebra, and assume that there is a normal faithful state* $\varphi \in \mathcal{M}_* \cap \mathcal{A}'$. *It follows that*

$$\operatorname{Fix}\Phi_* = \mathcal{M}_* \cap \mathcal{A}'. \tag{FP*}$$

Moreover, we then have $\sum_i a_i a_i^* = \mathbf{1}$.

Proof. Indeed, since $\operatorname{Fix}\Phi \subset \mathcal{A}'$, we have for all $a' \in \operatorname{Fix}\Phi$ and all $a \in \mathcal{A}$

$$a(a'\varphi) = a'(a\varphi) = a'(\varphi a) = (a'\varphi)a,$$

which means that all $a'\varphi$ commute with $\mathcal{A}$. From the denseness of the set of all $a'\varphi$ in $\operatorname{Fix}\Phi_*$, it follows that all elements in $\operatorname{Fix}\Phi_*$ commute with $\mathcal{A}$, showing that equality (FP*) holds.

Moreover, we have

$$\varphi = \sum_i a_i\varphi a_i^* = \varphi \sum_i a_i a_i^* = \varphi c,$$

hence for each $x \in \mathcal{M}$, we get

$$(x\varphi)(\mathbf{1}) = \varphi(x) = (\varphi c)(x) = \varphi(cx) = (x\varphi)(c),$$

and since the elements $x\varphi$ for $x \in \mathcal{M}$ lie densely in $\mathcal{M}_*$ (cf. the reasoning in the proof of Lemma 2.1), we obtain

$$\rho(\mathbf{1}) = \rho(c)$$

for each $\rho \in \mathcal{M}_*$, which yields $c = \mathbf{1}$. □

From the theorem above, we get

Corollary. *Let $\mathcal{M}$ be a finite von Neumann algebra. Then equality (FP*) holds.*

Indeed, if $\mathcal{M}$ is σ-finite and τ is a normal finite faithful trace on $\mathcal{M}$, then it is immediately seen that $\tau \in \mathcal{M}_* \cap \mathcal{A}'$. If $\mathcal{M}$ is not σ-finite, then the reasoning above applies to a faithful family of normal finite traces on $\mathcal{M}$ in place of a single normal finite faithful trace.

Now, assume that $\mathcal{M}$ is semifinite, and let τ be a normal semifinite faithful trace on $\mathcal{M}$. An isometric isomorphism between $\mathcal{M}_*$ and $L^1(\mathcal{M}, \tau)$ is given by

$$\mathcal{M}_* \ni \rho \longleftrightarrow h_\rho \in L^1(\mathcal{M}, \tau), \qquad \rho(x) = \tau(xh_\rho), \quad x \in \mathcal{M}.$$

h_ρ is called the *density of* ρ. In line with our previous terminology, we adopt the notation $\rho = h_\rho\tau$, thus

$$\mathcal{M}_* = \{h\tau : h \in L^1(\mathcal{M}, \tau)\}.$$

Let $h_+ \in L^1_+(\mathcal{M}, \tau)$. There is $\varphi \in \mathcal{M}_*^+$ such that h_+ is the density of φ. We have $a_i h_+ \in L^1(\mathcal{M}, \tau)$, and thus

$$\begin{aligned}\sum_i \|a_i h_+ a_i^*\|_1 &= \sum_i \tau(a_i h_+ a_i^*) = \sum_i \tau(a_i^* a_i h_+) = \sum_i \varphi(a_i^* a_i) \\ &= \varphi(\mathbf{1}) = \tau(h_+) = \|h_+\|_1 < \infty,\end{aligned}$$

which shows that the series $\sum_i a_i h a_i^*$ converges in the norm $\|\cdot\|_1$ for each $h_+ \in L^1_+(\mathcal{M}, \tau)$. Since every $h \in L^1(\mathcal{M}, \tau)$ is a linear combination of four elements from $L^1_+(\mathcal{M}, \tau)$, the series $\sum_i \|a_i h a_i^*\|_1$ converges for each $h \in L^1(\mathcal{M}, \tau)$. By the same token, we obtain that the series $\sum_i \|a_i^* h a_i\|_1$ converges for each $h \in L^1(\mathcal{M}, \tau)$.

Proposition 2.3. *Assume that for* Φ_* *condition* (1.2) *is satisfied. Then the following assertions hold:*

(i) $\widetilde{\Phi}$ *and* Φ *can be extended to bounded linear maps of norm one from* $L^1(\mathcal{M},\tau)$ *to* $L^1(\mathcal{M},\tau)$ *(denoted by the same symbol) such that* $\|\widetilde{\Phi}\|_1 = 1$, $\|\Phi\|_1 \leqslant \|c\|_\infty$, *and*

$$\tau \circ \widetilde{\Phi} = \tau, \qquad \tau \circ \Phi \leqslant \tau.$$

(It will be shown later that if $\operatorname{Fix}\Phi_* \neq \{0\}$, *then* $\|\Phi\|_1 = 1$*);*

(ii) *for any* $x, h \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$ *we have*

$$\tau(x\widetilde{\Phi}(h)) = \tau(\Phi(x)h); \tag{2.1}$$

(iii) *for each* $\rho \in \mathcal{M}_*$, *we have*

$$h_{\Phi_*(\rho)} = \widetilde{\Phi}(h_\rho).$$

Proof. First, take an arbitrary $h \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$. There is $\rho \in \mathcal{M}_*$ such that $h = h_\rho$. For each $x \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$, we have as before

$$\sum_i \|xa_i h_\rho a_i^*\|_1 \leqslant \sum_i \|x\|_\infty \|a_i h_\rho a_i^*\|_1 = \|x\|_\infty \sum_i \|a_i h_\rho a_i^*\|_1 < \infty,$$

and

$$\sum_i \|a_i^* x a_i h_\rho\|_1 \leqslant \sum_i \|a_i^* x a_i\|_1 \|h_\rho\|_\infty = \|h_\rho\|_\infty \sum_i \|a_i^* x a_i\|_1 < \infty,$$

in particular, the series $\sum\limits_i xa_i h_\rho a_i^*$, $\sum\limits_i a_i^* x a_i h_\rho$, $\sum\limits_i a_i h_\rho a_i^*$ and $\sum\limits_i a_i^* x a_i$ converge in $\|\cdot\|_1$-norm. On the other hand, we have

$$\widetilde{\Phi}(x) = \sum_i a_i x a_i^*, \quad x \in \mathcal{M},$$

where the series on the right-hand side converges σ-weakly. Thus

$$\widetilde{\Phi}(x)h_\rho = \Big(\sum_i a_i x a_i^*\Big) h_\rho = \sum_i a_i x a_i^* h_\rho,$$

and

$$x \sum_i a_i h_\rho a_i^* = \sum_i x a_i h_\rho a_i^*,$$

where the series on the right-hand sides of the equalities above converge in $\|\cdot\|_1$-norm. Consequently, we obtain

$$\begin{aligned} \tau(x\widetilde{\Phi}(h_\rho)) &= \tau\Big(x \sum_i a_i h_\rho a_i^*\Big) = \tau\Big(\sum_i x a_i h_\rho a_i^*\Big) \\ &= \sum_i \tau(x a_i h_\rho a_i^*) = \sum_i \tau(a_i^* x a_i h_\rho) \\ &= \tau\Big(\sum_i a_i^* x a_i h_\rho\Big) = \tau(\Phi(x)h_\rho), \end{aligned} \tag{2.2}$$

showing (ii). On the other hand, we have

$$\tau\big(x h_{\Phi_*(\rho)}\big) = (\Phi_*(\rho))(x) = \rho(\Phi(x)) = \tau\big(\Phi(x)h_\rho\big),$$

which yields

$$\tau\big(xh_{\Phi_*(\rho)}\big) = \tau\big(x\widetilde{\Phi}(h_\rho)\big)$$

for $x \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Since $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ is dense in $L^1(\mathcal{M}, \tau)$ in $\|\cdot\|_1$-norm, the equality above holds for all $x \in L^1(\mathcal{M}, \tau)$, which means that

$$h_{\Phi_*(\rho)} = \widetilde{\Phi}(h_\rho)$$

for all $h_\rho \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Taking into account isometric isomorphism between $\mathcal{M}_*$ and $L^1(\mathcal{M}, \tau)$, we obtain for such h_ρ

$$\|\widetilde{\Phi}(h_\rho)\|_1 = \|h_{\Phi_*(\rho)}\|_1 = \|\Phi_*(\rho)\| \leqslant \|\rho\| = \|h_\rho\|_1,$$

and the denseness of $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ in $L^1(\mathcal{M}, \tau)$ in $\|\cdot\|_1$-norm yields the extension of $\widetilde{\Phi}$. For $h_\rho \in L^1_+(\mathcal{M}, \tau)$, we have as before

$$\|\widetilde{\Phi}(h_\rho)\|_1 = \tau(\widetilde{\Phi}(h_\rho)) = \sum_i \tau(a_i h_\rho a_i^*) = \tau(h_\rho) = \|h_\rho\|_1,$$

showing that $\|\widetilde{\Phi}\|_1 = 1$. Moreover, the calculations above yield the relation

$$\tau \circ \widetilde{\Phi} = \tau$$

on $L^1_+(\mathcal{M}, \tau)$, and hence on the whole of $L^1(\mathcal{M}, \tau)$. Observe that since for each $h \in L^1(\mathcal{M}, \tau)$ the series $\sum_i a_i h a_i^*$ is convergent in the norm $\|\cdot\|_1$, we have the formula

$$\widetilde{\Phi}(h) = \sum_i a_i h a_i^*, \quad h \in L^1(\mathcal{M}, \tau).$$

The reasoning above may be repeated almost verbatim for Φ taking into account relation (2.2) and the equality

$$\tau(xh_{\widetilde{\Phi}_*(\rho)}) = (\widetilde{\Phi}_*(\rho))(x) = \rho(\widetilde{\Phi}(x)) = \tau(\widetilde{\Phi}(x)h_\rho) = \tau(x\Phi(h_\rho)),$$

giving the formula

$$h_{\widetilde{\Phi}_*(\rho)} = \Phi(h_\rho)$$

for $h_\rho \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Further, we have

$$\begin{aligned}\|\Phi(h_\rho)\|_1 &= \|h_{\widetilde{\Phi}_*(\rho)}\|_1 = \|\widetilde{\Phi}_*(\rho)\| \leqslant \|\widetilde{\Phi}_*\|\|\rho\| \\ &= \|c\|_\infty \|\rho\| = \|c\|_\infty \|h_\rho\|_1,\end{aligned}$$

and thus $\|\Phi\|_1 \leqslant \|c\|_\infty$.

Let $h \in L^1_+(\mathcal{M}, \tau)$, and let $\varphi \in \mathcal{M}_*^+$ be such that $h_\varphi = h$. Then

$$\begin{aligned}\tau(\Phi(h)) &= \tau\Big(\sum_i a_i^* h a_i\Big) = \sum_i \tau(a_i a_i^* h) \\ &= \sum_i \varphi(a_i a_i^*) = \varphi(c) \leqslant \varphi(\mathbf{1}) = \tau(h),\end{aligned}$$

showing the Φ-subinvariance of τ.

Let now h_ρ be an arbitrary element of $L^1(\mathcal{M}, \tau)$. For a given $\varepsilon > 0$, choose $h_\varphi \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$ such that

$$\|h_\rho - h_\varphi\|_1 < \varepsilon.$$

We have

$$\|\widetilde{\Phi}(h_\rho) - \widetilde{\Phi}(h_\varphi)\|_1 \leqslant \|h_\rho - h_\varphi\|_1 < \varepsilon,$$

and on account of isometric isomorphism between $\mathcal{M}_*$ and $L^1(\mathcal{M}, \tau)$ we obtain

$$\begin{aligned}\|h_{\Phi_*(\rho)} - h_{\Phi_*(\varphi)}\|_1 &= \|\Phi_*(\rho) - \Phi_*(\varphi)\| \\ &\leqslant \|\rho - \varphi\| = \|h_\rho - h_\varphi\|_1 < \varepsilon.\end{aligned}$$

Since

$$h_{\Phi_*(\varphi)} = \widetilde{\Phi}(h_\varphi),$$

we get

$$\begin{aligned}\|h_{\Phi_*(\rho)} - \widetilde{\Phi}(h_\rho)\|_1 &\leqslant \|h_{\Phi_*(\rho)} - h_{\Phi_*(\varphi)}\|_1 + \|h_{\Phi_*(\varphi)} - \widetilde{\Phi}(h_\varphi)\|_1 \\ &\quad + \|\widetilde{\Phi}(h_\varphi) - \widetilde{\Phi}(h_\rho)\|_1 < \varepsilon + 0 + \varepsilon = 2\varepsilon,\end{aligned}$$

showing that

$$h_{\Phi_*(\rho)} = \widetilde{\Phi}(h_\rho).$$

□

Remark. It can be shown that equality (2.1) still holds true if one of the elements, say x, is in $\mathcal{M}$ and the other, h, is in $L^1(\mathcal{M}, \tau)$, but we shall not need such generality.

As a consequence of Proposition 2.3, we obtain

Lemma 2.4. *Assume that for Φ_* condition* (1.2) *holds. Then for each $\rho \in \mathcal{M}_*$, each positive integer k, and any $x, h \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$ the following formulae hold*

$$h_{\Phi_*^k(\rho)} = \widetilde{\Phi}^k(h_\rho), \qquad \tau\big(x\widetilde{\Phi}^k(h)\big) = \tau\big(\Phi^k(x)h\big).$$

Proof. Indeed, we have

$$h_{\Phi_*^2(\rho)} = h_{\Phi_*(\Phi_*(\rho))} = \widetilde{\Phi}\big(h_{\Phi_*(\rho)}\big) = \widetilde{\Phi}(\widetilde{\Phi}(h_\rho)) = \widetilde{\Phi}^2(h_\rho),$$

and the general case follows by induction. The second equality is proven in virtually the same way. □

Assume that there is a normal faithful state $\varphi \in \operatorname{Fix}\Phi_*$, and take an arbitrary $\rho \in \mathcal{M}_*$. By the Mean Ergodic Theorem as referred to in Section 1, it follows that

$$\mathit{norm} - \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi_*^k(\rho) = \rho \circ \mathbb{E},$$

where by "$\mathit{norm} - \lim$" is meant the limit in the norm in $\mathcal{M}_*$. On account of isometric isomorphism between $\mathcal{M}_*$ and $L^1(\mathcal{M}, \tau)$, we get

$$\|\cdot\|_1 - \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_{\Phi_*^k(\rho)} = h_{\rho\circ\mathbb{E}},$$

and from Lemma 2.4 we get

$$\|\cdot\|_1 - \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \widetilde{\Phi}^k(h_\rho) = h_{\rho\circ\mathbb{E}}, \quad h_\rho \in L^1(\mathcal{M}, \tau).$$

Denoting the limit above by $\widetilde{\mathbb{E}}h_\rho$, we obtain that $\widetilde{\mathbb{E}}$ is a positive projection of norm one from $L^1(\mathcal{M},\tau)$ onto $\operatorname{Fix}\widetilde{\Phi}$ such that

$$\tau \circ \widetilde{\mathbb{E}} = \tau.$$

The next lemma is an obvious result in the theory of noncommutative L^p-spaces but for the sake of completeness we present its simple proof here.

Lemma 2.5. *For an arbitrary semifinite von Neumann algebra $\mathcal{N}$ with normal semifinite faithful trace ψ, the space $\{a\psi : a \in \mathcal{N} \cap L^1(\mathcal{N},\psi)\}$ is norm-dense in $\mathcal{N}_*$.*

Proof. We have

$$\mathcal{N}_* = \{a\psi : a \in L^1(\mathcal{N},\psi)\},$$

and the isometric isomorphism between $L^1(\mathcal{N},\psi)$ and $\mathcal{N}_*$ given by

$$L^1(\mathcal{N},\psi) \ni a \longleftrightarrow a\psi \in \mathcal{N}_*,$$

so the result follows from the $\|\cdot\|_1$-norm denseness of $\mathcal{N} \cap L^1(\mathcal{N},\psi)$ in $L^1(\mathcal{N},\psi)$. □

Lemma 2.6. *Assume that for Φ_* condition (1.2) holds, and that there is a normal faithful state $\varphi \in \operatorname{Fix}\Phi_*$. Then for arbitrary $h_\rho \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$, we have $\widetilde{\mathbb{E}}h_\rho \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$. Moreover, for any $a \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$, the following formula holds*

$$\tau((\mathbb{E}a)h_\rho) = \tau(a\widetilde{\mathbb{E}}h_\rho). \tag{2.3}$$

Proof. First note that for $h_\rho \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$, we have

$$\Big\|\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{\Phi}^k(h_\rho)\Big\|_\infty \leqslant \frac{1}{n}\sum_{k=0}^{n-1}\|\widetilde{\Phi}^k(h_\rho)\|_\infty \leqslant \|h_\rho\|_\infty.$$

For any $a \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$, equality (2.1) yields

$$\begin{aligned}(a\tau)\Big(\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{\Phi}^k(h_\rho)\Big) &= \frac{1}{n}\sum_{k=0}^{n-1}\tau\big(a\widetilde{\Phi}^k(h_\rho)\big) = \frac{1}{n}\sum_{k=0}^{n-1}\tau\big(\Phi^k(a)h_\rho\big)\\ &= \frac{1}{n}\sum_{k=0}^{n-1}\rho\big(\Phi^k(a)\big) \to \rho(\mathbb{E}a).\end{aligned} \tag{2.4}$$

Since the functionals $a\tau$ for $a \in \mathcal{M} \cap L^1(\mathcal{M},\tau)$ lie densely in $\mathcal{M}_*$, and the sequence $\Big(\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{\Phi}^k(h_\rho)\Big)$ is bounded in norm in $\mathcal{M}$, it follows that it converges σ-weakly to an element in $\mathcal{M}$. But this sequence converges in the norm $\|\cdot\|_1$ to $\widetilde{\mathbb{E}}h_\rho$, which means that $\widetilde{\mathbb{E}}h_\rho \in \mathcal{M}$.

We have

$$\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{\Phi}^k(h_\rho) \to \widetilde{\mathbb{E}}h_\rho \quad \textit{in } \|\cdot\|_1 - \textit{norm},$$

so

$$\frac{1}{n}\sum_{k=0}^{n-1}a\widetilde{\Phi}^k(h_\rho) \to a\widetilde{\mathbb{E}}h_\rho \quad \textit{in } \|\cdot\|_1 - \textit{norm},$$

hence

$$\frac{1}{n}\sum_{k=0}^{n-1}\tau\big(a\widetilde{\Phi}^k(h_\rho)\big)\to\tau(\widetilde{\mathbb{E}}h_\rho).$$

Relation (2.4) yields

$$\frac{1}{n}\sum_{k=0}^{n-1}\tau\big(a\widetilde{\Phi}^k(h_\rho)\big)=(a\tau)\bigg(\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{\Phi}^k(h_\rho)\bigg)\to\rho(\mathbb{E}a),$$

thus we obtain

$$\tau((\mathbb{E}a)h_\rho)=\rho(\mathbb{E}a)=\tau(a\widetilde{\mathbb{E}}h_\rho). \qquad \square$$

The proposition below shows an important relation between the fixed-point spaces of the maps Φ and $\widetilde{\Phi}$.

Proposition 2.7. *Assume that for Φ_* condition* (1.2) *holds, and that there is a normal faithful state $\varphi\in\operatorname{Fix}\Phi_*$. Then*

$$\operatorname{Fix}\widetilde{\Phi}\cap L^1(\mathcal{M},\tau)=\operatorname{Fix}\Phi\cap L^1(\mathcal{M},\tau).$$

Proof. Let $h\in\operatorname{Fix}\widetilde{\Phi}\cap L^1(\mathcal{M},\tau)$. First, observe that since $\widetilde{\Phi}$ is positive, the following equality holds true

$$\widetilde{\Phi}(h^*)=\widetilde{\Phi}(h)^*=h^*.$$

By virtue of relations (1.3), (2.1) and the Φ-subinvariance of τ, we have for h as above

$$\begin{aligned}0&\leqslant\tau((\Phi(h)-h)^*(\Phi(h)-h))=\tau(\Phi(h^*)\Phi(h))-\tau((\Phi(h^*)h)+\\&-\tau(h^*\Phi(h))+\tau(h^*h)=\tau(\Phi(h^*)\Phi(h))-\tau(h^*\widetilde{\Phi}(h))+\\&-\tau(\widetilde{\Phi}(h^*)h))+\tau(h^*h)=\tau(\Phi(h^*)\Phi(h))-\tau(h^*h)\\&\leqslant\tau(\Phi(h^*h))-\tau(h^*h)\leqslant\tau(h^*h)-\tau(h^*h)=0,\end{aligned}$$

and the faithfulness of τ yields

$$\Phi(h)=h.$$

In the same way, we show that for $h\in\operatorname{Fix}\Phi\cap L^1(\mathcal{M},\tau)$, we have

$$\widetilde{\Phi}(h)=h. \qquad \square$$

The next result may be compared with Theorem 2.2, however for a semifinite algebra we may assume a little less.

Proposition 2.8. *Let $\mathcal{M}$ be a von Neumann algebra with normal semifinite faithful trace τ. Assume that for Φ_* condition* (1.2) *holds, and that there is a normal faithful state $\varphi\in\operatorname{Fix}\Phi_*$. Then* $\operatorname{Fix}\Phi_*\subset\mathcal{M}_*\cap\mathcal{A}'$.

Proof. First, we shall show that $\tau|\operatorname{Fix}\Phi$ is semifinite. Take an arbitrary $x\in\operatorname{Fix}\Phi$, $x\geqslant 0$. Since $\operatorname{Fix}\Phi=\mathcal{M}\cap\mathcal{A}'$, we have

$$\widetilde{\Phi}(x)=\sum_i a_ixa_i^*=x\sum_i a_ia_i^*=xc=cx\leqslant x,$$

which yields that for arbitrary n the following inequalities hold true

$$x\geqslant\widetilde{\Phi}(x)\geqslant\widetilde{\Phi}^2(x)\geqslant\ldots\geqslant\widetilde{\Phi}^n(x).$$

Since τ is semifinite, there is $0 \leqslant z \in \mathcal{M}$ such that $z \leqslant x$ and $0 < \tau(z) < \infty$, and the relation above yields

$$\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{\Phi}^k(z) \leqslant \frac{1}{n}\sum_{k=0}^{n-1}\widetilde{\Phi}^k(x) \leqslant x.$$

Passing to the limit, we obtain

$$\widetilde{\mathbb{E}}z \leqslant x.$$

From Proposition 2.3 we get

$$\tau(z) = \tau(\widetilde{\mathbb{E}}z),$$

hence $0 < \tau(\widetilde{\mathbb{E}}z) < \infty$. Lemmas 2.6 and 2.7 yield $\widetilde{\mathbb{E}}z \in \operatorname{Fix}\Phi$, so the semifiniteness of $\tau|\operatorname{Fix}\Phi$ follows. This fact allows us to consider the space $L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)$.

(Observe that we have $\Phi(\widetilde{\mathbb{E}}z) = \widetilde{\mathbb{E}}z$, hence

$$\|\Phi(\widetilde{\mathbb{E}}z)\|_1 = \|\widetilde{\mathbb{E}}z\|_1 = \tau(\widetilde{\mathbb{E}}z) > 0,$$

showing that $\|\Phi\|_1 = 1$.)

Now we shall show that for any $a' \in L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)$, we have $a'\tau \in \operatorname{Fix}\Phi_*$. Assume first that $a' \in \operatorname{Fix}\Phi \cap L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi) = \operatorname{Fix}\Phi \cap L^1(\mathcal{M}, \tau)$. Then by Lemma 2.7, $a' \in \operatorname{Fix}\widetilde{\Phi}$, which means that $\widetilde{\mathbb{E}}a' = a'$. For any $x \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$, it follows from relation (2.4) that

$$(a'\tau)(\mathbb{E}x) = \tau((\mathbb{E}x)a') = \tau(x\widetilde{\mathbb{E}}a') = \tau(xa') = (a'\tau)(x),$$

and the σ-weak denseness of $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ in $\mathcal{M}$ yields

$$(a'\tau) \circ \mathbb{E} = a'\tau.$$

Since $\operatorname{Fix}\Phi \cap L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)$ is dense in $L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)$ in the norm $\|\cdot\|_1$, the equality above holds for all $a' \in L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)$ meaning that $a'\tau \in \operatorname{Fix}\Phi_*$.

Further, we have

$$(\operatorname{Fix}\Phi)_* = \{a'(\tau|\operatorname{Fix}\Phi) : a' \in L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)\},$$

and the isometric isomorphism

$$\operatorname{Fix}\Phi_* \ni \rho \longleftrightarrow \rho|\operatorname{Fix}\Phi \in (\operatorname{Fix}\Phi)_*$$

between $\operatorname{Fix}\Phi_*$ and $(\operatorname{Fix}\Phi)_*$ yields the equality

$$\operatorname{Fix}\Phi_* = \{a'\tau : a' \in L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)\}.$$

Since

$$\{a'(\tau|\operatorname{Fix}\Phi) : a' \in \operatorname{Fix}\Phi \cap L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)\}$$

is norm-dense in $(\operatorname{Fix}\Phi)_*$, it follows that

$$\{a'\tau : a' \in \operatorname{Fix}\Phi \cap L^1(\operatorname{Fix}\Phi, \tau|\operatorname{Fix}\Phi)\}$$

is norm-dense in $\operatorname{Fix}\Phi_*$. For any $a \in \mathcal{A}$, $a' \in \operatorname{Fix}\Phi \cap L^1(\mathcal{M}, \tau)$, and $x \in \mathcal{M}$, we have as $a' \in \mathcal{M} \cap \mathcal{A}'$

$$(a(a'\tau))(x) = (a'\tau)(xa) = \tau(xaa') = \tau(xa'a),$$

and

$$((a'\tau)a)(x) = (a'\tau)(ax) = \tau(axa').$$

Since $xa' \in L^1(\mathcal{M}, \tau)$, it follows that

$$\tau(axa') = \tau(xa'a),$$

giving

$$a(a'\tau) = (a'\tau)a$$

for any $a \in \mathcal{A}$, $a' \in \operatorname{Fix}\Phi \cap L^1(\mathcal{M}, \tau)$. Since the functionals $a'\tau$ for $a' \in \operatorname{Fix}\Phi \cap L^1(\mathcal{M}, \tau)$ lie densely in $\operatorname{Fix}\Phi_*$, we get

$$a\rho = \rho a$$

for every $\rho \in \operatorname{Fix}\Phi_*$ and $a \in \mathcal{A}$, showing the claim. □

Observe that the existence of a normal faithful state in $\operatorname{Fix}\Phi_*$, and thus by the proposition above in $\mathcal{M}_* \cap \mathcal{A}'$, yields, as in the proof of Theorem 2.2, that we must have $\sum_i a_i a_i^* = \mathbf{1}$.

Before proving our main result, we show a property of the support of a normal invariant state.

Lemma 2.9. *Let $0 \leqslant \rho \in \operatorname{Fix}\Phi_*$. Then $a_i \boldsymbol{s}(\rho) = \boldsymbol{s}(\rho) a_i \boldsymbol{s}(\rho)$.*

Proof. On account of [8, Lemma 1], we have

$$\Phi(\mathbf{s}(\rho)) \geqslant \mathbf{s}(\rho),$$

hence

$$\sum_i a_i^* \mathbf{s}(\rho)^\perp a_i = \Phi(\mathbf{s}(\rho)^\perp) \leqslant \mathbf{s}(\rho)^\perp.$$

Multiplying both sides of the above inequality by $\mathbf{s}(\rho)$ on the left and right, we get

$$\sum_i \mathbf{s}(\rho) a_i^* \mathbf{s}(\rho)^\perp a_i \mathbf{s}(\rho) = 0,$$

i.e.

$$\sum_i (\mathbf{s}(\rho)^\perp a_i \mathbf{s}(\rho))^* (\mathbf{s}(\rho)^\perp a_i \mathbf{s}(\rho)) = 0$$

which yields

$$\mathbf{s}(\rho)^\perp a_i \mathbf{s}(\rho) = 0$$

for each i. Consequently, we obtain

$$a_i \mathbf{s}(\rho) = \mathbf{s}(\rho) a_i \mathbf{s}(\rho).$$ □

Now we are in a position to characterise the set $\operatorname{Fix}\Phi_*$ for a semifinite von Neumann algebra.

Theorem 2.10. *Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful semifinite trace τ, and let Φ_* be a Lüders operation for which condition* (1.1) *holds. Then*

$$\operatorname{Fix}\Phi_* = \mathcal{M}_* \cap \mathcal{A}'. \tag{FP*}$$

Proof. As we have already seen, condition (1.1) guarantees that

$$\mathcal{M}_* \cap \mathcal{A}' \subset \operatorname{Fix} \Phi_*,$$

so we only need to show the converse inclusion.

First, take an arbitrary $0 \leqslant \varphi \in \operatorname{Fix} \Phi_*$, and consider the reduced von Neumann algebra

$$\mathcal{M}_{\mathbf{s}(\varphi)} = \{\mathbf{s}(\varphi)x|\mathcal{K} : x \in \mathcal{M}\},$$

where

$$\mathcal{K} = \mathbf{s}(\varphi)(\mathcal{H}),$$

with trace $\widehat{\tau}$ defined on positive elements $\mathbf{s}(\varphi)x|\mathcal{K}$ as

$$\widehat{\tau}(\mathbf{s}(\varphi)x|\mathcal{K}) = \tau(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)).$$

In what follows, we shall consider all operators on the subspace $\mathcal{K}$ only, so to simplify the notation, we shall sometimes drop the symbol of restriction $\cdot|\mathcal{K}$. First we shall show that $\widehat{\tau}$ is semifinite. Indeed, for any positive $\mathbf{s}(\varphi)x\mathbf{s}(\varphi)$ there is $0 \leqslant z \in \mathcal{M}$ such that $z \leqslant \mathbf{s}(\varphi)x\mathbf{s}(\varphi)$ and

$$0 < \tau(z) < \infty.$$

But then

$$z = \mathbf{s}(\varphi)z\mathbf{s}(\varphi),$$

which means that $z|\mathcal{K} \in \mathcal{M}_{\mathbf{s}(\varphi)}$, and

$$0 < \widehat{\tau}(z|\mathcal{K})) = \tau(\mathbf{s}(\varphi)z\mathbf{s}(\varphi)) = \tau(z) < \infty,$$

showing the claim.

Denoting by $\widehat{\varphi}$ the obvious restriction of φ to $\mathcal{M}_{\mathbf{s}(\varphi)}$, i.e.

$$\widehat{\varphi}(\mathbf{s}(\varphi)x|\mathcal{K}) = \varphi(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) = \varphi(x),$$

we infer that $\widehat{\varphi}$ is a normal positive faithful functional. Consider the map $\widehat{\Phi}_*$ defined on $\big(\mathcal{M}_{\mathbf{s}(\varphi)}\big)_*$ by the formula

$$\widehat{\Phi}_*(\widehat{\rho}) = \sum_i \mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)\widehat{\rho}\mathbf{s}(\varphi)a_i^*\mathbf{s}(\varphi), \quad \widehat{\rho} \in \big(\mathcal{M}_{\mathbf{s}(\varphi)}\big)_*.$$

The relation $\Phi(\mathbf{s}(\varphi)) \geqslant \mathbf{s}(\varphi)$ yields

$$\sum_i \mathbf{s}(\varphi)a_i^*\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi) = \mathbf{s}(\varphi)\Phi(\mathbf{s}(\varphi))\mathbf{s}(\varphi) = \mathbf{s}(\varphi),$$

and further

$$\begin{aligned}\sum_i \mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)a_i^*\mathbf{s}(\varphi) &= \mathbf{s}(\varphi)\widetilde{\Phi}(\mathbf{s}(\varphi))\mathbf{s}(\varphi)\\ &\leqslant \mathbf{s}(\varphi)\widetilde{\Phi}(\mathbf{1})\mathbf{s}(\varphi) \leqslant \mathbf{s}(\varphi),\end{aligned}$$

thus, since $\mathbf{s}(\varphi)$ is the identity of $\mathcal{M}_{\mathbf{s}(\varphi)}$, our previous considerations may be applied to the von Neumann algebra $\mathcal{M}_{\mathbf{s}(\varphi)}$ and the Lüders operation $\widehat{\Phi}$ satisfying conditions (1.2).

For the functional $\widehat{\varphi}$ we have

$$\sum_i(\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)\widehat{\varphi}\mathbf{s}(\varphi)a_i^*\mathbf{s}(\varphi))(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) = \sum_i \varphi(a_i^*\mathbf{s}(\varphi)x\mathbf{s}(\varphi)a_i)$$
$$= \sum_i(a_i\varphi a_i^*)(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) = \varphi(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) = \widehat{\varphi}(\mathbf{s}(\varphi)x\mathbf{s}(\varphi))$$

showing that $\widehat{\Phi}_*(\widehat{\varphi}) = \widehat{\varphi}$. Since $\widehat{\varphi}$ is faithful, we have, as observed after Proposition 2.8,

$$\mathbf{s}(\varphi)\widetilde{\Phi}(\mathbf{s}(\varphi))\mathbf{s}(\varphi) = \sum_i \mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)a_i^*\mathbf{s}(\varphi) = \mathbf{s}(\varphi), \tag{2.5}$$

giving

$$\mathbf{s}(\varphi)(\mathbf{1} - \widetilde{\Phi}(\mathbf{s}(\varphi)))\mathbf{s}(\varphi) = 0.$$

From this we obtain

$$\mathbf{s}(\varphi)(\mathbf{1} - \widetilde{\Phi}(\mathbf{s}(\varphi))) = 0,$$

which means that

$$\mathbf{s}(\varphi) = \mathbf{s}(\varphi)\widetilde{\Phi}(\mathbf{s}(\varphi)) = \widetilde{\Phi}(\mathbf{s}(\varphi))\mathbf{s}(\varphi),$$

showing the inequality

$$\mathbf{s}(\varphi) \leqslant \widetilde{\Phi}(\mathbf{s}(\varphi)).$$

Further, we have

$$\widetilde{\Phi}(\mathbf{s}(\varphi)^{\perp}) = \widetilde{\Phi}(\mathbf{1}) - \widetilde{\Phi}(\mathbf{s}(\varphi)) \leqslant \mathbf{1} - \mathbf{s}(\varphi) = \mathbf{s}(\varphi)^{\perp},$$

which means that

$$\sum_i a_i\mathbf{s}(\varphi)^{\perp}a_i^* \leqslant \mathbf{s}(\varphi)^{\perp}.$$

This yields the relation

$$\sum_i \mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)^{\perp}a_i^*\mathbf{s}(\varphi) = 0,$$

i.e.

$$\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)^{\perp} = 0.$$

The last equality means that

$$\mathbf{s}(\varphi)a_i = \mathbf{s}(\varphi)a_i\mathbf{s}(\varphi),$$

which together with Lemma 2.9 gives

$$\mathbf{s}(\varphi)a_i = a_i\mathbf{s}(\varphi). \tag{2.6}$$

Taking into account Proposition 2.8, we obtain

$$\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)\widehat{\varphi} = \widehat{\varphi}\,\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi). \tag{2.7}$$

For each $x \in \mathcal{M}$, we have by virtue of (2.6)

$$\begin{aligned}(\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)\,\widehat{\varphi})(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) &= \widehat{\varphi}(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi))\\ &= \varphi(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)) = \varphi(x\mathbf{s}(\varphi)a_i)\\ &= \varphi(xa_i\mathbf{s}(\varphi)) = \varphi(xa_i),\end{aligned}$$

and

$$\begin{aligned}(\widehat{\varphi}\,\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi))(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) &= \widehat{\varphi}(\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) = \varphi(a_i\mathbf{s}(\varphi)x) \\ &= \varphi(\mathbf{s}(\varphi)a_ix) = \varphi(a_ix),\end{aligned}$$

which means that

$$\begin{aligned}(a_i\varphi)(x) = \varphi(xa_i) &= (\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi)\,\widehat{\varphi})(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) \\ &= (\widehat{\varphi}\,\mathbf{s}(\varphi)a_i\mathbf{s}(\varphi))(\mathbf{s}(\varphi)x\mathbf{s}(\varphi)) = \varphi(a_ix) = (\varphi a_i)(x).\end{aligned}$$

As is easily seen, the last equality is equivalent to the relation

$$\varphi \in \mathcal{M}_* \cap \mathcal{A}'.$$

Now, let $\varphi \in \operatorname{Fix}\Phi_*$ be arbitrary. We have

$$\varphi = \varphi_1 + i\varphi_2,$$

with φ_1 and φ_2 hermitian. It is clear that $\varphi_1, \varphi_2 \in \operatorname{Fix}\Phi_*$. For the Jordan decomposition

$$\varphi_k = \varphi_k^+ - \varphi_k^-, \quad k = 1, 2,$$

we have

$$\varphi_k = \Phi_*(\varphi_k) = \Phi_*(\varphi_k^+) - \Phi_*(\varphi_k^-),$$

and

$$\|\Phi_*(\varphi_k^+)\| = \varphi_k^+(\Phi(\mathbf{1})) = \varphi_k^+(\mathbf{1}) = \|\varphi_k^+\|,$$

and similarly for φ_k^-, so the uniqueness of the Jordan decomposition yields

$$\Phi_*(\varphi_k^+) = \varphi_k^+ \qquad \text{and} \qquad \Phi_*(\varphi_k^-) = \varphi_k^-,$$

i.e. $\varphi_k^+, \varphi_k^- \in \operatorname{Fix}\Phi_*$. From the first part of the proof, it follows that $\varphi_k^+, \varphi_k^- \in \mathcal{M}_* \cap \mathcal{A}'$, and thus $\varphi \in \mathcal{M}_* \cap \mathcal{A}'$ which ends the proof. □

The theorem above, together with some earlier results, yields a number of interesting consequences, namely:

1. Assume that for the Lüders operation Φ_* condition (1.1) holds. Then we have
$$\operatorname{Fix}\Phi_* = \operatorname{Fix}\widetilde{\Phi}_* = \mathcal{M} \cap \mathcal{A}'.$$
2. Assume that for the Lüders operation Φ_* condition (1.2) holds with $c \neq \mathbf{1}$. Then there is no faithful Φ-invariant normal state.
3. Assume that for the Lüders operation Φ_* condition (1.2) holds. Let p_r be the *recurrence projection* for Φ, i.e.
$$p_r = \sup\{\mathbf{s}(\rho) : 0 \leqslant \rho \in \operatorname{Fix}\Phi_*\}.$$
Then by [8, Lemma 1], we have $\Phi(p_r) \geqslant p_r$, and exactly as in the proof of Lemma 2.9 we obtain
$$a_ip_r = p_ra_ip_r,$$
and equivalently
$$p_ra_i^* = p_ra_i^*p_r.$$
Further, we have
$$a_ip_ra_i^* = p_ra_ip_ra_i^* = p_ra_ip_ra_i^*p_r,$$

and summing up on i we get

$$\widetilde{\Phi}(p_r) = p_r\widetilde{\Phi}(p_r)p_r.$$

Now, following the lines of the proof of Theorem 2.10 from equation (2.5) on, we first obtain

$$\widetilde{\Phi}(p_r^\perp) \leqslant p_r^\perp,$$

and then

$$p_r a_i = p_r a_i p_r,$$

which gives

$$p_r a_i = a_i p_r.$$

Consequently,

$$c \geqslant \sum_i a_i p_r a_i^* = p_r c = c p_r,$$

showing that

$$p_r \leqslant c,$$

and thus

$$\sum_i a_i p_r a_i^* = p_r c = p_r.$$

In particular, if $\operatorname{Fix}\Phi_* \neq \{0\}$, i.e. $p_r \neq 0$, then we must have $\|c\|_\infty = 1$.

4. Assume that for the Lüders operation Φ_* condition (1.1) holds. It is easily seen that the relation

$$\sum_i a_i \rho a_i^* = \rho$$

for $\rho \in \mathcal{M}_*$ is equivalent to the relation

$$\sum_i a_i h_\rho a_i^* = h_\rho$$

for the densities $h_\rho \in L^1(\mathcal{M}, \tau)$, and that the relation

$$a\rho = \rho a, \quad a \in \mathcal{A}$$

is equivalent to the relation

$$a h_\rho = h_\rho a, \quad a \in \mathcal{A}.$$

Thus Theorem 2.10 gives the description of the fixed points of the operation $\widetilde{\Phi}\colon L^1(\mathcal{M}, \tau) \to L^1(\mathcal{M}, \tau)$ as those elements in $L^1(\mathcal{M}, \tau)$ that commute with $\mathcal{A}$.

To complete our considerations, we shall show the equality (FP*) for an arbitrary algebra but with a more restrictive assumption about the a_i.

Theorem 2.11. *Let $\mathcal{M}$ be an arbitrary von Neumann algebra, and let Φ_* be a Lüders operation for which condition* (1.1) *holds. Assume that $\mathcal{A}$ is abelian. Then*

$$\operatorname{Fix}\Phi_* = \mathcal{M}_* \cap \mathcal{A}'. \tag{FP*}$$

Proof. By the argument used in the proof of the preceding theorem, it is enough to show that for any $0 \leqslant \varphi \in \operatorname{Fix}\Phi_*$ we have $\varphi \in \mathcal{M}_* \cap \mathcal{A}'$.

Assume first that φ is faithful. Let $\mathbb{E}\colon \mathcal{M} \to \operatorname{Fix}\Phi = \mathcal{M} \cap \mathcal{A}'$ be the conditional expectation such that

$$\varphi \circ \mathbb{E} = \varphi$$

as described before. Since $\mathcal{A} \subset \mathcal{A}'$, we have for any $x \in \mathcal{M}$ and $a \in \mathcal{A}$

$$\begin{aligned}(a\varphi)(x) &= \varphi(xa) = \varphi(\mathbb{E}(xa)) = \varphi((\mathbb{E}x)a)\\ &= \varphi(a\mathbb{E}x) = \varphi(\mathbb{E}(ax)) = \varphi(ax) = (\varphi a)(x),\end{aligned}$$

showing that $\varphi \in \mathcal{M}_* \cap \mathcal{A}'$.

For arbitrary $0 \leqslant \varphi \in \operatorname{Fix}\Phi_*$, we consider the reduced von Neumann algebra $\mathcal{M}_{\mathbf{s}(\varphi)}$ and proceed exactly as in the proof of Theorem 2.10. □

From the result above, we obtain

Corollary. *Let Φ be the classical Lüders operation defined on the predual of an arbitrary von Neumann algebra $\mathcal{M}$ as*

$$\Phi(\rho) = \sum_i e_i \rho e_i, \quad \rho \in \mathcal{M}_*,$$

where $\{e_i\}$ is a partition of identity into orthogonal projections from $\mathcal{M}$. Then equality (FP) holds.*

3. Concluding remarks

The object of our interest in this paper has been the equality

$$\operatorname{Fix}\Phi_* = \mathcal{M}_* \cap \mathcal{A}' \tag{FP*}$$

for a generalized Lüders operation Φ_*. A "dual" equality is

$$\operatorname{Fix}\Phi = \mathcal{M} \cap \mathcal{A}' \tag{FP}$$

but even in the case $a_i = a_i^*$, despite the same form of Φ_* and Φ, there are situations in which only one of these equalities hold and situations in which hold both. For instance, for semifinite algebras we have (FP*) but not always (FP) (example: $\mathcal{M} = \mathbb{B}(\mathcal{H})$, see [1]), while for finite algebras we have both. The same is true if the a_i and a_i^* in the definition of Φ_* commute. If $\operatorname{Fix}\Phi_*$ contains a faithful state, then we have (FP) but whether (FP*) holds is an open problem even though $\operatorname{Fix}\Phi$ is then a von Neumann algebra and we have the relation

$$(\operatorname{Fix}\Phi)_* = \operatorname{Fix}\Phi_*.$$

In general, the relation between the two equalities (FP*) and (FP) is not clear, and let us finish with showing the extreme situation

$$\operatorname{Fix}\Phi_* = \{0\},$$

and $\operatorname{Fix}\Phi$ — a non-trivial von Neumann algebra. Put

$$\mathcal{H} = L^2([0,1], m),$$

where m is Lebesgue measure, $\mathcal{M} = \mathbb{B}(\mathcal{H})$, and consider the von Neumann algebra $L^\infty([0,1], m)$. This algebra is generated by the spectral measure e defined as

$$e(\Delta) = \chi_\Delta, \quad \Delta \in \mathcal{B}([0,1]),$$

where χ_Δ is the characteristic function of Borel set Δ. Define $a_1 \in L^\infty([0,1], m)$ by the formula

$$a_1 = \int_0^1 \sqrt{\lambda}\, e(d\lambda),$$

and put $a_2 = (\mathbf{1} - a_1^2)^{1/2}$. We have $a_1^2 + a_2^2 = \mathbf{1}$, and

$$\mathcal{A} = W^*(\{a_1, a_2\}) = L^\infty([0,1], m);$$

moreover, $\mathcal{A} = \mathcal{A}'$ since $L^\infty([0,1], m)$ is maximal abelian. Let Φ_* be the Lüders operation defined by a_1, a_2. Assume that $0 \leqslant \rho \in \operatorname{Fix}\Phi_*$. Then, as is easily seen, for its density h_ρ we have

$$h_\rho = \sum_{i=1}^{2} a_i h_\rho a_i,$$

which on account of [2] yields $h_\rho \in \mathcal{A}' = \mathcal{A}$. In particular, the spectral projections of h_ρ belong to $\mathcal{A}$. But since h_ρ is tracial, these projections are of finite rank, and $L^\infty([0,1], m)$ has no finite rank projections except 0. Consequently, $h_\rho = 0$, i.e. $\rho = 0$, and the argument used in the proof of Theorem 2.10 shows that for arbitrary $\rho \in \operatorname{Fix}\Phi_*$ we have $\rho = 0$. On the other hand, as shown in [2], we have

$$\operatorname{Fix}\Phi = \mathcal{A}' = L^\infty([0,1], m).$$

As indicated in the Introduction, the investigations of $\operatorname{Fix}\Phi_*$ have, in principle, hardly begun, and we hope that the present paper will be a starting point in this direction.

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