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POSITIVE SOLUTIONS FOR A SYSTEM OF SINGULAR SECOND-ORDER INTEGRAL BOUNDARY VALUE PROBLEMS

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Abstract. We investigate the existence of positive solutions of a system of second-order nonlinear differential equations subject to integral boundary conditions, where the nonlinearities do not possess any sublinear or superlinear growth conditions and may be singular.
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1. Introduction

Boundary value problems with positive solutions describe many phenomena in the applied sciences such as the nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, and concentration in chemical or biological problems (see [4], [5], [9], [11], [12], [26], [29]). Integral boundary conditions arise in thermal conduction, semiconductor and hydrodynamic problems (see for example [6], [7], [23], [35]). In the last decades, many authors investigated scalar problems with integral boundary conditions (see for example [1], [3], [24], [25], [28], [33], [38], [40]). We also mention the papers [8], [10], [13], [21], [22], [27], [30], [36], [39], [41], [42], where the authors studied the existence of positive solutions for some systems of differential equations with integral boundary conditions.

In this paper, we consider the system of nonlinear second-order ordinary differential equations

$$(S) \qquad \begin{cases} (a(t)u'(t))' - b(t)u(t) + f(t, v(t)) = 0, \;\; 0 < t < 1, \\ (c(t)v'(t))' - d(t)v(t) + g(t, u(t)) = 0, \;\; 0 < t < 1, \end{cases}$$

with the integral boundary conditions

$$(BC)\quad \begin{cases} \alpha u(0)-\beta a(0)u'(0)=\displaystyle\int_0^1 u(s)dH_1(s),\ \gamma u(1)+\delta a(1)u'(1)=\int_0^1 u(s)dH_2(s),\\ \widetilde{\alpha}v(0)-\widetilde{\beta}c(0)v'(0)=\displaystyle\int_0^1 v(s)dK_1(s),\ \widetilde{\gamma}v(1)+\widetilde{\delta}c(1)v'(1)=\int_0^1 v(s)dK_2(s),\end{cases}$$

where the above integrals are Riemann-Stieltjes integrals.

We present some weaker assumptions on the functions f and g, which do not possess any sublinear or superlinear growth conditions and may be singular at $t=0$ and/or $t=1$, such that positive solutions for problem $(S)-(BC)$ exist. By a positive solution of $(S)-(BC)$ we understand a pair of functions $(u,v)\in (C([0,1],\mathbb{R}_+)\cap C^2(0,1))^2$ satisfying (S) and (BC) with $\sup_{t\in[0,1]}u(t)>0$, $\sup_{t\in[0,1]}v(t)>0$. This problem is a generalization of the problem studied in [14], where in (S) we have $a(t)=1$, $c(t)=1$, $b(t)=0$, $d(t)=0$ for all $t\in(0,1)$ (denoted by $(\widetilde{S})$), and $\alpha=\widetilde{\alpha}=1$, $\beta=\widetilde{\beta}=0$, $\gamma=\widetilde{\gamma}=1$, $\delta=\widetilde{\delta}=0$, H_1 and K_1 are constant functions, and H_2 and K_2 are step functions. Problem $(\widetilde{S})-(BC)$ also generalizes the problem investigated in [31], where the authors studied the existence of positive solutions for system $(\widetilde{S})$ with the boundary conditions $u(0)=0$, $u(1)=\alpha u(\eta)$, $v(0)=0$, $v(1)=\alpha v(\eta)$ with $\eta\in(0,1)$, $0<\alpha\eta<1$. The existence and multiplicity of positive solutions for problem $(S)-(BC)$ when the nonlinearities f and g are nonsingular functions were studied in [19] by using some theorems from the fixed point index theory. Some integral boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were investigated in recent years by using the Guo-Krasnosel'skii fixed point theorem. For example, in [15], we give sufficient conditions for λ, μ, f and g such that the system

$$(S_1)\qquad \begin{cases} (a(t)u'(t))'-b(t)u(t)+\lambda p(t)f(t,u(t),v(t))=0,\ \ 0<t<1,\\ (c(t)v'(t))'-d(t)v(t)+\mu q(t)g(t,u(t),v(t))=0,\ \ 0<t<1,\end{cases}$$

with the boundary conditions (BC) has positive solutions ($u(t)\geq 0$, $v(t)\geq 0$ for all $t\in[0,1]$ and $(u,v)\neq(0,0)$). For some higher-order multi-point boundary value problems we mention the papers [16], [17], [18], [32], [37], and the book [20].

In Section 2, we shall present some auxiliary results which investigate a boundary value problem for second-order equations. In Section 3, we shall prove two existence results for the positive solutions with respect to a cone for our problem $(S)-(BC)$, which are based on the Guo-Krasnosel'skii fixed point theorem (see [11]) which we present now.

Theorem 1.1. *Let X be a Banach space and let $C\subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0\in\Omega_1\subset\overline{\Omega_1}\subset\Omega_2$ and let $A: C\cap(\overline{\Omega_2}\setminus\Omega_1)\to C$ be a completely continuous operator (continuous, and compact, that is, it maps bounded sets into relatively compact sets) such that, either*

i) $\|Au\|\leq\|u\|$, $u\in C\cap\partial\Omega_1$, and $\|Au\|\geq\|u\|$, $u\in C\cap\partial\Omega_2$, or

ii) $\|Au\|\geq\|u\|$, $u\in C\cap\partial\Omega_1$, and $\|Au\|\leq\|u\|$, $u\in C\cap\partial\Omega_2$.

Then A has a fixed point in $C\cap(\overline{\Omega_2}\setminus\Omega_1)$.

Finally, in Section 4, two examples are given to support our main results.

2. Auxiliary results

In this section, we present some auxiliary results from [15] related to the following second-order differential equation with integral boundary conditions

$$(a(t)u'(t))' - b(t)u(t) + y(t) = 0, \ \ t \in (0,1), \tag{2.1}$$

$$\alpha u(0) - \beta a(0)u'(0) = \int_0^1 u(s)\, dH_1(s), \ \ \gamma u(1) + \delta a(1)u'(1) = \int_0^1 u(s)\, dH_2(s). \tag{2.2}$$

For $a \in C^1([0,1],(0,\infty))$, $b \in C([0,1],[0,\infty))$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$, we denote by ψ and ϕ the solutions of the following linear problems

$$\begin{cases} (a(t)\psi'(t))' - b(t)\psi(t) = 0, \ \ 0 < t < 1, \\ \psi(0) = \beta, \ \ a(0)\psi'(0) = \alpha, \end{cases}$$

$$\begin{cases} (a(t)\phi'(t))' - b(t)\phi(t) = 0, \ \ 0 < t < 1, \\ \phi(1) = \delta, \ \ a(1)\phi'(1) = -\gamma, \end{cases}$$

respectively.

We denote by θ_1 the function $\theta_1(t) = a(t)(\phi(t)\psi'(t) - \phi'(t)\psi(t))$ for $t \in [0,1]$. By using the equations above, we deduce that $\theta_1'(t) = 0$, that is $\theta_1(t) = const.$, for all $t \in [0,1]$. We denote this constant by τ_1.

Lemma 2.1. *([15]) We assume that $a \in C^1([0,1],(0,\infty))$, $b \in C([0,1],[0,\infty))$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$, and H_1, $H_2 : [0,1] \to \mathbb{R}$ are functions of bounded variation. If $\tau_1 \neq 0$,*

$$\Delta_1 = \left(\tau_1 - \int_0^1 \psi(s)\, dH_2(s)\right)\left(\tau_1 - \int_0^1 \phi(s)\, dH_1(s)\right) - \left(\int_0^1 \psi(s)\, dH_1(s)\right)\left(\int_0^1 \phi(s)\, dH_2(s)\right) \neq 0,$$

and $y \in C(0,1) \cap L^1(0,1)$, then the unique solution of (2.1)-(2.2) is given by

$$u(t) = \int_0^1 G_1(t,s)y(s)\, ds,$$

where the Green's function G_1 is defined by

$$\begin{aligned} G_1(t,s) &= g_1(t,s) \\ &+ \frac{1}{\Delta_1}\left[\psi(t)\left(\int_0^1 \phi(s)\, dH_2(s)\right) + \phi(t)\left(\tau_1 - \int_0^1 \psi(s)\, dH_2(s)\right)\right]\int_0^1 g_1(\tau,s)\, dH_1(\tau) \\ &+ \frac{1}{\Delta_1}\left[\psi(t)\left(\tau_1 - \int_0^1 \phi(s)\, dH_1(s)\right) + \phi(t)\left(\int_0^1 \psi(s)\, dH_1(s)\right)\right]\int_0^1 g_1(\tau,s)\, dH_2(\tau), \end{aligned} \tag{2.3}$$

for all $(t,s) \in [0,1] \times [0,1]$, and

$$g_1(t,s) = \frac{1}{\tau_1}\begin{cases} \phi(t)\psi(s), & 0 \le s \le t \le 1, \\ \phi(s)\psi(t), & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Now, we introduce the assumptions

$(A1)$ $a \in C^1([0,1],(0,\infty))$, $b \in C([0,1],[0,\infty))$.
$(A2)$ $\alpha,\ \beta,\ \gamma,\ \delta \in [0,\infty)$ with $\alpha+\beta>0$ and $\gamma+\delta>0$.
$(A3)$ If $b(t)\equiv 0$, then $\alpha+\gamma>0$.
$(A4)$ $H_1,\ H_2 : [0,1]\to\mathbb{R}$ are nondecreasing functions.
$(A5)$ $\tau_1-\int_0^1 \phi(s)\,dH_1(s)>0$, $\tau_1-\int_0^1 \psi(s)\,dH_2(s)>0$ and $\Delta_1>0$.

Lemma 2.2. *([2], [34]) Let $(A1)-(A3)$ hold. Then the function g_1 given by (2.4) has the properties*

a) g_1 is a continuous function on $[0,1]\times[0,1]$.
b) $g_1(t,s)\ge 0$ for all $t,\ s\in[0,1]$, and $g_1(t,s)>0$ for all $t,\ s\in(0,1)$.
c) For any $\sigma\in(0,1/2)$, we have $\min_{t\in[\sigma,1-\sigma]} g_1(t,s)\ge \nu_1 g_1(s,s)$ for all $s\in[0,1]$, where $\nu_1=\min\left\{\dfrac{\phi(1-\sigma)}{\phi(0)},\dfrac{\psi(\sigma)}{\psi(1)}\right\}$.

Lemma 2.3. *([15]) Let $(A1)-(A5)$ hold. Then the Green's function G_1 of problem (2.1)-(2.2) given by (2.3) is continuous on $[0,1]\times[0,1]$ and satisfies $G_1(t,s)\ge 0$ for all $(t,s)\in[0,1]\times[0,1]$. Moreover, if $y\in C(0,1)\cap L^1(0,1)$ satisfies $y(t)\ge 0$ for all $t\in(0,1)$, then the solution u of problem (2.1)-(2.2) satisfies $u(t)\ge 0$ for all $t\in[0,1]$.*

Lemma 2.4. *([15]) Assume that $(A1)-(A5)$ hold. Then the Green's function G_1 of problem (2.1)-(2.2) satisfies the inequalities*

a) $G_1(t,s)\le J_1(s)$, $\forall\,(t,s)\in[0,1]\times[0,1]$, where

$$
\begin{aligned}
J_1(s) &= g_1(s,s)\\
&+\frac{1}{\Delta_1}\left[\psi(T)\left(\int_0^1 \phi(s)\,dH_2(s)\right)+\phi(0)\left(\tau_1-\int_0^1 \psi(s)\,dH_2(s)\right)\right]\int_0^1 g_1(\tau,s)\,dH_1(\tau)\\
&+\frac{1}{\Delta_1}\left[\psi(T)\left(\tau_1-\int_0^1 \phi(s)\,dH_1(s)\right)+\phi(0)\left(\int_0^1 \psi(s)\,dH_1(s)\right)\right]\int_0^1 g_1(\tau,s)\,dH_2(\tau).
\end{aligned}
$$

b) For every $\sigma\in(0,1/2)$, we have

$$\min_{t\in[\sigma,1-\sigma]} G_1(t,s)\ge \nu_1 J_1(s)\ge \nu_1 G_1(t',s),\ \ \forall\, t',\ s\in[0,1],$$

where ν_1 is given in Lemma 2.2.

Lemma 2.5. *([15]) Assume that $(A1)-(A5)$ hold and let $\sigma\in(0,1/2)$. If $y\in C(0,1)\cap L^1(0,1)$, $y(t)\ge 0$ for all $t\in(0,1)$, then the solution $u(t)$, $t\in[0,1]$ of problem (2.1)-(2.2) satisfies the inequality $\inf_{t\in[\sigma,1-\sigma]} u(t)\ge \nu_1 \sup_{t'\in[0,1]} u(t')$.*

We can also formulate similar results as Lemmas 2.1-2.5 above for the boundary value problem

$$(c(t)v'(t))'-d(t)v(t)+h(t)=0,\ \ 0<t<1, \tag{2.5}$$

$$\widetilde{\alpha}v(0)-\widetilde{\beta}c(0)v'(0)=\int_0^1 v(s)\,dK_1(s),\ \ \widetilde{\gamma}v(1)+\widetilde{\delta}c(1)v'(1)=\int_0^1 v(s)\,dK_2(s), \tag{2.6}$$

under similar assumptions as $(A1)-(A5)$ and $h\in C(0,1)\cap L^1(0,1)$. We denote by $\widetilde{\psi}$, $\widetilde{\phi}$, θ_2, τ_2, Δ_2, g_2, G_2, ν_2 and J_2 the corresponding constants and functions for problem (2.5)-(2.6) defined in a similar manner as ψ, ϕ, θ_1, τ_1, Δ_1, g_1, G_1, ν_1 and J_1, respectively.

3. Main results

In this section, we shall investigate the existence of positive solutions for our problem $(S)-(BC)$, under various assumptions on the singular functions f and g.

We present the assumptions that we shall use in the sequel.

$(L1)$ The functions $a,\, c \in C^1([0,1],(0,\infty))$ and $b,\, d \in C([0,1],[0,\infty))$.

$(L2)$ $\alpha,\, \beta,\, \gamma,\, \delta,\, \widetilde{\alpha},\, \widetilde{\beta},\, \widetilde{\gamma},\, \widetilde{\delta} \in [0,\infty)$ with $\alpha+\beta>0$, $\gamma+\delta>0$, $\widetilde{\alpha}+\widetilde{\beta}>0$, $\widetilde{\gamma}+\widetilde{\delta}>0$; if $b \equiv 0$ then $\alpha+\gamma>0$; if $d \equiv 0$ then $\widetilde{\alpha}+\widetilde{\gamma}>0$.

$(L3)$ $H_1,\, H_2,\, K_1,\, K_2 : [0,1] \to \mathbb{R}$ are nondecreasing functions.

$(L4)$ $\tau_1 - \int_0^1 \phi(s)\, dH_1(s) > 0$, $\tau_1 - \int_0^1 \psi(s)\, dH_2(s) > 0$, $\tau_2 - \int_0^1 \widetilde{\phi}(s)\, dK_1(s) > 0$, $\tau_2 - \int_0^1 \widetilde{\psi}(s)\, dK_2(s) > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$, where $\tau_1,\, \tau_2,\, \Delta_1,\, \Delta_2$ are defined in Section 2.

$(L5)$ The functions $f,\, g \in C((0,1)\times\mathbb{R}_+,\mathbb{R}_+)$ and there exist $p_i \in C((0,1),\mathbb{R}_+)$, $q_i \in C(\mathbb{R}_+,\mathbb{R}_+)$, $i=1,2$, with $0 < \int_0^1 p_i(t)\, dt < \infty$, $i=1,2$, $q_1(0)=0$, $q_2(0)=0$ such that

$$f(t,x) \le p_1(t)q_1(x),\;\; g(t,x) \le p_2(t)q_2(x),\;\; \forall\, t \in (0,1),\;\; x \in \mathbb{R}_+.$$

The pair of functions $(u,v) \in (C([0,1]) \cap C^2(0,1))^2$ is a solution for our problem $(S)-(BC)$ if and only if $(u,v) \in (C([0,1]))^2$ is a solution for the nonlinear integral equations

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,\tau) g(\tau, u(\tau))\, d\tau\right) ds,\;\; t \in [0,1], \\ v(t) = \displaystyle\int_0^1 G_2(t,s) g(s,u(s))\, ds,\;\; t \in [0,1]. \end{cases}$$

We consider the Banach space $X = C([0,1])$ with the supremum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and the cone $P \subset X$ by $P = \{u \in X,\;\; u(t) \ge 0,\;\; \forall\, t \in [0,1]\}$.

We also define the operator $D : P \to X$ by

$$D(u)(t) = \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,\tau) g(\tau,u(\tau))\, d\tau\right) ds.$$

Lemma 3.1. *Assume that (L1)-(L5) hold. Then $D : P \to P$ is completely continuous.*

Proof. We denote by $\alpha_0 = \int_0^1 J_1(s)p_1(s)\, ds$ and $\beta_0 = \int_0^1 J_2(s)p_2(s)\, ds$. Using $(L5)$, we deduce that $0 < \alpha_0 < \infty$ and $0 < \beta_0 < \infty$. By Lemma 2.3, we deduce that D maps P into P. We shall prove that D maps bounded sets into relatively compact sets. Suppose $E \subset P$ is an arbitrary bounded set. First, we prove that $D(E)$ is a bounded set. Because E is bounded, then there exists $M_1 > 0$ such that $\|u\| \le M_1$ for all $u \in E$. By the continuity of q_2, there exists $M_2 > 0$ such that $M_2 = \sup_{x\in[0,M_1]} q_2(x)$. By using Lemma 2.4, for any $u \in E$ and $s \in [0,1]$, we obtain

$$\int_0^1 G_2(s,\tau) g(\tau,u(\tau))\, d\tau \le \int_0^1 G_2(s,\tau) p_2(\tau) q_2(u(\tau))\, d\tau \le \beta_0 M_2. \tag{3.1}$$

Because q_1 is continuous, there exists $M_3 > 0$ such that $M_3 = \sup_{x\in[0,\beta_0 M_2]} q_1(x)$. Therefore, from (3.1), ($L5$) and Lemma 2.4, we deduce

$$\begin{aligned}(Du)(t) &\le \int_0^1 G_1(t,s)p_1(s)q_1\left(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds \\ &\le M_3\int_0^1 J_1(s)p_1(s)\,ds = \alpha_0 M_3, \ \ \forall\, t\in[0,1].\end{aligned} \tag{3.2}$$

So, $\|Du\| \le \alpha_0 M_3$ for all $u \in E$. Therefore, $D(E)$ is a bounded set.

In what follows, we shall prove that $D(E)$ is equicontinuous. By using (2.5) from Lemma 2.1, we have for all $t \in [0,1]$

$$\begin{aligned}(Du)(t) &= \int_0^1 G_1(t,s)f\left(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds \\ &= \int_0^1 \left\{g_1(t,s) + \frac{1}{\Delta_1}\left[\psi(t)\left(\int_0^1 \phi(\tau)dH_2(\tau)\right)\right.\right. \\ &+ \left.\phi(t)\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right)\right]\left(\int_0^1 g_1(\tau,s)dH_1(\tau)\right) \\ &+ \frac{1}{\Delta_1}\left[\psi(t)\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right) + \phi(t)\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\right] \\ &\times \left.\left(\int_0^1 g_1(\tau,s)dH_2(\tau)\right)\right\} f\left(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds \\ &= \int_0^t \frac{1}{\tau_1}\phi(t)\psi(s)f\left(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds \\ &+ \int_t^1 \frac{1}{\tau_1}\phi(s)\psi(t)f\left(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds \\ &+ \frac{1}{\Delta_1}\int_0^1\left[\psi(t)\left(\int_0^1 \phi(\tau)dH_2(\tau)\right) + \phi(t)\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right)\right] \\ &\times \left(\int_0^1 g_1(\tau,s)dH_1(\tau)\right) f\left(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds \\ &+ \frac{1}{\Delta_1}\int_0^1\left[\psi(t)\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right) + \phi(t)\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\right] \\ &\times \left(\int_0^1 g_1(\tau,s)dH_2(\tau)\right) f\left(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds.\end{aligned}$$

Therefore, we obtain for any $t \in (0,1)$

$$
\begin{aligned}
&(Du)'(t) = \frac{1}{\tau_1}\phi(t)\psi(t)f\left(t, \int_0^1 G_2(t,\tau)g(\tau,u(\tau))\,d\tau\right)\\
&+ \frac{1}{\tau_1}\int_0^t \phi'(t)\psi(s)f\left(s, \int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds\\
&- \frac{1}{\tau_1}\phi(t)\psi(t)f\left(t, \int_0^1 G_2(t,\tau)g(\tau,u(\tau))\,d\tau\right)\\
&+ \frac{1}{\tau_1}\int_t^1 \psi'(t)\phi(s)f\left(s, \int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds\\
&+ \frac{1}{\Delta_1}\int_0^1 \left[\psi'(t)\left(\int_0^1 \phi(\tau)dH_2(\tau)\right) + \phi'(t)\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right)\right]\\
&\times \left(\int_0^1 g_1(\tau,s)dH_1(\tau)\right) f\left(s, \int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds\\
&+ \frac{1}{\Delta_1}\int_0^1 \left[\psi'(t)\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right) + \phi'(t)\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\right]\\
&\times \left(\int_0^1 g_1(\tau,s)dH_2(\tau)\right) f\left(s, \int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds.
\end{aligned}
$$

So, for any $t \in (0,1)$, we deduce

$$
\begin{aligned}
|(Du)'(t)| \le\ & \frac{1}{\tau_1}\int_0^t |\phi'(t)\psi(s)|p_1(s)q_1\left(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds\\
&+\frac{1}{\tau_1}\int_t^1 |\psi'(t)\phi(s)|p_1(s)q_1\left(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds\\
&+\frac{1}{\Delta_1}\int_0^1 \left[|\psi'(t)|\left(\int_0^1 \phi(\tau)dH_2(\tau)\right) + |\phi'(t)|\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right)\right]\\
&\times \left(\int_0^1 g_1(\tau,s)dH_1(\tau)\right) p_1(s)q_1\left(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds\\
&+\frac{1}{\Delta_1}\int_0^1 \left[|\psi'(t)|\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right) + |\phi'(t)|\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\right]\\
&\times \left(\int_0^1 g_1(\tau,s)dH_2(\tau)\right) p_1(s)q_1\left(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds.
\end{aligned}
$$

Hence, we obtain for any $t \in (0,1)$

$$\begin{aligned}|(Du)'(t)| &\leq M_3 \left\{ -\frac{1}{\tau_1}\int_0^t \phi'(t)\psi(s)p_1(s)\,ds + \frac{1}{\tau_1}\int_t^1 \psi'(t)\phi(s)p_1(s)\,ds \right. \\ &+ \frac{1}{\Delta_1}\int_0^1 \left[\psi'(t)\left(\int_0^1 \phi(\tau)dH_2(\tau)\right) - \phi'(t)\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right)\right] \\ &\times \left(\int_0^1 g_1(\tau,s)dH_1(\tau)\right)p_1(s)\,ds \\ &+ \frac{1}{\Delta_1}\int_0^1 \left[\psi'(t)\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right) - \phi'(t)\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\right] \\ &\left. \times \left(\int_0^1 g_1(\tau,s)dH_2(\tau)\right)p_1(s)\,ds \right\}.\end{aligned} \tag{3.3}$$

We denote

$$\begin{aligned}h(t) &= -\frac{1}{\tau_1}\int_0^t \phi'(t)\psi(s)p_1(s)\,ds + \frac{1}{\tau_1}\int_t^1 \psi'(t)\phi(s)p_1(s)\,ds, \ \ t \in (0,1), \\ \mu(t) &= h(t) + \frac{1}{\Delta_1}\int_0^1 \left[\psi'(t)\left(\int_0^1 \phi(\tau)dH_2(\tau)\right) - \phi'(t)\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right)\right] \\ &\times \left(\int_0^1 g_1(\tau,s)dH_1(\tau)\right)p_1(s)\,ds + \frac{1}{\Delta_1}\int_0^1 \left[\psi'(t)\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right)\right. \\ &\left. - \phi'(t)\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\right]\left(\int_0^1 g_1(\tau,s)dH_2(\tau)\right)p_1(s)\,ds, \ \ t \in (0,1).\end{aligned}$$

For the integral of the function h, by exchanging the order of integration, we obtain

$$\begin{aligned}\int_0^1 h(t)\,dt &= \frac{1}{\tau_1}\int_0^1 \left(\int_0^t (-\phi'(t))\psi(s)p_1(s)\,ds\right)dt \\ &+ \frac{1}{\tau_1}\int_0^1 \left(\int_t^1 \psi'(t)\phi(s)p_1(s)\,ds\right)dt \\ &= \frac{1}{\tau_1}\int_0^1 \left(\int_s^1 (-\phi'(t))\psi(s)p_1(s)\,dt\right)ds \\ &+ \frac{1}{\tau_1}\int_0^1 \left(\int_0^s \psi'(t)\phi(s)p_1(s)\,dt\right)ds \\ &= \frac{1}{\tau_1}\int_0^1 \psi(s)(\phi(s)-\phi(1))p_1(s)\,ds + \frac{1}{\tau_1}\int_0^1 \phi(s)(\psi(s)-\psi(0))p_1(s)\,ds \\ &\leq \frac{1}{\tau_1}[\psi(1)(\phi(0)-\phi(1)) + \phi(0)(\psi(1)-\psi(0))]\int_0^1 p_1(s)\,ds \\ &= \widetilde{M}_0 \int_0^1 p_1(s)\,ds < \infty,\end{aligned}$$

where $\widetilde{M_0} = \frac{1}{\tau_1}[\psi(1)(\phi(0)-\phi(1))+\phi(0)(\psi(1)-\psi(0))]$. For the integral of the function μ, we have

$$\begin{aligned}\int_0^1 \mu(t)\,dt \quad &\leq \widetilde{M_0}\int_0^1 p_1(s)\,ds + \frac{1}{\Delta_1}\left[\left(\int_0^1 \psi'(t)\,dt\right)\left(\int_0^1 \phi(\tau)dH_2(\tau)\right)\right.\\ &\quad \left.-\left(\int_0^1 \phi'(t)\,dt\right)\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right)\right]\\ &\quad \times\left(\int_0^1\left(\int_0^1 g_1(\tau,s)dH_1(\tau)\right)p_1(s)\,ds\right)\\ &\quad +\frac{1}{\Delta_1}\left[\left(\int_0^1 \psi'(t)\,dt\right)\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right) - \left(\int_0^1 \phi'(t)\,dt\right)\right.\\ &\quad \left.\times\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\right]\left(\int_0^1\left(\int_0^1 g_1(\tau,s)dH_2(\tau)\right)p_1(s)\,ds\right)\end{aligned}$$

$$\begin{aligned}&\leq \widetilde{M_0}\int_0^1 p_1(s)\,ds + \frac{1}{\Delta_1}\left[(\psi(1)-\psi(0))\left(\int_0^1 \phi(\tau)dH_2(\tau)\right)\right.\\ &\times\left(\int_0^1 dH_1(\tau)\right) + (\phi(0)-\phi(1))\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right)\\ &\times\left(\int_0^1 dH_1(\tau)\right) + (\psi(1)-\psi(0))\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right)\\ &\times\left(\int_0^1 dH_2(\tau)\right) + (\phi(0)-\phi(1))\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\\ &\left.\times\left(\int_0^1 dH_2(\tau)\right)\right]\left(\int_0^1 g_1(s,s)p_1(s)\,ds\right)\\ &\leq \widetilde{M_0}\int_0^1 p_1(s)\,ds + \frac{1}{\tau_1\Delta_1}\phi(0)\psi(1)\Big[(\psi(1)-\psi(0))(H_1(1)-H_1(0))\\ &\times\left(\int_0^1 \phi(\tau)dH_2(\tau)\right) + (\phi(0)-\phi(1))(H_1(1)-H_1(0))\\ &\times\left(\tau_1 - \int_0^1 \psi(\tau)dH_2(\tau)\right) + (\psi(1)-\psi(0))(H_2(1)-H_2(0))\\ &\times\left(\tau_1 - \int_0^1 \phi(\tau)dH_1(\tau)\right) + (\phi(0)-\phi(1))(H_2(1)-H_2(0))\\ &\times\left(\int_0^1 \psi(\tau)dH_1(\tau)\right)\Big]\int_0^1 p_1(s)\,ds < \infty.\end{aligned} \tag{3.4}$$

We deduce that $\mu \in L^1(0,1)$. Thus for any given $t_1,\, t_2 \in [0,1]$ with $t_1 \leq t_2$ and $u \in E$, by (3.3), we obtain

$$|(Du)(t_1) - (Du)(t_2)| = \left|\int_{t_1}^{t_2} (Du)'(t)\,dt\right| \leq M_3\int_{t_1}^{t_2} \mu(t)\,dt. \tag{3.5}$$

From (3.4), (3.5) and the absolute continuity of the integral function, we obtain that $D(E)$ is equicontinuous. This conclusion, together with (3.2) and Ascoli-Arzelà theorem, yields that $D(E)$ is relatively compact. Therefore D is a compact operator.

We show now that the operator D is continuous. Suppose that $u_p,\, u \in E$ for $p \in \mathbb{N}$ and $\|u_p - u\| \to 0$, as $p \to \infty$. Then there exists $M_4 > 0$ such that $\|u_p\| \leq M_4$

and $\|u\| \le M_4$. From the first part of this proof we know that $\{Du_p,\ p \in \mathbb{N}\}$ is relatively compact. We shall prove that $\|Du_p - Du\| \to 0$, as $p \to \infty$. If we suppose that this is not true, then there exists $\varepsilon_0 > 0$ and a subsequence $(u_{p_k})_k \subset (u_k)_k$ such that $\|Du_{p_k} - Du\| \ge \varepsilon_0,\ k = 1, 2, \ldots$. Since $\{Du_{p_k},\ k = 1, 2, \ldots\}$ is relatively compact, there exists a subsequence of $(Du_{p_k})_k$ which converges in P to some $u^* \in P$. Without loss of generality, we assume that $(Du_{p_k})_k$ itself converges to u^*, that is, $\lim\limits_{k\to\infty} \|Du_{p_k} - u^*\| = 0$. From the above relation, we deduce that $(Du_{p_k})(t) \to u^*(t)$, as $k \to \infty$ for all $t \in [0, 1]$. By $(L5)$ and Lemma 2.4, we obtain

$$G_2(s,\tau)g(\tau, u_{p_k}(\tau)) \le J_2(\tau)p_2(\tau)q_2(u_{p_k}(\tau)) \le M_5 J_2(\tau)p_2(\tau),$$

for all $s,\ \tau \in [0, 1]$, where $M_5 = \sup\limits_{x\in[0,M_4]} q_2(x) < \infty$. Therefore we obtain

$$\begin{aligned} &G_1(t,s)f\left(s, \int_0^1 G_2(s,\tau)g(\tau, u_{p_k}(\tau))\,d\tau\right) \\ &\quad\le J_1(s)p_1(s)q_1\left(\int_0^1 G_2(s,\tau)g(\tau, u_{p_k}(\tau))\,d\tau\right) \le M_6 J_1(s)p_1(s), \end{aligned} \tag{3.6}$$

where $M_6 = \sup\limits_{x\in[0,\beta_0 M_5]} q_1(x)$.

By $(L5)$, (3.6) and the Lebesgue's Dominated Convergence Theorem, we obtain

$$u^*(t) = \lim_{k\to\infty} (Du_{p_k})(t) = (Du)(t),\ \ \forall\, t \in [0, 1],$$

that is, $u^* = Du$. This relation contradicts the inequality $\|Du_{p_k} - u^*\| \ge \varepsilon_0,\ k = 1, 2, \ldots$. Therefore, D is continuous in u, and in general on P. Lemma 3.1 is completely proved. □

For $\sigma \in (0, 1/2)$ we define the cone

$$P_0 = \{u \in X,\ \ u(t) \ge 0,\ \ \forall\, t \in [0, 1],\ \ \inf_{t\in[\sigma,1-\sigma]} u(t) \ge \nu\|u\|\} \subset P,$$

where $\nu = \min\{\nu_1, \nu_2\}$, and ν_1 and ν_2 are defined in Section 2 (Lemma 2.2). Under the assumptions $(L1) - (L5)$, we have $D(P) \subset P_0$. Indeed, for $u \in P$, let $v = D(u)$. By Lemma 2.5, we have $\inf_{t\in[\sigma,1-\sigma]} v(t) \ge \nu_1\|v\| \ge \nu\|v\|$, that is $v \in P_0$.

Theorem 3.2. *Assume that $(L1) - (L5)$ hold. If the functions f and g also satisfy the conditions*

$(L6)$ *There exist $r_1,\ r_2 \in (0, \infty)$ with $r_1 r_2 \ge 1$ such that*

$$i)\ \ q_{10}^s = \limsup_{x\to 0^+} \frac{q_1(x)}{x^{r_1}} \in [0, \infty);\ \ \ ii)\ \ q_{20}^s = \limsup_{x\to 0^+} \frac{q_2(x)}{x^{r_2}} = 0,$$

$(L7)$ *There exist $l_1,\ l_2 \in (0, \infty)$ with $l_1 l_2 \ge 1$ and $\sigma \in (0, 1/2)$ such that*

$$i)\ \ f_\infty^i = \liminf_{x\to\infty} \inf_{t\in[\sigma,1-\sigma]} \frac{f(t,x)}{x^{l_1}} \in (0, \infty];\ \ \ ii)\ \ g_\infty^i = \liminf_{x\to\infty} \inf_{t\in[\sigma,1-\sigma]} \frac{g(t,x)}{x^{l_2}} = \infty,$$

then problem $(S) - (BC)$ has at least one positive solution $(u(t), v(t)),\ t \in [0, 1]$.

Proof. We consider the cone P_0 with σ given in $(L7)$. From $(L6)$ i) and $(L5)$, we deduce that there exists $C_1 > 0$ such that

$$q_1(x) \le C_1 x^{r_1},\ \ \forall\, x \in [0, 1]. \tag{3.7}$$

From $(L6)$ ii) and $(L5)$, for $C_2 = \min\left\{(1/(C_1\alpha_0\beta_0^{r_1}))^{1/r_1}, 1/\beta_0\right\} > 0$ with α_0, β_0 defined in the proof of Lemma 3.1, we conclude that there exists $\delta_1 \in (0,1)$ such that

$$q_2(x) \leq C_2 x^{r_2}, \ \ \forall\, x \in [0, \delta_1]. \tag{3.8}$$

From (3.8), $(L5)$ and Lemma 2.4, for any $u \in \partial B_{\delta_1} \cap P_0$ and $s \in [0,1]$, we obtain

$$\int_0^1 G_2(s,\tau) g(\tau, u(\tau))\, d\tau \leq C_2 \int_0^1 J_2(\tau) p_2(\tau)\, d\tau \cdot \|u\|^{r_2} = C_2 \beta_0 \delta_1^{r_2} \leq \delta_1^{r_2} < 1. \tag{3.9}$$

By using (3.7)-(3.9) and $(L5)$, for any $u \in \partial B_{\delta_1} \cap P_0$ and $t \in [0,1]$, we deduce

$$\begin{aligned} (Du)(t) \ \ &\leq C_1 \int_0^1 G_1(t,s) p_1(s) \left(\int_0^1 G_2(s,\tau) g(\tau, u(\tau))\, d\tau \right)^{r_1} ds \\ &\leq C_1 \int_0^1 G_1(t,s) p_1(s) \left(C_2 \int_0^1 G_2(s,\tau) p_2(\tau) (u(\tau))^{r_2}\, d\tau \right)^{r_1} ds \\ &\leq C_1 \left(\int_0^1 J_1(s) p_1(s)\, ds \right) \left(C_2 \int_0^1 J_2(\tau) p_2(\tau)\, d\tau \right)^{r_1} \|u\|^{r_1 r_2} \leq \|u\|. \end{aligned}$$

Therefore

$$\|Du\| \leq \|u\|, \ \ \forall\, u \in \partial B_{\delta_1} \cap P_0. \tag{3.10}$$

From $(L7)$ i), we conclude that there exist $C_3 > 0$ and $x_1 > 0$ such that

$$f(t,x) \geq C_3 x^{l_1}, \ \ \forall\, x \geq x_1, \ \ \forall\, t \in [\sigma, 1-\sigma]. \tag{3.11}$$

We consider now $C_4 = \max\left\{(\nu_2 \nu^{l_2} \theta_2)^{-1}, (C_3 \nu_1 \nu_2^{l_1} \nu^{l_1 l_2} \theta_1 \theta_2^{l_1})^{-1/l_1}\right\} > 0$, where

$$\theta_1 = \int_\sigma^{1-\sigma} J_1(s) ds > 0 \text{ and } \theta_2 = \int_\sigma^{1-\sigma} J_2(s)\, ds > 0.$$

From $(L7)$ ii), we deduce that there exists $x_2 \geq 1$ such that

$$g(t,x) \geq C_4 x^{l_2}, \ \ \forall\, x \geq x_2, \ \ \forall\, t \in [\sigma, 1-\sigma]. \tag{3.12}$$

We choose $R_0 = \max\{x_1, x_2\}$ and $R > \max\left\{R_0/\nu, R_0^{1/l_2}\right\}$. Then for any $u \in \partial B_R \cap P_0$, we have $\inf_{t \in [\sigma, 1-\sigma]} u(t) \geq \nu \|u\| = \nu R > R_0$.

By using (3.11) and (3.12), for any $u \in \partial B_R \cap P_0$ and $s \in [\sigma, 1-\sigma]$, we obtain

$$\begin{aligned} \int_0^1 G_2(s,\tau) g(\tau, u(\tau))\, d\tau \ \ &\geq \nu_2 C_4 \int_\sigma^{1-\sigma} J_2(\tau) (u(\tau))^{l_2}\, d\tau \\ &\geq \nu_2 C_4 \nu^{l_2} \int_\sigma^{1-\sigma} J_2(\tau)\, d\tau \cdot \|u\|^{l_2} \geq \|u\|^{l_2} = R^{l_2} > R_0. \end{aligned}$$

Then for any $u \in \partial B_R \cap P_0$ and $t \in [\sigma, 1-\sigma]$, we have

$$\begin{aligned}(Du)(t) &\geq \int_\sigma^{1-\sigma} G_1(t,s) f\left(s, \int_0^1 G_2(s,\tau) g(\tau, u(\tau))\, d\tau\right) ds \\ &\geq C_3 \int_\sigma^{1-\sigma} G_1(t,s) \left(\nu_2 \int_\sigma^{1-\sigma} J_2(\tau) C_4 (u(\tau))^{l_2}\, d\tau\right)^{l_1} ds \\ &\geq C_3 C_4^{l_1} \nu_2^{l_1} \int_\sigma^{1-\sigma} G_1(t,s) \nu^{l_1 l_2} \|u\|^{l_1 l_2} \left(\int_\sigma^{1-\sigma} J_2(\tau)\, d\tau\right)^{l_1} ds \\ &\geq C_3 C_4^{l_1} \nu_2^{l_1} \nu_1 \nu^{l_1 l_2} \left(\int_\sigma^{1-\sigma} J_1(s)\, ds\right) \left(\int_\sigma^{1-\sigma} J_2(\tau)\, d\tau\right)^{l_1} \|u\|^{l_1 l_2} \geq \|u\|.\end{aligned}$$

Therefore we obtain

$$\|Du\| \geq \|u\|, \ \ \forall\, u \in \partial B_R \cap P_0. \tag{3.13}$$

By (3.10), (3.13), Lemma 3.1 and Theorem 1.1 i), we conclude that D has a fixed point $u_1 \in (\bar{B}_R \setminus B_{\delta_1}) \cap P_0$, that is $\delta_1 \leq \|u_1\| \leq R$. Let

$$v_1(t) = \int_0^1 G_2(t,s) g(s, u_1(s))\, ds.$$

Then $(u_1, v_1) \in P_0 \times P_0$ is a positive solution of $(S) - (BC)$. In addition $\|v_1\| > 0$. Indeed, if we suppose that $v_1(t) = 0$ for all $t \in [0,1]$, then by using $(L5)$ we have $f(s, v_1(s)) = f(s, 0) = 0$ for all $s \in [0,1]$. This implies $u_1(t) = 0$ for all $t \in [0,1]$, which contradicts $\|u_1\| > 0$. The proof of Theorem 3.2 is completed. □

Theorem 3.3 *Assume that $(L1) - (L5)$ hold. If the functions f and g also satisfy the conditions*

$(L8)$ *There exist α_1, $\alpha_2 \in (0,\infty)$ with $\alpha_1\alpha_2 \leq 1$ such that*

$$i)\ \ q_{1\infty}^s = \limsup_{x\to\infty} \frac{q_1(x)}{x^{\alpha_1}} \in [0,\infty); \quad ii)\ \ q_{2\infty}^s = \limsup_{x\to\infty} \frac{q_2(x)}{x^{\alpha_2}} = 0,$$

$(L9)$ *There exist β_1, $\beta_2 \in (0,\infty)$ with $\beta_1\beta_2 \leq 1$ and $\sigma \in (0, 1/2)$ such that*

$$i)\ \ f_0^i = \liminf_{x\to 0^+} \inf_{t\in[\sigma,1-\sigma]} \frac{f(t,x)}{x^{\beta_1}} \in (0,\infty]; \quad ii)\ \ g_0^i = \liminf_{x\to 0^+} \inf_{t\in[\sigma,1-\sigma]} \frac{g(t,x)}{x^{\beta_2}} = \infty,$$

then problem $(S) - (BC)$ has at least one positive solution $(u(t), v(t))$, $t \in [0,1]$.

Proof. We consider the cone P_0 with σ given in $(L9)$. By $(L8)$ i) we deduce that there exist $C_5 > 0$ and $C_6 > 0$ such that

$$q_1(x) \leq C_5 x^{\alpha_1} + C_6, \ \ \forall\, x \in [0,\infty). \tag{3.14}$$

From $(L8)$ ii), for $\varepsilon_0 > 0$, $\varepsilon_0 < (2^{\alpha_1} C_5 \alpha_0 \beta_0^{\alpha_1})^{-1/\alpha_1}$, we conclude that there exists $C_7 > 0$ such that

$$q_2(x) \leq \varepsilon_0 x^{\alpha_2} + C_7, \ \ \forall\, x \in [0,\infty). \tag{3.15}$$

By using (3.14), (3.15) and ($L5$), for any $u \in P_0$, we obtain

$$\begin{aligned}(Du)(t) &\le \int_0^1 G_1(t,s)p_1(s)q_1\left(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds\\ &\le C_5\int_0^1 G_1(t,s)p_1(s)\left(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)^{\alpha_1}ds\\ &\quad +C_6\int_0^1 J_1(s)p_1(s)\,ds\\ &\le C_5\int_0^1 J_1(s)p_1(s)\,ds\left(\int_0^1 J_2(\tau)p_2(\tau)\,d\tau\right)^{\alpha_1}(\varepsilon_0\|u\|^{\alpha_2}+C_7)^{\alpha_1}+\alpha_0C_6\\ &\le C_5 2^{\alpha_1}\varepsilon_0^{\alpha_1}\alpha_0\beta_0^{\alpha_1}\|u\|^{\alpha_1\alpha_2}+C_5 2^{\alpha_1}\alpha_0\beta_0^{\alpha_1}C_7^{\alpha_1}+\alpha_0C_6,\ \ \forall\, t\in[0,1].\end{aligned}$$

By definition of ε_0, we can choose sufficiently large $R_1 > 0$ such that

$$\|Du\| \le \|u\|,\ \ \forall\, u \in \partial B_{R_1} \cap P_0. \tag{3.16}$$

From ($L9$) i), we deduce that there exist positive constants $C_8 > 0$ and $x_3 > 0$ such that $f(t,x) \ge C_8 x^{\beta_1}$, for all $x \in [0,x_3]$ and $t \in [\sigma, 1-\sigma]$. From ($L9$) ii), for $\varepsilon_1 = \left(C_8\nu_1\nu_2^{\beta_1}\nu^{\beta_1\beta_2}\theta_1\theta_2^{\beta_1}\right)^{-1/\beta_1} > 0$, we conclude that there exists $x_4 > 0$ such that $g(t,x) \ge \varepsilon_1 x^{\beta_2}$ for all $x \in [0,x_4]$ and $t \in [\sigma,1-\sigma]$.

We consider $x_5 = \min\{x_3, x_4\}$. So we obtain

$$f(t,x) \ge C_8 x^{\beta_1},\quad g(t,x) \ge \varepsilon_1 x^{\beta_2},\ \ \forall\,(t,x) \in [\sigma,1-\sigma]\times[0,x_5]. \tag{3.17}$$

From assumption $q_2(0) = 0$ and the continuity of q_2, we deduce that there exists sufficiently small $\varepsilon_2 \in (0, \min\{x_5, 1\})$ such that $q_2(x) \le \beta_0^{-1}x_5$ for all $x \in [0,\varepsilon_2]$.

Therefore for any $u \in \partial B_{\varepsilon_2} \cap P_0$ and $s \in [0,1]$, we have

$$\int_0^1 G_2(s,\tau)g(\tau,u(\tau))\,d\tau \le \beta_0^{-1}x_5\int_0^1 J_2(\tau)p_2(\tau)\,d\tau = x_5. \tag{3.18}$$

By (3.17), (3.18), Lemma 2.4 and Lemma 2.5, for any $t \in [\sigma, 1-\sigma]$, we obtain

$$\begin{aligned}(Du)(t) &\ge C_8\int_\sigma^{1-\sigma} G_1(t,s)\left(\int_\sigma^{1-\sigma} G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)^{\beta_1}ds\\ &\ge C_8\nu_1\int_\sigma^{1-\sigma} J_1(s)\left[(\varepsilon_1\nu_2)^{\beta_1}\left(\int_\sigma^{1-\sigma} J_2(\tau)(u(\tau))^{\beta_2}\,d\tau\right)^{\beta_1}\right]ds\\ &\ge C_8\nu_1\nu_2^{\beta_1}\varepsilon_1^{\beta_1}\nu^{\beta_1\beta_2}\theta_1\theta_2^{\beta_1}\|u\|^{\beta_1\beta_2} \ge \|u\|.\end{aligned}$$

Therefore

$$\|Du\| \ge \|u\|,\ \ \forall\, u \in \partial B_{\varepsilon_2} \cap P_0. \tag{3.19}$$

By (3.16), (3.19), Lemma 3.1 and Theorem 1.1 ii), we deduce that D has at least one fixed point $u_2 \in (\bar{B}_{R_1} \setminus B_{\varepsilon_2}) \cap P_0$. Then our problem $(S)-(BC)$ has at least one positive solution $(u_2, v_2) \in P_0 \times P_0$ where $v_2(t) = \int_0^1 G_2(t,s)g(s,u_2(s))\,ds$. This completes the proof of Theorem 3.3. $\square$

4. Examples

In this section, we shall present two examples which illustrate our main results.
Example 4.1. Let

$$f(t,x) = \frac{x^a}{t^{\gamma_1}(1-t)^{\delta_1}}, \quad g(t,x) = \frac{x^b}{t^{\gamma_2}(1-t)^{\delta_2}}, \quad \forall\, t \in (0,1), \quad x \in [0,\infty),$$

with $a,\, b > 1$ and $\gamma_1,\, \delta_1,\, \gamma_2,\, \delta_2 \in (0,1)$. Here $f(t,x) = p_1(t)q_1(x)$ and $g(t,x) = p_2(t)q_2(x)$, where

$$p_1(t) = \frac{1}{t^{\gamma_1}(1-t)^{\delta_1}}, \quad p_2(t) = \frac{1}{t^{\gamma_2}(1-t)^{\delta_2}}, \quad q_1(x) = x^a, \quad q_2(x) = x^b.$$

We have $0 < \int_0^1 p_1(s)\,ds < \infty$, $0 < \int_0^1 p_2(s)\,ds < \infty$.
In $(L6)$, for $r_1 < a,\ r_2 < b$ and $r_1 r_2 \geq 1$, we have

$$\limsup_{x\to 0^+} \frac{q_1(x)}{x^{r_1}} = \lim_{x\to 0^+} x^{a-r_1} = 0, \quad \limsup_{x\to 0^+} \frac{q_2(x)}{x^{r_2}} = \lim_{x\to 0^+} x^{b-r_2} = 0.$$

In $(L7)$, for $l_1 < a,\ l_2 < b,\ l_1 l_2 \geq 1$ and $\sigma \in \left(0, \frac{1}{2}\right)$, we have

$$\begin{aligned} &\liminf_{x\to\infty} \inf_{t\in[\sigma,1-\sigma]} \frac{f(t,x)}{x^{l_1}} = \liminf_{x\to\infty} \inf_{t\in[\sigma,1-\sigma]} \frac{x^{a-l_1}}{t^{\gamma_1}(1-t)^{\delta_1}} \\ &= \left(\max\left\{ \frac{\gamma_1^{\gamma_1}\delta_1^{\delta_1}}{(\gamma_1+\delta_1)^{\gamma_1+\delta_1}},\, \sigma^{\gamma_1}(1-\sigma)^{\delta_1}, \sigma^{\delta_1}(1-\sigma)^{\gamma_1} \right\} \right)^{-1} \cdot \lim_{x\to\infty} x^{a-l_1} = \infty. \end{aligned}$$

In a similar manner, we have $\liminf_{x\to\infty} \inf_{t\in[\sigma,1-\sigma]} \frac{g(t,x)}{x^{l_2}} = \infty$.

For example, if $a = 2,\ b = 3/2,\ r_1 = 1,\ r_2 = 4/3,\ l_1 = 3/2,\ l_2 = 1$, the above conditions are satisfied. Under the assumptions $(L1) - (L4)$, by Theorem 3.2, we deduce that problem $(S) - (BC)$ has at least one positive solution.
Example 4.2. Let

$$f(t,x) = \frac{x^a(2+\cos x)}{t^{\gamma_1}}, \quad g(t,x) = \frac{x^b(1+\sin x)}{(1-t)^{\delta_1}}, \quad \forall\, t \in (0,1), \quad x \in [0,\infty),$$

with $a,\, b \in (0,1)$ and $\gamma_1,\, \delta_1 \in (0,1)$. Here $f(t,x) = p_1(t)q_1(x)$ and $g(t,x) = p_2(t)q_2(x)$, where

$$p_1(t) = \frac{1}{t^{\gamma_1}}, \quad p_2(t) = \frac{1}{(1-t)^{\delta_1}}, \quad q_1(x) = x^a(2+\cos x), \quad q_2(x) = x^b(1+\sin x).$$

We have $0 < \int_0^1 p_1(s)\,ds < \infty$, $0 < \int_0^1 p_2(s)\,ds < \infty$.
In $(L8)$, for $\alpha_1 = a,\ \alpha_2 > b$ and $\alpha_1\alpha_2 \leq 1$, we have

$$\limsup_{x\to\infty} \frac{q_1(x)}{x^{\alpha_1}} = \limsup_{x\to\infty} \frac{x^a(2+\cos x)}{x^{\alpha_1}} = 3, \quad \limsup_{x\to\infty} \frac{q_2(x)}{x^{\alpha_2}} = \limsup_{x\to\infty} \frac{x^b(1+\sin x)}{x^{\alpha_2}} = 0.$$

In $(L9)$, for $\beta_1 = a$, $\beta_2 > b$, $\beta_1\beta_2 \le 1$ and $\sigma \in \left(0, \frac{1}{2}\right)$, we have

$$\liminf_{x\to 0^+} \inf_{t\in[\sigma,1-\sigma]} \frac{f(t,x)}{x^{\beta_1}} = \liminf_{x\to 0^+} \inf_{t\in[\sigma,1-\sigma]} \frac{x^a(2+\cos x)}{t^{\gamma_1}x^{\beta_1}} = \frac{3}{(1-\sigma)^{\gamma_1}} > 0,$$

$$\liminf_{x\to 0^+} \inf_{t\in[\sigma,1-\sigma]} \frac{g(t,x)}{x^{\beta_2}} = \liminf_{x\to 0^+} \inf_{t\in[\sigma,1-\sigma]} \frac{x^b(1+\sin x)}{(1-t)^{\delta_1}x^{\beta_2}} = \frac{1}{(1-\sigma)^{\delta_1}} \lim_{x\to 0^+} x^{b-\beta_2} = \infty.$$

For example, if $a = 1/3$, $b = 1/2$, $\alpha_1 = 1/3$, $\alpha_2 = 1$, $\beta_1 = 1/3$, $\beta_2 = 1$, the above conditions are satisfied. Under the assumptions $(L1) - (L4)$, by Theorem 3.3, we deduce that problem $(S) - (BC)$ has at least one positive solution.

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