Fixed Point Theory, 19(2018), No. 1, 265-274
DOI 10.24193/fpt-ro.2018.1.21
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SOLUTION OF A PAIR OF NONLINEAR MATRIX EQUATIONS

SK MONOWAR HOSSEIN*, SNEHASISH BOSE** AND KALLOL PAUL**

*Department of Mathematics, Aliah University
IIA/27 Newtown, Kolkata - 156, West Bengal, India
E-mail: sami_milu@yahoo.co.uk

**Department of Mathematics, Jadavpur University
Jadavpur-32, West Bengal, India
E-mail: bosesonay@gmail.com, kalloldada@gmail.com

Abstract. In this paper we consider a pair of nonlinear matrix equations of the form $X = Q_1 + (Y^*XY)^{r_1},\ Y = Q_2 + (X^*YX)^{r_2}$, where Q_1, Q_2 are $n \times n$ Hermitian positive definite matrices, $r_1, r_2 \in \mathbb{R}$ and prove the existence and uniqueness of positive definite solutions of these equations. We provide an algorithm to approach the solution. We present a coupled fixed point theorem for non-decreasing mapping and show that a particular case of our nonlinear matrix equations also can be solved by using the derived coupled fixed point theorem. Also we show that by replacing Y with Y^{-1} in first equation and X with X^{-1} in second equation and taking $Q_1 = Q_2$ and $r_1 = r_2$, the reduced system can be solved using the coupled fixed point theorem of Berinde [5].
Key Words and Phrases: Fixed point, partially ordered set, matrix equation, Thompson metric.
2010 Mathematics Subject Classification: 15A24, 47H10, 47H09.

1. Introduction and preliminaries

Ran and Reurings [1] extended the Banach contraction principle in partially ordered metric space and presented an application to solve a nonlinear matrix equation of the form

$$X \pm \sum_{i=1}^{m} {A_i}^* X A_i = Q, \tag{1.1}$$

where Q is a positive definite matrix and $A_1, .., A_m$ arbitrary $n \times n$ matrices. Since then, there has been a constantly increasing interest in developing theory and investigating solution of matrix equation. Later Nieto and López [2] extended it a little further in partial ordered set and also presented its application in ordinary differential equation. They established the following result.

Theorem 1.1. ([2]) *Let $(\mathbb{X}, \leq)$ be a partially ordered set and suppose that there exists a metric d in $\mathbb{X}$ such that $(\mathbb{X}, d)$ is a complete metric space. Assume that $\mathbb{X}$ satisfies the property that, if a non-decreasing sequence $x_n \to x$ in $\mathbb{X}$, then $x_n \leq x,\ \forall\ n$. Let*

$f : \mathbb{X} \to \mathbb{X}$ be a monotone non-decreasing mapping such that there exists $k \in [0,1)$ with

$$d\left(f(x), f(y)\right) \leq k\ d(x,y),\ \forall\ x \geq y. \tag{1.2}$$

If there exists $x_0 \in \mathbb{X}$ with $x_0 \leq f(x_0)$, then f has a fixed point. In addition if for any two $x, y \in \mathbb{X}$ there exists $z \in \mathbb{X}$ comparable with both x, y, then f has a unique fixed point $\bar{x}$ and the sequence x_n defined by $x_{n+1} = f(x_n),\ \forall\ n \geq 0$ converges to $\bar{x}$ and we have the following estimate:

$$d(\bar{x}, x_n) \leq \frac{k^n}{1-k} d(x_1, x_0). \tag{1.3}$$

Next we mention some definitions related to coupled fixed point theory. Let $(\mathbb{X}, \leq)$ be a partially ordered set. Endow $\mathbb{X} \times \mathbb{X}$ with partial order:

$$\text{for } (x,y),(u,v) \in \mathbb{X} \times \mathbb{X},\ (x,y) \preceq_p (u,v) \ \text{ iff } x \leq u, y \geq v. \tag{1.4}$$

Definition 1.2. A mapping $F : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ has a non-decreasing property if $F(x,y)$ is monotone non-decreasing in x and also in y, that is, for any $x, y \in \mathbb{X}$,

$$x_1, x_2 \in \mathbb{X},\ x_1 \leq x_2 \ \Rightarrow\ F(x_1, y) \leq F(x_2, y)$$

and, respectively,

$$y_1, y_2 \in \mathbb{X},\ y_1 \leq y_2 \ \Rightarrow\ F(x, y_1) \leq F(x, y_2).$$

Definition 1.3. ([4]) A mapping $F : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ has a mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in \mathbb{X}$,

$$x_1, x_2 \in \mathbb{X},\ x_1 \leq x_2 \ \Rightarrow\ F(x_1, y) \leq F(x_2, y)$$

and, respectively,

$$y_1, y_2 \in \mathbb{X},\ y_1 \leq y_2 \ \Rightarrow\ F(x, y_1) \geq F(x, y_2).$$

Definition 1.4. ([4]) A pair $(x,y) \in \mathbb{X} \times \mathbb{X}$ is called a coupled fixed point of a mapping $F : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ if $F(x,y) = x$ and $F(y,x) = y$.

Coupled fixed point theory have been studied by many mathematicians like Guo and Lakshmikantham [3], Bhaskar and Lakshmikantham [4], Berinde [5], Bota et al [6] and many more. Using the mixed monotone property Bhaskar and Lakshmikantham [4] presented a coupled fixed point theorem in partial ordered metric spaces and applied it to solve periodic boundary value problem. Later in 2011, Berinde [5] generalized the result of Bhaskar-Lakshmikantham [4]. He established the following result.

Theorem 1.5. ([5]) *Let $(\mathbb{X}, \leq)$ be a partially ordered set and suppose there is a metric d on $\mathbb{X}$ such that $(\mathbb{X}, d)$ is a complete metric space. Let $F : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ be a mixed monotone mapping for which there exists a constant $k \in [0,1)$ such that for each $u \geq x, v \leq y$,*

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \leq k[d(x,u) + d(y,v)]. \tag{1.5}$$

If there exist $x_0, y_0 \in \mathbb{X}$ such that

$$x_0 \leq F(x_0, y_0) \ \text{ and } \ y_0 \geq F(y_0, x_0), \tag{1.6}$$

or

$$x_0 \geq F(x_0, y_0) \quad and \quad y_0 \leq F(y_0, x_0), \tag{1.7}$$

then there exist $\bar{x}, \bar{y} \in \mathbb{X}$ *such that* $\bar{x} = F(\bar{x}, \bar{y})$ *and* $\bar{y} = F(\bar{y}, \bar{x})$. *In addition if for all* $(x, y), (u, v) \in \mathbb{X}^2$, *there exists* $(z_1, z_2) \in \mathbb{X}^2$ *comparable with* (x, y) *and* (u, v) *with respect to partial ordering* $\preceq_p$ *defined in (1.4), then* F *has a unique coupled fixed point.*

If we endow $\mathbb{X} \times \mathbb{X}$ with a different partial ordering $\preceq$ defined by $(x_1, y_1) \preceq (x_2, y_2) \implies x_1 \leq x_2$ and $y_1 \leq y_2$, then we can establish an analogous result for a non-decreasing map $F : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ to have a coupled fixed point, the proof of which is similar to the proof of Theorem 1.5. We give the statement here, which we will use later.

Theorem 1.6. *Let* $(\mathbb{X}, \leq)$ *be a partially ordered set and suppose there is a metric* $\tilde{d}$ *on* $\mathbb{X}$ *such that* $(\mathbb{X}, \tilde{d})$ *is a complete metric space. Let* $F : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ *be a non-decreasing mapping for which there exists a constant* $k \in [0, 1)$ *such that for each* $x \leq u, y \leq v$,

$$\tilde{d}(F(x, y), F(u, v)) + \tilde{d}(F(y, x), F(v, u)) \leq k[\tilde{d}(x, u) + \tilde{d}(y, v)]. \tag{1.8}$$

If there exist $x_0, y_0 \in \mathbb{X}$ *such that*

$$x_0 \leq F(x_0, y_0) \quad and \quad y_0 \leq F(y_0, x_0),$$

then there exist $\bar{x}, \bar{y} \in \mathbb{X}$ *such that* $\bar{x} = F(\bar{x}, \bar{y})$ *and* $\bar{y} = F(\bar{y}, \bar{x})$. *In addition if for all* $(x, y), (u, v) \in \mathbb{X}^2$, *there exists* $(z_1, z_2) \in \mathbb{X}^2$ *comparable with* (x, y) *and* (u, v) *with respect to partial ordering* $\preceq$, *then* F *has a unique coupled fixed point.*

In the last few years many authors have presented Hermitian positive definite solutions of different classes of matrix equations via fixed point and coupled fixed point theorems (See [7, 8, 9, 10, 11, 12, 13, 14] and the references therein). Among them, in 2012, Berzig [10] solved a system of nonlinear matrix equation $X = Q + \sum_{i=1}^{m} A_i{}^* X A_i - \sum_{i=1}^{m} B_i{}^* Y B_i$ using Bhaskar-Lakshmikantam coupled fixed point theorem.

In this paper, we consider the following nonlinear matrix equations

$$X = Q_1 + (Y^* X Y)^{r_1}, \; Y = Q_2 + (X^* Y X)^{r_2}, \tag{1.9}$$

where, Q_1, Q_2 are Hermitian positive definite matrices and $r_1, r_2 \in \mathbb{R}$. We here establish existence and uniqueness of a positive definite solution of this class of matrix equations and also provide an algorithm for approaching the solution. We show that a particular case of our equations (taking $Q_1 = Q_2 = Q$ and $r_1 = r_2 = r$ in (1.9)) can be solved using Theorem 1.6. We also show that by taking $Q_1 = Q_2$ and $r_1 = r_2$ and replacing Y with Y^{-1} in first equation and X with X^{-1} in second equation, the derive system can be solved using Theorem 1.5.

Let $H(n)$ $\big(P(n)\big)$ be the set of all $n \times n$ Hermitian (Hermitian positive definite) matrices and $Q \in P(n)$. So, there exists a non-singular matrix S such that $S^{-1}QS$ is diagonal. Let $K(n, S, Q)$ be the set of all $n \times n$ Hermitian positive semi-definite matrices A such that $S^{-1}AS$ is diagonal. Then $K(n, S, Q)$ is closed convex cone of $(H(n), \| \,.\, \|_1)$. Recall that $\| \,.\, \|_1$ is Ky Fan norm defined as $\| A \|_1 = \sum_{j=1}^{n} s_j(A)$, where $s_j(A)$, $j = 1, ..., n$, are the singular values of A. We write $A \leq B$ iff $B - A \in K(n, S, Q)$. Then $K(n, S, Q)$ is normal cone with normal constant 1.

Definition 1.7. ([15]) Elements A and B belonging to $K(n,S,Q)$ but not both zero are said to be *linked* if and only if there exist finite (positive) real numbers δ and μ with $A \le \delta B$ and $B \le \mu A$.

This equivalence relation splits $K(n,S,Q)$ into a set of mutually exclusive constituents, **C**.

Now we endow each **C** of $K(n,S,Q)$ with Thompson metric $d(\ ,\ )$ defined by,

$$d(A,B) = \log\big\{\max\ \{\alpha,\beta\}\big\}, \tag{1.10}$$

where $\alpha = \inf\big\{\delta\ : A \le \delta B\big\}$ and $\beta = \inf\big\{\mu\ : B \le \mu A\big\}$.

Lemma 1.8. ([15]) *Each constituent,* ***C*** *of* $K(n,S,Q)$ *is complete with respect to* $d(\ ,\ )$*.*

Therefore $P(n,S,Q)$, the set of all $n \times n$ Hermitian positive definite matrices A such that $S^{-1}AS$ is diagonal, is complete with respect to Thompson metric $d(\ ,\ )$. It turns out if $A,B \in P(n,S,Q)$, $\alpha = \inf\big\{\delta\ : A \le \delta B\big\} = \lambda^+(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$, maximum eigenvalue of $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ while $\beta = \inf\big\{\mu\ : B \le \mu A\big\} = \lambda^+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$, maximum eigenvalue of $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Therefore it is convenient here to use Thompson metric. Also it has some useful properties for positive definite Hermitian matrices.

Lemma 1.9. ([8]) $d(X,Y) = d(X^{-1},Y^{-1}) = d(MXM^*,MYM^*)$*, for any* $X,Y \in P(n)$ *and non-singular matrix* M*.*

Lemma 1.10. ([8]) $d(X^r,Y^r) \le rd(X,Y)$*, for any* $X,Y \in P(n)$ *and* $r \in [0,1]$*.*

Lemma 1.11. ([8]) *For all* $A,B,C,D \in P(n)$*, we have* $d(A+B,C+D) \le max\ \big\{d(A,C),d(B,D)\big\}$*. In particular* $d(A+B,A+C) \le d(B,C)$*.*

Lemma 1.12. *For all* $X,Y,U,V \in P(n,S,Q)$ *with* $X \le U$ *and* $Y \le V$*,* $d(Y^*XY,V^*UV) \le 2\ d(Y,V) + d(X,U)$*.*

Proof. Since $X,Y,U,V \in P(n,S,Q)$, they are commutative with each other. Now as $X \le U$ and $Y \le V$, there exist $\alpha,\beta \ge 1$ such that $U \le \alpha X$ and $V \le \beta Y$. Here α,β are infima of all scalars satisfying the respected inequalities. Therefore $Y^*XY \le V^*UV$ and $V^*UV \le \beta^2\alpha\ Y^*XY$. So $d(Y^*XY,V^*UV) \le \log(\beta^2\alpha) = 2\ d(Y,V) + d(X,U)$.

Note that Thompson metric on $P(n,S,Q)(\subset P(n))$ is the restriction of Thompson metric on $P(n)$ over $P(n,S,Q)$. So if a sequence $(X_n)_{n\ge0}$ converges to X and X' in $P(n,S,Q)$ and $P(n,L,Q)$ respectively, then $X = X'$.

2. Main results

Theorem 2.1. *Suppose that* Q_1,Q_2 *are commutative and* $r_1,r_2 \in [0,\frac{1}{3})$*. Then equation (1.9) has a unique solution* $(\bar{X},\bar{Y}) \in P(n,S,Q_1)^2$*, where* $S \in GL(n,\mathbb{R})$ *such that* $S^{-1}Q_1S$ *and* $S^{-1}Q_2S$ *are diagonal matrices. Moreover for* $X(0) = Q_1, Y(0) = Q_2 \in P(n,S,Q_1)$*, the sequence* $\big(X(k),Y(k)\big)$*,* $k \ge 0$ *defined by*

$$X(k+1) = Q_1 + \big(Y(k)^*X(k)Y(k)\big)^{r_1},\ Y(k+1) = Q_2 + \big(X(k)^*Y(k)X(k)\big)^{r_2} \tag{2.1}$$

converges to $(\bar{X},\bar{Y})$ *and the error estimation is given by,*

$$max\ \big\{d(X(k),\bar{X}),d(Y(k),\bar{Y})\big\} \le \frac{\delta^k}{1-\delta} max\ \big\{d(X(1),X(0)),d(Y(1),Y(0))\big\}, \tag{2.2}$$

where δ *is a certain constant in* $[0,1)$*.*

Proof. Since $Q_1, Q_2 \in P(n)$ are commutative, they are simultaneously diagonalizable i.e, there exists an $S \in GL(n, \mathbb{R})$ such that $S^{-1}Q_1S$ and $S^{-1}Q_2S$ are diagonal. Therefore $Q_1, Q_2 \in P(n, S, Q_1) = P(n, S, Q_2)$.
Now we endow a metric d_m on $P(n, S, Q_1) \times P(n, S, Q_1)$ defined by,

$$d_m\big((X,Y),(U,V)\big) = max\ \big\{d(X,U), d(Y,V)\big\}, \tag{2.3}$$

where $d(\ ,\ )$ is Thompson metric in $P(n, S, Q_1)$. We also endow a partial order $\preceq$ on $P(n, S, Q_1) \times P(n, S, Q_1)$ defined by, $(X, Y) \preceq (U, V)$ iff $X \leq U$ and $Y \leq V$. Therefore, $\big(P(n, S, Q_1) \times P(n, S, Q_1), \preceq, d_m\big)$ is a complete partially ordered, regular non-decreasing metric space (i.e, if $(X_n)_{n\geq 0}$ is a non-decreasing sequence converges to X, then $X_n \preceq X, \forall\ n \geq 0$).
Consider a function $f : P(n, S, Q_1) \times P(n, S, Q_1) \to P(n, S, Q_1) \times P(n, S, Q_1)$ defined by,

$$f(X,Y) = \big(Q_1 + (Y^*XY)^{r_1}, Q_2 + (X^*YX)^{r_2}\big). \tag{2.4}$$

Then f is order preserving mapping and also $(Q_1, Q_2) \preceq f(Q_1, Q_2)$. So for $(X, Y) \preceq (U, V)$ we have

$$\begin{aligned}
&d_m\big(f(X,Y), f(U,V)\big)\\
&\quad= d_m\Big(\big(Q_1 + (Y^*XY)^{r_1}, Q_2 + (X^*YX)^{r_2}\big),\\
&\qquad\quad \big(Q_1 + (V^*UV)^{r_1}, Q_2 + (U^*VU)^{r_2}\big)\Big)\\
&\quad= max\ \Big\{d\big(Q_1 + (Y^*XY)^{r_1}, Q_1 + (V^*UV)^{r_1}\big),\\
&\qquad\quad d\big(Q_2 + (X^*YX)^{r_2}, Q_2 + (U^*VU)^{r_2}\big)\Big\}\\
&\quad\leq\ max\ \Big\{d\big((Y^*XY)^{r_1}, (V^*UV)^{r_1}\big), d\big((X^*YX)^{r_2}, (U^*VU)^{r_2}\big)\Big\}\\
&\quad\leq max\ \Big\{r_1\big(d(Y^*XY, V^*UV)\big), r_2\big(d(X^*YX, U^*VU)\big)\Big\}\\
&\quad\leq\ max\ \Big\{r_1\big(d(X,U) + 2d(Y,V)\big), r_2\big(2d(X,U) + d(Y,V)\big)\Big\}\\
&\quad\leq \delta\ max\ \big\{d(X,U), d(Y,V)\big\} = \delta\ d_m\big((X,Y),(U,V)\big),
\end{aligned}$$

where, $\delta = 3\ max\ \{r_1, r_2\} < 1$.
Therefore, by using Theorem 1.1 we conclude that f has a fixed point $(\bar{X}, \bar{Y})$ in $P(n, S, Q_1) \times P(n, S, Q_1)$. In fact the fixed point $(\bar{X}, \bar{Y}) = \lim_{n\to\infty} f^n(Q_1, Q_2)$.
Again for any $X, Y, U, V \in P(n, S, Q_1)$ there exist $\alpha, \beta > 0$ such that $X \leq \alpha\ U$ and $Y \leq \beta\ V$, so $(X, Y), (U, V) \preceq \big((1 + \alpha)U, (1 + \beta)V\big) = Z \in P(n, S, Q_1) \times P(n, S, Q_1)$. Therefore the fixed point $(\bar{X}, \bar{Y})$ is unique in $P(n, S, Q_1) \times P(n, S, Q_1)$.

From last theorem it follows that the equation (1.9) has a unique solution $(\bar{X}, \bar{Y})$ in $P(n, S, Q)^2$ where S is the non-singular matrix for which $S^{-1}Q_1S$ and $S^{-1}Q_2S$ are both diagonal. Since $Q_1Q_2 = Q_2Q_1$ so for the solution $(\bar{X}, \bar{Y})$ we get $\bar{X}\bar{Y} = \bar{Y}\bar{X}$. Now let $(\hat{X}, \hat{Y}) \in P(n)^2$ be a solution of (1.9) with $\hat{X}\hat{Y} = \hat{Y}\hat{X}$. Then $Q_1, Q_2, \hat{X}, \hat{Y}$ are all commute with each other. So there exists a non-singular matrix T such that $T^{-1}XT$ is diagonal for all $X \in \{Q_1, Q_2, \hat{X}, \hat{Y}\}$. As in Theorem 2.1 considering $P(n, T, Q_1)$ we can show that $(\hat{X}, \hat{Y}) = \lim_{n\to\infty} f^n(Q_1, Q_2)$, which is again same as $(\bar{X}, \bar{Y})$. Thus $\bar{X} = \hat{X}, \bar{Y} = \hat{Y}$. Therefore we have the following theorem.

Theorem 2.2. *Suppose that Q_1, Q_2 are commutative and $r_1, r_2 \in [0, \frac{1}{3})$. Then equation (1.9) has a unique solution $(\bar{X}, \bar{Y}) \in P(n)^2$ such that $\bar{X}\bar{Y} = \bar{Y}\bar{X}$.*

If we take $Q_1 = Q_2 = Q \in P(n)$ and $r_1 = r_2 = r$, then (1.9) reduces to

$$X = Q + (Y^*XY)^r,\ Y = Q + (X^*YX)^r. \tag{2.5}$$

Now using Theorem 1.6 we show that the system (2.5) has a unique solution.

Theorem 2.3. *Suppose that $Q \in P(n)$ and $r \in [0, \frac{1}{3})$. Then equation (2.5) has a unique solution $(\bar{X}, \bar{Y}) \in P(n)^2$ such that $\bar{X} = \bar{Y}$.*

Proof. Consider a function $F : P(n, S, Q) \times P(n, S, Q) \to P(n, S, Q)$ defined by $F(X, Y) = Q + (Y^*XY)^r$. Then F is non-decreasing and $Q \le F(Q, Q)$. So for $X \le U$ and $Y \le V$ we have

$$\begin{aligned}
&d\big(F(X,Y), F(U,V)\big) + d\big(F(Y,X), F(V,U)\big)\\
&\quad = d\big(Q + (Y^*XY)^r, Q + (V^*UV)^r\big) + d\big(Q + (X^*YX)^r, Q + (U^*VU)^r)\big)\\
&\quad \le d((Y^*XY)^r, (V^*UV)^r) + d((X^*YX)^r, (U^*VU)^r)\\
&\quad \le r\big(d(Y^*XY, V^*UV)\big) + r\big(d(X^*YX, U^*VU)\big)\\
&\quad \le r\big(d(X,U) + 2d(Y,V)\big) + r\big(2d(X,U) + d(Y,V)\big)\\
&\quad \le 3r\ [d(X,U) + d(Y,V)].
\end{aligned}$$

Therefore by using Theorem 1.6 we conclude that F has a unique coupled fixed point $(\bar{X}, \bar{Y}) = \lim_{n\to\infty} \big(F^n(Q,Q), F^n(Q,Q)\big)$ in $P(n, S, Q)^2$. So, $\bar{X} = \bar{Y}$. Thus using the same argument in Theorem 2.2 we conclude that equation (2.5) has a unique solution $(\bar{X}, \bar{Y}) \in P(n)^2$ with $\bar{X} = \bar{Y}$.

Now if we take $Q_1 = Q_2 = Q \in P(n)$ and $r_1 = r_2 = r$, and replace Y with Y^{-1} in the first equation and X with X^{-1} in the second equation of (1.9) then the equations in (1.9) reduces to

$$X = Q + \big((Y^{-1})^*XY^{-1}\big)^r,\ Y = Q + \big((X^{-1})^*YX^{-1}\big)^r. \tag{2.6}$$

Notice that if $F : P(n, S, Q) \times P(n, S, Q) \to P(n, S, Q)$ is a function defined by $F(X, Y) = Q + \big((Y^{-1})^*XY^{-1}\big)^r$ then F has mixed monotone property. Therefore we can use Theorem 1.5 to obtain a solution of (2.6). Thus we have the following result.

Theorem 2.4. *Suppose that $Q \in P(n)$ and $r \in [0, \frac{1}{3})$. If $Q + I \le Q^2$ then equation (2.6) has a unique solution $(\bar{X}, \bar{Y}) \in P(n)^2$ such that $\bar{X} = \bar{Y}$.*

Proof. Consider a function $F : P(n, S, Q) \times P(n, S, Q) \to P(n, S, Q)$ defined by $F(X, Y) = Q + \big((Y^{-1})^*XY^{-1}\big)^r$. Then F has mixed monotone property with $Q \le$

$F(Q,Q^2)$ and $Q^2 \geq F(Q^2,Q)$. So for $X \leq U$ and $Y \geq V$ we have

$$\begin{aligned}
&d\big(F(X,Y),F(U,V)\big)+d\big(F(Y,X),F(V,U)\big)\\
&\quad= d\Big(Q+\big((Y^{-1})^*XY^{-1}\big)^r, Q+\big((V^{-1})^*UV^{-1}\big)^r\Big)+\\
&\qquad d\Big(Q+\big((X^{-1})^*YX^{-1}\big)^r, Q+\big((U^{-1})^*VU^{-1}\big)^r)\Big)\\
&\quad\leq d\Big(\big((Y^{-1})^*XY^{-1}\big)^r, \big((V^{-1})^*UV^{-1}\big)^r\Big)+\\
&\qquad d\Big(\big((X^{-1})^*YX^{-1}\big)^r, \big((U^{-1})^*VU^{-1}\big)^r\Big)\\
&\quad\leq r\Big(d\big((Y^{-1})^*XY^{-1}, (V^{-1})^*UV^{-1}\big)\Big)+\\
&\qquad r\Big(d\big((X^{-1})^*YX^{-1}, (U^{-1})^*VU^{-1}\big)\Big)\\
&\quad\leq r\big(d(X,U)+2d(Y^{-1},V^{-1})\big)+r\big(2d(X^{-1},U^{-1})+d(Y,V)\big)\\
&\quad\leq r\big(d(X,U)+2d(Y,V)\big)+r\big(2d(X,U)+d(Y,V)\big)\\
&\quad\leq 3r\ [d(X,U)+d(Y,V)].
\end{aligned}$$

Therefore by using Theorem 1.5 we conclude that F has a unique coupled fixed point $(\bar{X},\bar{Y}) = \lim_{n\to\infty}\big(F^n(Q,Q^2),F^n(Q^2,Q)\big)$ in $P(n,S,Q)^2$. Also since for every pair of elements of $P(n,S,Q)$ has an upper bound or lower bound in $P(n,S,Q)$, so we have $\bar{X}=\bar{Y}$. Thus using the same argument in Theorem 2.2 we conclude that equation (2.6) has a unique solution $(\bar{X},\bar{Y}) \in P(n)^2$ with $\bar{X}=\bar{Y}$.

Now we give a numerical example to illustrate our obtained result.

Example. Consider the pair of matrix equations

$$X = Q_1 + (Y^*XY)^{\frac{1}{6}},\ Y = Q_2 + (X^*YX)^{\frac{1}{6}}, \tag{2.7}$$

where, $Q_1 = \begin{pmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{pmatrix}$ and $Q_2 = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$.

Then Q_1 and Q_2 are commutative and $3r_1 = 3r_2 = \frac{1}{2} < 1$. We are interested in approximating positive definite solutions of (2.7). Using iteration in (2.1) with $X(0) = Q_1$ and $Y(0) = Q_2$, after 25th iteration we get unique pair of commutative solution $(\bar{X},\bar{Y})$ given by

$$\bar{X} = X(25) = \begin{pmatrix} 2.59306 & 0.36202 & 0.36202 \\ 0.36202 & 2.59306 & 0.36202 \\ 0.36202 & 0.36202 & 2.59206 \end{pmatrix},$$

$$\bar{Y} = Y(25) = \begin{pmatrix} 2.59754 & 0.63552 & 0.63552 \\ 0.63552 & 2.59754 & 0.63552 \\ 0.63552 & 0.63552 & 2.59754 \end{pmatrix}$$

and the residue error is $R(25) = d_m\big((\bar{X},\bar{Y}), f(\bar{X},\bar{Y})\big) = 2.15938 \times 10^{-12}$. The convergence history is given by figure 1. Here Curve 1 corresponds to $d(X(k),X(k+1))$ and Curve 2 corresponds to $d(Y(k),Y(k+1))$.

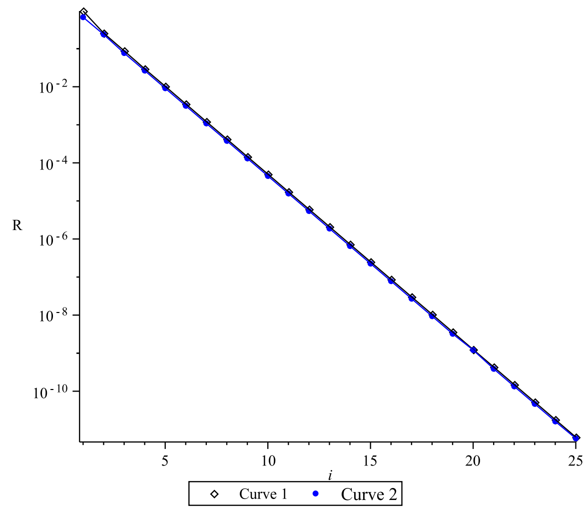

FIGURE 1. Convergence history of (2.7)

Acknowledgments. Sk. Monowar Hossein would like to thank IMSC, Chennai, India for providing research facilities. Snehasish Bose gratefully acknowledges the financial support of CSIR, Govt. of India. The authors also gratefully acknowledge the reviewer for his/her valuable suggestions.

REFERENCES

[1] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), no. 5, 1435-1443.

[2] J.J. Nieto, R. Rodriguez-López, *Contractive Mapping Theorems in Partially Ordered Sets and Applications to Ordinary Differential Equations*, Order, **22**(2005), no. 3, 223-239.

[3] D. Guo, V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., **11**(1987), no. 5, 623-632.

[4] T.G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(2006), no. 7, 1379-1393.

[5] V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **74**(2011), no. 18, 7347-7355.

[6] M. Bota, A. Petruşel, G. Petruşel, B. Samet, *Coupled fixed point theorems for single-valued operators in b-metric spaces*, Fixed Point Theory Appl., **2015**(2015), no. 1, 231.

[7] X. Duan, A. Liao, B. Tang, *On the nonlinear matrix equation* $X - \sum_{i=1}^{m} A_i{}^* X^{\delta_i} A_i = Q$, Linear Algebra Appl., **429**(2008), no. 1, 110–121.

[8] Y. Lim, *Solving the nonlinear matrix equation* $X = Q + \sum_{i=1}^{m} M_i X^{\delta_i} M_i{}^*$ *via a contraction principle*, Linear Algebra Appl., **430**(2009), no. 4, 1380-1383.

[9] X. Duan, A. Liao, *On Hermitian positive definite solution of the matrix equation* $X = Q + \sum_{i=1}^{m} M_i X^r M_i{}^*$, J. Comput. Appl. Math., **229**(2009), no. 1, 27-36.

[10] M. Berzig, B. Samet, *Solving systems of nonlinear matrix equations involving Lipschitzian mappings*, Fixed Point Theory Appl., **2011**(2011), no. 1, 89.

[11] M. Berzig, *Solving a class of matrix equations via the Bhaskar–Lakshmikantham coupled fixed point theorem*, Appl. Math. Lett., **25**(2012), no. 11, 1638-1643.

[12] V.I. Hasanov, *Positive definite solutions of the matrix equations* $X \pm A^*X^{-q}A = Q$, Linear Algebra Appl., **404**(2005), 166-182.
[13] Y. Peretz, *On applications of Schauder's fixed-point theorem for the solution of the non-symmetric algebraic Riccati equation*, Linear Algebra Appl., **445**(2014), 29–55.
[14] A. Sanand, A. Dilip, H.K. Pillai *Yet another characterization of solutions of the algebraic Riccati equation*, Linear Algebra Appl., **481**(2015), 1-35.
[15] A.C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Proc. Amer. Math. Soc., **14**(1963), no. 3, 438-443.

Received: September 2, 2015; Accepted: February 21, 2016.