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THE MONOTONE MINORANT METHOD AND EIGENVALUE PROBLEM FOR MULTIVALUED OPERATORS IN CONES

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Abstract. The main aim of this paper is to obtain a general theorem on existence of continuous branch of solutions of equations which depend on a parameter by using the monotone minorant method in conjunction with the theory of fixed point index. As an application, we apply this theorem to prove the existence of a positive eigen-pair of multivalued homogeneous increasing operators. The simplicity and uniqueness of the eigen-pair are also investigated in this paper.
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1. Introduction

The monotone minorant method is a powerful tool for studying equations in ordered spaces. It has been used in conjunction with the theory of fixed point index to prove the existence of continuum of solutions for equations, which depends on a parameter (see [1, 3, 8, 11, 19, 21, 22] and references therein). Recently, this method has been employed by J. Mallet-Paret, R. Nussbaum [17], R. Mahadevan [15] and K.C. Chang [7] to generalize Krein-Rutman's result on eigenvalues of positive linear operators to homogeneous increasing operators.

The fixed point index for multivalued operators in cones was introduced by P.M. Fitzpatrick and W.V. Pettryshn in [9], and has been applied to proving Krasnoselskii theorem on cone expansion and compression, and Leggett-Williams fixed point theorem for multivalued operators (see, for example [2, 18, 20]). However, to the best of our knowledge, the monotone minorant method has not been applied in the settings of multivalued operators in literature. The main difficulty lies on the lack of a suitable notion of monotonicity for multivalued operators. Nevertheless, by using a natural notion of monotonicity and some general principles on ordered sets, S. Carl-S. Heikila, A. Bucur-L. Guran-A. Petrusel in [4, 5, 6] and N.B. Huy-N.H. Khanh in [12, 13] obtained fixed point theorems for increasing multivalued operators, which may be discontinuous and have nonconvex values. These results shed the light of applying the monotone minorant method in studying equations with multivalued operators.

In the present paper, we shall use the monotone minorant method in combination with the theory of fixed point index to obtain a general theorem on existence of continuous branch of solutions for multivalued operators depending on a parameter (see Section 2.1). Using this result allow us to prove the existence of a positive eigenpair of multivalued homogeneous increasing operators. We also generalize the notion of u_0-positiveness of Krasnoselskii, the notion of semi-strongly positiveness and some quantities of the type of spectral radius in [7] to multivalued operators and then apply them to investigate some estimates involving the eigenvalue, and prove its simplicity and uniqueness (see Section 2.2).

2. Main results

2.1. **The monotone minorant method.** Firstly, we review some preliminaries about the continuity and the theory of fixed point index for multivalued operators.

Definition 2.1. (see e.g. [8]) Let X, Y be Banach spaces and $F : D \subset X \to 2^Y \setminus \{\varnothing\}$ be a multivalued operator.

1. The operator F is said to be *upper* (resp. *lower*) *semi-continuous* in D if the set

$$\{x \in D : F(x) \subset V\} \text{ (resp. } \{x \in D : F(x) \cap V \neq \varnothing\})$$

is open in D for every open subset $V \subset Y$.

2. The operator F is called *compact* if for any bounded subset $B \subset D$, the set $F(B) = \cup_{x \in B} F(x)$ is relatively compact.

The following result gives necessary conditions for the continuity of multivalued operators.

Proposition 2.2. (see e.g. [10]) 1. *Assume that* F is an upper semi-continuous multivalued operator in D with closed values and $x_n \to x$, $y_n \in F(x_n)$, and $y_n \to y$. Then $y \in F(x)$.

2. *If the multivalued operator* F is lower semi-continuous in D, then for any sequence $x_n \to x$, and for every $y \in F(x)$, *there is subsequence* $\{x_{n_k}\}$ and a sequence $\{y_k\}$such that $y_k \in F(x_{n_k})$ and $y_k \to y$.

Proposition 2.3. (see [16]) *Let* $F : D \subset X \to 2^Y \setminus \{\varnothing\}$ *be a lower semi-continuous operator with closed convex values. Then* F *admits a continuous selection, that is, there is a continuous single-valued operator* $f : D \to Y$ *such that* $f(x) \in F(x)$ *for all* $x \in D$.

Let X be a Banach space and K be a cone in X. In X we define an order by $x \leq y$ iff $y - x \in K$ and we call the pair (X, K) an ordered Banach space. We also define

$\dot{K} = K \setminus \{\theta\}$ and the dual cone $K^* = \{f \in X^* : f(x) \geq 0$ for all $x \in K\}$ where θ is the zero element in X. It can be verified that $x \geq \theta$ iff $f(x) \geq 0$ for all $f \in K^*$. Moreover, if $x \in \mathrm{int}K$ then $f(x) > 0$ for all $f \in K^* \setminus \left\{\widehat{\theta}\right\}$ ($\widehat{\theta}$ is the zero element in X^*).

Let $D \subset X$ be an open bounded subset and $F : K \cap \overline{D} \to 2^K \setminus \{\varnothing\}$ be an upper semi-continuous compact operator with closed convex values. If $x \notin F(x)$ for all $x \in K \cap \partial D$ then the fixed point index (or relatively topological degree) of the operator F in D with respect to K is defined, which is an integer denoted by $i_K(F, D)$ (see e.g. [9]).

This index has all useful properties of topological degree of a single compact operator. The following results on the computation of the index were taken in [9, proof of Theorem 3.2].

Proposition 2.4. *Let D be a bounded open subset and $F : \overline{D} \cap K \to 2^K \setminus \{\varnothing\}$ be an upper semi-continuous compact operator with closed convex values such that $x \notin F(x)$ for all $x \in K \cap \partial D$. Then*

1. $i_K(F, D) = 0$ *if there is* $x_0 \in K \setminus \{\theta\}$ *such that* $x \notin F(x) + tx_0$ *for all* $t > 0,\ x \in K \cap \partial D$.
2. $i_K(F, D) = 1$ *if* $tx \notin F(x)$ *for all* $t > 1,\ x \in K \cap \partial D$.

Now we introduce some orders between the two subsets.

Definition 2.5. Let (X, K) be an ordered Banach space.

1. For subsets $A,\ B \in 2^X \setminus \{\varnothing\}$ we define

(a) $A \leq_{(1)} B$ iff ($\forall x \in A$, $\exists y \in B$ such that $x \leq y$).

(b) $A \leq_{(2)} B$ iff ($\forall y \in B$, $\exists x \in A$ such that $x \leq y$).

(c) $A \leq_{(3)} B$ iff ($x \in A$ and $y \in B$ imply that $x \leq y$).

Clearly, the above relations are transitive and coincident with the order defined in (X, K) if the sets A and B are singletons.

2. An operator $F : M \subset X \to 2^X \setminus \{\varnothing\}$ is said to be (k)-*increasing*, $k = 1, 2$, if $x, y \in M$ and $x \leq y$ imply that $F(x) \leq_{(k)} F(y)$; moreover, it is said to be (3)-*increasing* if $x, y \in M$ and $x < y$ imply that $F(x) \leq_{(3)} F(y)$.

Example. It is easy to see that if $A : X \to X$ is a single-valued linear operator then A is increasing iff $A(x) \in K$ for every $x \in K$ (or equivalently $x \geq \theta$ implies $A(x) \geq \theta$). In multivalued analysis, a replacement of linear operator which is so-called convex processes, is an operator $A : X \to 2^X \setminus \{\varnothing\}$ satisfying

(i) $A(x) + A(y) \subset A(x + y)$ for all $x, y \in X$,

(ii) $A(tx) = tA(x)$ for every $t > 0, x \in X$ (we also say A is positively 1-homogeneous).

If A is a convex process then we have

1. A is (1)-increasing if $\{\theta\} \leq_{(1)} A(x)$ for all $x \geq \theta$,
2. A is (2)-increasing if $A(x) \leq_{(2)} \{\theta\}$ for all $x \leq \theta$.

Let us prove the first assertion. For $x \leq y$ we have $\{\theta\} \leq_{(1)} A(y - x)$ and hence $\exists u \in A(y - x) : \theta \leq u$. From $A(y - x) + A(x) \subset A(y)$, we deduce that

$$\forall v \in A(x)\ \ \exists w \in A(y) : u + v = w \text{ and } v \leq w \text{ as well.}$$

Therefore, $A(x) \leq_{(1)} A(y)$.

Theorem 2.6. *Let (X, K) be an ordered Banach space and $F : K \to 2^K \setminus \{\varnothing\}$ be an upper semi-continuous compact operator with closed convex values. Assume that there is (2)-increasing operator $B : K \to 2^K \setminus \{\varnothing\}$ satisfying*

(i) $B(x) \leq_{(2)} F(x)$ for every $x \in K$,

(ii) there are positive numbers a, b and an element $u \in K \setminus \{\theta\}$ such that $\{btu\} \leq_{(2)} B(tu)$ for all $t \in [0, a]$.

Then the solution set $S = \left\{ x \in \dot{K} : \exists \lambda > 0,\ x \in \lambda F(x) \right\}$ forms an unbounded continuous branch emanating from θ, that is, $S \cap \partial G \neq \varnothing$ for any bounded open subset $G \ni \theta$.

Proof. Let $G \subset X$ be a bounded open subset and $G \ni \theta$. We now claim that $S \cap \partial G \neq \varnothing$. Indeed, assume by contradiction that $x \notin \lambda F(x)$ for all $x \in K \cap \partial G$, $\forall \lambda > 0$. Then, by the homotopy-invariant property, the index $i_K(\lambda F, G)$ is a constant for $\lambda \in (0, \infty)$.

We shall show $i_K(\lambda F, G) = 1$ for sufficiently small λ by proving that

$$tx \notin \lambda F(x) \ \ \forall x \in K \cap \partial G, \ \forall t > 1. \tag{2.1}$$

In fact, since G is open with $\theta \in G$ and $F(K \cap \partial G)$ is relatively compact, there are numbers $\alpha > 0$ and $\beta > 0$ such that $\|x\| \geq \alpha \ \forall x \in K \cap \partial G$ and $\|y\| \leq \beta$ $\forall y \in F(K \cap \partial G)$. If $tx \in \lambda F(x)$ for some $x \in K \cap \partial G$ and $t, \lambda > 0$ then we have

$$t\alpha \leq t\|x\| \leq \lambda\beta. \tag{2.2}$$

Therefore, (2.1) holds for $\lambda < \dfrac{\alpha}{\beta}$.

To obtain a contradiction we will show $i_K(\lambda F, G) = 0$ for large enough λ. Let us prove that

$$\exists \lambda_0 > 0 : x \notin \lambda F(x) + tu, \ \forall x \in K \cap \partial G, \ \forall \lambda \geq \lambda_0. \tag{2.3}$$

Indeed, if (2.3) is not true, we can find sequences $\{x_n\} \subset K \cap \partial G$, $\{t_n\} \subset (0, \infty)$ and $\{\lambda_n\} \subset (0, \infty)$ such that

$$\lambda_n \to \infty, \ x_n \in \lambda_n F(x_n) + t_n u\ . \tag{2.4}$$

Let s_n be a maximal number satisfying $x_n \geq s_n u$. From (2.4), we get $s_n \geq t_n$ and hence $s_n > 0$. Set $N_1 = \{n \in \mathbb{N}^* : s_n \leq a\}$ and $N_2 = \{n \in \mathbb{N}^* : s_n > a\}$. We shall show that both N_1 and N_2 are finite, thus we get a contradiction. Indeed, for $n \in N_1$ we have

$$\{bs_n u\} \leq_{(2)} B(s_n u) \leq_{(2)} B(x_n) \leq_{(2)} F(x_n). \tag{2.5}$$

It follows from (2.4) that $\lambda_n F(x_n) \leq_{(2)} \{x_n\}$ which togeter with (2.5) yields $x_n \geq \lambda_n b s_n u$. By the maximality of s_n, we deduce $\lambda_n b \leq 1$. Therefore, N_1 is finite. For $n \in N_2$, by the same argument used to obtain (2.5), we arrive at

$$\{\lambda_n abu\} \leq_{(2)} \lambda_n B(au) \leq_{(2)} \lambda_n B(s_n u) \leq_{(2)} \lambda_n F(x_n) \leq_{(2)} \{x_n\}. \tag{2.6}$$

If N_2 is infinite then from (2.6), the boundeness of $\{x_n\}$ and $\lambda_n \to \infty$ we obtain $u \leq \theta$. This is a contradiction. Thus (2.3) holds and $i_K(\lambda F, G) = 0$ for sufficiently large λ. The proof is complete. □

Theorem 2.7. *If hypothesis on "upper semi-continuous" of Theorem 2.6 is replaced by "lower semi-continuous" then the conclusion is still true.*

Proof. Let f be a continuous selection of F. Then f is completely continuous, $B(x) \leq_{(2)} f(x)$ and $\{x \in K\setminus\{\theta\} : x = \lambda f(x)\} \subset \{x \in K\setminus\{\theta\} : x \in \lambda F(x)\}$. By applying Theorem 2.6 to f, we obtain the conclusion. □

2.2. **Application to eigenvalue problems.** In what follows, we consider an ordered Banach space (X, K). The pair (λ_0, x_0) is called a *positive eigen-pair* of the operator $A : K \to 2^K\setminus\{\varnothing\}$ if $x_0 \in K\setminus\{\theta\}$, $\lambda_0 > 0$ and $\lambda_0 x_0 \in A(x_0)$.

2.2.1. **Existence of a positive eigen-pair.**

Theorem 2.8. *Let $A : K \to 2^K\setminus\{\varnothing\}$ be a positively 1-homogeneous, compact, upper semi-continuous operator with closed convex values, such that*
(i) A is (2)-increasing,
(ii) $\exists u \in K\setminus\{\theta\}$, $\exists \alpha > 0 : \{\alpha u\} \leq_{(2)} A(u)$.
Then A admits a positive eigen-pair (λ_0, x_0) with $\lambda_0 \geq \alpha$ and $\|x_0\| = 1$.

Proof. Applying Theorem 2.6 to the operator $A(x) + \frac{u}{n}$ with A playing the role of a minorant, we find two sequences $\{x_n\}$ and $\{\lambda_n\}$ such that

$$\lambda_n > 0,\ x_n \in K,\ \ \|x_n\| = 1 \text{ and } \lambda_n x_n \in A(x_n) + \frac{u}{n},$$

or equivalently, $\lambda_n x_n = y_n + \frac{u}{n}$ for some $y_n \in A(x_n)$. Since A is compact, we can assume that $\{y_n\}$ is convergent to some $y_0 \in K$. We shall show $\lambda_n \geq \alpha$ for all n. Let t_n be a maximal number such that $x_n \geq t_n u$. Then we have $t_n \geq \frac{1}{n\lambda_n}$ and

$$\{\alpha t_n u\} \leq_{(2)} A(t_n u) \leq_{(2)} A(x_n) \leq_{(2)} \{\lambda_n x_n\}.$$

Therefore, $t_n \geq \frac{\alpha t_n}{\lambda_n}$ so $\lambda_n \geq \alpha$. We can assume that $\lambda_n \to \lambda_0 \geq \alpha$. Hence, $x_n = \frac{1}{\lambda_n} y_n + \frac{1}{n\lambda_n} u$ converges to some $x_0 \in K$, and so $\lambda_0 x_0 = y_0$ and $\|x_0\| = 1$. Since A is upper semi-continuous, we have $y_0 \in A(x_0)$. Thus (λ_0, x_0) is a positive eigen-pair of A. □

Theorem 2.9. *Let $A : K \to 2^K\setminus\{\varnothing\}$ be a positively 1-homogeneous, compact, upper semi-continuous operator with closed convex values such that*
(i) A is (2)-increasing,
(ii) The number $\rho(A) = \sup_{u\in K, \|u\|=1}\left\{\inf\left\{\lambda > 0 : \exists x \geq u,\ A(x) \leq_{(2)} \lambda x\right\}\right\}$ is positive.
Then A has a positive eigen-pair (λ_0, x_0) with $\lambda_0 \geq \rho(A)$. Moreover, if A is (3)-increasing then $\lambda_0 = \rho(A)$.

Proof. From the definition of $\rho(A)$, there is a sequence $\{u_n\} \subset K$ such that $\|u_n\| = 1$ and the sequence $t_n = \inf\left\{\lambda > 0 : \exists x \geq u_n,\ A(x) \leq_{(2)} \lambda x\right\}$ converges to $\rho(A)$. The application of Theorem 2.6 to the operators $A(x) + \frac{u_n}{n}$ gives us the existence of sequences $\{x_n\} \subset K$ and $\{\lambda_n\}$ satisfying $\|x_n\| = 1$, $\lambda_n > 0$ and $\lambda_n x_n \in A(x_n) + \frac{u_n}{n}$ so $\lambda_n x_n = y_n + \frac{u_n}{n}$ for some $y_n \in A(x_n)$. First we shall prove that $\lambda_n \geq t_n$. Indeed, from $n\lambda_n x_n \in \frac{1}{\lambda_n} A(n\lambda_n x_n) + u_n$, we have $n\lambda_n x_n \geq u_n$ and $A(n\lambda_n x_n) \leq_{(2)} \lambda_n (n\lambda_n x_n)$. Therefore, $\lambda_n \geq t_n$ according to the definition of t_n. Following the same arguments as in the proof of Theorem 2.8, we can assume that $\lambda_n \to \lambda_0 \geq \rho(A)$; $x_n \to x_0$; $y_n \to y_0 \in A(x_0)$ and we also deduce that $\lambda_0 x_0 \in A(x_0)$.

Now, let A be (3)-increasing, we shall prove that $\lambda_0 \leq \rho(A)$. Consider an element x such that $x \geq x_0$ and $A(x) \leq_{(2)} \lambda x$. Let t be a maximal number such that $x \geq tx_0$. Clearly, $t \geq 1$ and

$$t\lambda_0 x_0 \in A(tx_0) \leq_{(3)} A(x) \leq_{(2)} \{\lambda x\}.$$

It follows from Definition 2.5 that $t\lambda_0 x_0 \leq \lambda x$. By the maximality of t, we get $t \geq t\frac{\lambda_0}{\lambda}$, and so $\lambda \geq \lambda_0$. Thus $\inf\left\{\lambda > 0 : \exists x \geq x_0,\ A(x) \leq_{(2)} \lambda x\right\} \geq \lambda_0$ and $\rho(A) \geq \lambda_0$. The proof is complete. □

2.2.2. **Some Krein-Rutman's properties.** We first generalize the notion of u_0-positiveness, the notion of semi-strong positiveness and some quantities due to K.C.Chang [7] for multivalued operators.

Definition 2.10. Let K be a cone in Banach space X and $A : K \to 2^K \setminus \{\varnothing\}$, $u_0 \in K\setminus\{\theta\}$. We denote $\langle u_0 \rangle_+ = \{tu_0 : t > 0\}$.

1. A is said to be u_0- *positive* if $\forall x \in K\setminus\{\theta\}$ we have $\langle u_0\rangle_+ \leq_{(2)} A(x) \leq_{(1)} \langle u_0\rangle_+$ or equivalently

$$\forall x \in K\setminus\{\theta\}, \forall y \in A(x)\ \ \exists \alpha, \beta > 0 : \alpha u_0 \leq y \leq \beta u_0.$$

2. A is said to be *strongly* u_0- *positive* if $\forall x \in K\setminus\{\theta\}$ then $\exists \alpha, \beta > 0$ such that

$$\alpha u_0 \leq_{(2)} A(x) \leq_{(1)} \beta u_0.$$

Definition 2.11. Assume that $\mathrm{int}K \neq \varnothing$. An operator $A : K \to 2^K \setminus \{\varnothing\}$ is said to be *semi strongly positive* if $\exists g \in K^*$ such that $\forall x \in \dot{K}\setminus \mathrm{int}K$, we have $\langle g, x\rangle = 0$ and $\langle g, z\rangle > 0$ for all $z \in A(x)$.

Definition 2.12. Given an operator $A : K \to 2^K \setminus \{\varnothing\}$.

1. We associate every $x \in \dot{K}$ with the subsets of K^*:

$$K^*(x) = \{f \in K^* : f(x) > 0\}, S^*(x) = \{f \in K^* : f(x) = 1\}$$

and with the numbers

$$\begin{aligned} \mu_*(x) &= \inf\{\langle f, z\rangle : z \in A(x), f \in S^*(x)\} \text{ and} \\ \mu^*(x) &= \sup\{\langle f, z\rangle : z \in A(x), f \in S^*(x)\}. \end{aligned}$$

2. We define

$$r_*(A) = \sup_{x \in K\setminus\{\theta\}} \mu_*(x) \text{ and } r^*(A) = \inf_{x \in K\setminus\{\theta\}} \mu^*(x).$$

If int$K \neq \varnothing$ we define

$$or_*(A) = \sup_{x \in \mathrm{int} K} \mu_*(x) \text{ and } or^*(A) = \inf_{x \in \mathrm{int} K} \mu^*(x).$$

Lemma 2.13. 1. $\mu_*(x)\, x \leq_{(2)} A(x) \quad \forall x \in K \backslash \{\theta\}$.

2. *If* $\mu^*(x) < \infty$ *then* $A(x) \leq_{(1)} \mu^*(x)\, x$.

3. $\mu^*(x) < \infty$ *iff* $\exists \mu > 0 : A(x) \leq_{(1)} \mu x$. *If* $x \in$*int*K *and* $A(x)$ *is compact then* $\mu^*(x) < \infty$.

Proof. 1. It follows from Definition 2.12 that $\mu_*(x) \leq \langle f, z\rangle$ for all $z \in A(x)$, $f \in S^*(x)$. Hence

$$\mu_*(x)\langle f, x\rangle \leq \langle f, z\rangle \quad \text{for all } z \in A(x),\ f \in K^*(x). \tag{2.7}$$

Since $K^*(x)$ is dense in K^* (see [7, p.544]), we deduce from (2.7) that $\langle f, z - \mu_*(x)\, x\rangle \geq 0$, $\forall f \in K^*$. Hence, $\mu_*(x)\, x \leq z \quad \forall z \in A(x)$.

2. Similarly to (2.7), we have

$$\langle f, z\rangle \leq \mu^*(x)\langle f, x\rangle \quad \text{for all } z \in T(x),\ f \in K^*(x). \tag{2.8}$$

The set $K^*(x)$ is dense in K^* so (2.8) holds for $\forall f \in K^*$. This implies that $z \leq \mu^*(x)\, x$, $\forall z \in A(x)$, or equivalently, $A(x) \leq_{(1)} \mu^*(x)\, x$.

3. The first assertion follows from the definition of $\mu^*(x)$ and property 2. Let $x \in$int$K$ and $r > 0$ such that $B(x, r) \subset K$. For $y \in X$ with $\|y\| = 1$ we have $x \geq \pm ry$. Therefore, we have $1 \geq r\, |f(y)|$ for all $f \in S^*(x)$. Thus, $\|f\| \leq \dfrac{1}{r}$ for all $f \in S^*(x)$. The set $S^*(x)$ is bounded and $(*)$-weakly closed, hence, it is $(*)$-weakly compact. The set $S^*(x) \times A(x)$ is compact and the operator $(f, z) \mapsto \langle f, z\rangle$ is continuous with respect to the $(*)$-weak topology in X^* and the norm-topology in X. Therefore $\mu^*(x) < \infty$. □

Lemma 2.14. *Let* A *be a* $(k)-$*increasing and positively* $1-$*homogeneous operator,* $k = 1, 2$. *Let* $x, y \in K \backslash \{\theta\}$ *and* $\lambda, \mu > 0$ *satisfying* $A(x) \leq_{(k)} \lambda x$ *and* $\mu y \leq_{(k)} A(y)$. *Moreover, assume that one of the following conditions holds*

(i) A *is* u_0-*positive,*

(ii) $x \in$*int*K.

Then $\mu \leq \lambda$.

Proof. Let t be a maximal number such that $x \geq ty$. We shall prove $t > 0$. Clearly, it is true if $x \in$int$K$. By the definition of the relation "$\leq_{(k)}$", we conclude that $\exists u \in A(x)$, $\exists v \in A(y)$ such that $u \leq \lambda x$ and $\mu y \leq v$. Since A is u_0-positive, we can find $\alpha, \beta > 0$ such that $\alpha u_0 \leq u$ and $v \leq \beta u_0$. Therefore, $x \geq \dfrac{\alpha}{\lambda} u_0 \geq \dfrac{\alpha\mu}{\lambda\beta} y$ which proves $t > 0$. By the monotonicity of A we have

$$t\mu y \leq_{(k)} tA(y) = A(ty) \leq_{(k)} A(x) \leq_{(k)} \lambda x$$

which shows $\lambda \geq \mu$ from the maximality of t. □

Theorem 2.15. *Assume that the operator* A *is positively* 1*-homogeneous, compact, upper semi-continuous with closed convex values. In addition, let* A *be* $(2)-$*increasing and* $r_*(A) > 0$. *Then* A *admits a positive eigen-pair* (λ_0, x_0) *with* $\lambda_0 \geq r_*(A)$. *Moreover,*

1. *if A is $(1)-$increasing then*
   (a) $r_*(A) \leq \lambda_0 \leq r^*(A)$ if A is strongly u_0-positive.
   (b) $x_0 \in \text{int}K$ and $r_*(A) \leq \lambda_0 \leq or^*(A)$ if A is semi strong positive.
2. *If A is lower semi-continuous, semi strong positive and is $(3)-$increasing then $r_*(A) = \lambda_0 = r^*(A)$.*

Proof. Since $r^*(A) > 0$, there is a sequence $\{x_n\} \subset K$ such that $\|x_n\| = 1$ and $0 < r_*(A) - \dfrac{1}{n} \leq \mu_*(x_n)$. By Lemma 2.13, it follows that $\left(r_*(A) - \dfrac{1}{n}\right) x_n \leq_{(2)} A(x_n)$. On the other hand, we use Theorem 2.8, then there exist sequences $\{y_n\} \subset K$, and $\{\lambda_n\} \subset [0, \infty)$ such that $\lambda_n \geq r_*(A) - \dfrac{1}{n}$, $\|y_n\| = 1$ and $\lambda_n y_n \in A(y_n)$. At this stage, by using the same argument as in the proof of Theorem 2.8, it can be verified that A admits a positive eigen-pair (λ_0, x_0) with $\lambda_0 \geq r_*(A)$.

1a. Since A is strongly u_0-positive, there is $\beta > 0$ such that $A(u_0) \leq_{(1)} \beta u_0$. Therefore, by Lemma 2.13, we have $\mu^*(u_0) < \infty$ and $r^*(A) < \infty$. We now choose a sequence $\{y_n\} \subset K$ such that $\|y_n\| = 1$ and $\mu^*(y_n) \to r^*(A)$. We then have $\lambda_0 x_0 \leq_{(1)} A(x_0)$ and $A(y_n) \leq_{(1)} \mu^*(y_n) y_n$, which along with Lemma 2.14 yields that $\lambda_0 \leq \mu^*(y_n)$. Therefore, $\lambda_0 \leq r^*(A)$.

1b. We first prove $x_0 \in \text{int}K$. If $x_0 \in \dot{K} \backslash \text{int}K$ then for g as in Definition 2.11 we have

$$\langle g, x_0 \rangle = 0, \quad \langle g, z \rangle > 0 \text{ for all } z \in A(x_0)$$

which contradicts with $\lambda_0 x_0 \in A(x_0)$.

Since $x_0 \in \text{int}K$, by using Lemma 2.13, we have $\mu^*(x_0) < \infty$ and $or^*(A) < \infty$ as well. Let $\{y_n\} \subset \text{int}K$ such that $\|y_n\| = 1$ and $\lim_{n \to \infty} \mu^*(y_n) = or^*(A)$. Moreover, due to the fact that

$$\lambda_0 x_0 \leq_{(1)} A(x_0), \quad A(y_n) \leq_{(1)} \mu^*(y_n) y_n$$

and Lemma 2.14, we obtain $\lambda_0 \leq \mu^*(y_n)$, hence, $\lambda_0 \leq or^*(A)$.

2. Fix an element $u \in K \backslash \{\theta\}$. For every sufficiently small ε we have $x_0 \pm \varepsilon u \in \text{int}K$ and define $x_\varepsilon = x_0 + \varepsilon u$, $y_\varepsilon = x_0 - \varepsilon u$,

$$\begin{aligned} \beta(x_\varepsilon) &= \inf\{\langle f, z \rangle : f \in S^*(x_0),\ z \in A(x_\varepsilon)\} \text{ and} \\ \gamma(y_\varepsilon) &= \sup\{\langle f, z \rangle : f \in S^*(x_0),\ z \in A(y_\varepsilon)\}. \end{aligned}$$

We shall prove $\beta(x_\varepsilon) \to \mu_*(x_0)$ and $\gamma(y_\varepsilon) \to \mu^*(x_0)$ as $\varepsilon \to 0$.

Since $S^*(x_0) \times A(x_\varepsilon)$ is compact, there is $(f_\varepsilon, z_\varepsilon) \in S^*(x_0) \times A(x_\varepsilon)$ such that $\beta(x_\varepsilon) = \langle f_\varepsilon, z_\varepsilon \rangle$. We can assume that $f_\varepsilon \to f_0 \in S^*(x_0)$ $(*)$-weakly, $z_\varepsilon \to z_0 \in A(x_0)$. Therefore, $\beta(x_\varepsilon) \to \langle f_0, z_0 \rangle \geq \mu_*(x_0)$. On the other hand, since A is lower semicontinous and $x_\varepsilon \to x_0$, it follows from Proposition 2.2 that for every $v \in A(x_0)$ there exists $u_{\varepsilon'} \in A(x_{\varepsilon'})$ such that $u_{\varepsilon'} \to v$ ($\{x_{\varepsilon'}\}$ is a subsequence of $\{x_\varepsilon\}$). For any $f \in S^*(x_0)$ we have

$$\beta(x_{\varepsilon'}) \leq \langle f, u_{\varepsilon'} \rangle = \langle f, v \rangle + \langle f, u_{\varepsilon'} - v \rangle$$

which implies $\lim_{\varepsilon' \to 0} \beta(x_{\varepsilon'}) \leq \langle f, v \rangle$. Thus $\lim_{\varepsilon \to 0} \beta(x_\varepsilon) \leq \mu_*(x_0)$. Similarly, $\gamma(y_\varepsilon) \to \mu^*(x_0)$.

Finally, from $A(y_\varepsilon) \leq_{(3)} A(x_0) \leq_{(3)} A(x_\varepsilon)$, we have

$$v \leq \lambda_0 x_0 \leq w \text{ and } \langle f, v\rangle \leq \lambda_0 \leq \langle f, w\rangle \text{ for all } f \in S^*(x_0), v \in A(y_\varepsilon), w \in A(x_\varepsilon).$$

Therefore, $\gamma(y_\varepsilon) \leq \lambda_0 \leq \beta(x_\varepsilon)$ which implies $\mu^*(x_0) \leq \lambda_0 \leq \mu_*(x_0)$ and hence $\mu^*(x_0) = \lambda_0 = \mu_*(x_0)$. □

Definition 2.16. Given $A : K \to 2^K \setminus \{\varnothing\}$.

1. The operator A is said to be u_0- *increasing* if $x \leq y$ implies

$$\langle u_0\rangle_+ \leq_{(2)} [A(y) - A(x)] \cap \dot{K},$$

or equivalently, for all $v \in A(y)$, $u \in A(x)$ if $v - u \in \dot{K}$ then $\exists \alpha > 0$ such that $v - u \geq \alpha u_0$.

2. Let (λ_0, x_0) be a positive eigen-pair of A. Then λ_0 is said to be *geometrically simple* if from $\lambda_0 x \in A(x)$ with $x \in \dot{K}$, it follows $x \in \langle x_0\rangle_+$.

3. We say that the positive eigen-pair (λ_0, x_0) of the operator A is *unique* if for any positive eigen-pair (λ, x) of A we have $\lambda = \lambda_0$ and $x \in \langle x_0\rangle_+$.

Theorem 2.17. *Let $A : K \to 2^K \setminus \{\varnothing\}$ be a positively 1-homogeneous, u_0-positive, u_0-increasing operator and (λ_0, x_0) be a positive eigen-pair of A. Then*

1. *λ_0 is geometrically simple.*
2. *If A is $(3)-$increasing then (λ_0, x_0) is unique.*

Proof. 1. Suppose that $\lambda_0 x \in A(x)$ with $x \in \dot{K}$. We need to prove $x \in \langle x_0\rangle_+$. Since A is u_0-positive, it is easy to see that x_0 and x are comparable with u_0 and so they are comparable with each other. Therefore, there exists a maximal positive number t such that $x_0 \geq tx$. We shall prove $x_0 = tx$. Indeed, otherwise, if $x_0 \neq tx$ then we have

$$\lambda_0 x_0 \in A(x_0), \quad \lambda_0 tx \in A(tx), \quad \lambda_0 x_0 - \lambda_0 tx \in \dot{K}.$$

This implies that there exist α', $\alpha > 0$ such that $\lambda_0 (x_0 - tx) \geq \alpha u_0 \geq \alpha' x$. It follows $x_0 \geq \left(t + \alpha' \lambda_0^{-1}\right) x$, we obtain a contradiction with the maximality of t.

2. Suppose that $\lambda_1 x_1 \in A(x_1)$ with $x_1 \in \dot{K}$ and $\lambda_1 > 0$. We need to prove $\lambda_1 = \lambda_0$. Assume by contradiction that $\lambda_0 > \lambda_1$. Since x_0 and x_1 are comparable, there exists a maximal positive number t such that $x_1 \geq tx_0$. If $x_1 \neq tx_0$ then we have

$$\lambda_0 tx_0 \in A(tx_0), \quad \lambda_1 x_1 \in A(x_1), \ tx_0 < x_1,$$

which give $\lambda_1 x_1 \geq \lambda_0 tx_0$ by (3)-nonotonicity of A. By the maximality of t, this yields $\lambda_1 \geq \lambda_0$, which is a contradiction with $\lambda_0 > \lambda_1$. Thus, $x_1 = tx_0$. Taking $\lambda_0 = a^2 \lambda_1$ with $a > 1$, we obtain

$$ax_0 \in A\left(\frac{x_0}{a\lambda_1}\right), \ x_0 \in A\left(\frac{x_0}{\lambda_1}\right), \ \frac{x_0}{a\lambda_1} < \frac{x_0}{\lambda_1}.$$

This is a contradiction with the fact that A is $(3)-$increasing. Thus $\lambda_0 = \lambda_1$ and hence, $x_1 \in \langle x_0\rangle_+$. □

Theorem 2.18. *Let* $intK \neq \varnothing$, $A : K \to 2^K \setminus \{\varnothing\}$ *be a positively* 1*-homogeneous operator and* (λ_0, x_0) *be a positive eigen-pair of* A. *Then*

1. λ_0 *is geometrically simple if* A *is semi strongly increasing, this is,* $\exists g \in K^*$ *such that if* $x - y \in \dot{K} \setminus intK$ *then*

$$\langle g, x-y \rangle = 0 \text{ and } \langle g, u \rangle > 0 \text{ for all } u \in A(x) - A(y). \tag{2.9}$$

2. *If* A *is semi strongly increasing and is* (3) *−increasing then* (λ_0, x_0) *is unique.*

Proof. 1. First we shall prove that $x_0 \in \text{int}K$. Indeed, assume by the contradiction that $x_0 \in \dot{K} \setminus \text{int}K$. Taking $y = \theta$ in (2.9), we obtain $\langle g, x_0 \rangle = 0$ and $\langle g, \lambda_0 x_0 - v \rangle > 0$ for $v \in A(\theta)$. Hence, $0 = \langle g, \lambda_0 x_0 \rangle \geq \langle g, \lambda_0 x_0 - v \rangle > 0$, which is a contradiction.

Let $\lambda_0 x_1 \in A(x_1)$, $x_1 \in \dot{K}$. Since $x_0 \in \text{int}K$, there exists a maximal positive t such that $x_0 \geq t x_1$. If $x_0 \neq t x_1$, by the maximality of t we have $x_0 - t x_1 \in \dot{K} \setminus \text{int}K$. Therefore, it follows from $\lambda_0 x_0 \in A(x_0)$, $t\lambda_0 x_1 \in A(t x_1)$ and (2.9) that $g(x_0 - t x_1) = 0$ and $g(\lambda_0 x_0 - \lambda_0 t x_1) > 0$. This is a contradiction.
2. Argue by the contradiction that $\lambda_1 x_1 \in A(x_1)$ and $\lambda_0 > \lambda_1$. Since A is semi strongly increasing, we have $x_0 \in \text{int}K$ and $x_1 \in \text{int}K$. Let t be a maximal number such that $x_1 \geq t x_0$, then $t > 0$. If $x_1 \neq t x_0$ then $x_1 - t x_0 \in \partial K \setminus \{\theta\}$. Hence, $\exists g \in K^*$ such that $g(x_1 - t x_0) = 0$ and $g(\lambda_1 x_1 - t\lambda_0 x_0) > 0$. Since A is (3) −increasing, then $t\lambda_0 x_0 \leq \lambda_1 x_1$ and

$$\begin{aligned} 0 &< g(\lambda_1 x_1) - g(t\lambda_0 x_0) = \lambda_1 g(x_1) - \lambda_0 t g(x_0) \\ &= \lambda_1 t g(x_0) - \lambda_0 t g(x_0) = t(\lambda_1 - \lambda_0) g(x_0) \leq 0, \end{aligned}$$

which is a contradiction. Thus, $x_1 = t x_0$. By an argument analogous to that used for the proof of Theorem 2.17, we complete the proof. □

Remark 2.19. If A is a single positively 1-homogeneous increasing operator then our results in Section 2.2 coincide with those in Theorems 4.8, 4.13 of [7]. The condition of the (3) −increasing of the operator A seems to be strong and weakening this condition is still open and would be an interesting problem to be studied elsewhere.

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