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ON BANACH CONTRACTION PRINCIPLES IN FUZZY METRIC SPACES

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Abstract. In this paper we discuss the concept of Cauchy sequence due to Grabiec, that we call G-Cauchy, in the context of fuzzy metric spaces. It leads to introduce and study a concept of weak G-completeness in fuzzy and classical context. Then, we generalize the celebrated Grabiec's fuzzy Banach Contraction Principle. Also, we extend the Mihet's fixed point theorem given for weak B-contractive mappings.

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1. Introduction

Kramosil and Michalek [7] gave a notion of fuzzy metric space, that we denote KM-fuzzy metric space, which could be considered as a reformulation, in the fuzzy context, of the notion of PM-space (or more precisely, Menger space). In this paper we call fuzzy metric space $(X, M, *)$ the one defined by George and Veeramani [1] (Definition 2.4), which is a slight modification of the KM-fuzzy metric space. In both spaces, and in a similar way, a topology on X can be deduced on X from the fuzzy metric M. Many concepts given in PM-spaces have been adapted to the fuzzy context. That is the case of the concept of Cauchy sequence given by George and Veeramani [1] (Definition 3.14) that we adopt here. As usual, a fuzzy metric space is called complete if every Cauchy sequence is convergent.

In 1988 M. Grabiec [2] introduced in the context of KM-fuzzy metric spaces a weaker concept than the Cauchy sequence and in a natural way a stronger concept of completeness that we will call G-Cauchyness and G-completeness, respectively. So, he introduced the first fuzzy version of the Banach Contraction Principle for a class of contractive mappings defined on G-complete KM-fuzzy metric spaces. Unfortunately, its applicability is drastically reduced because the concept of G-completeness is so strong that even compact spaces are not necessarily G-complete (see Example 5.6 and [10] Example 3.7). The aim of this paper is, basically, to overcome this inconvenience introducing an appropriate concept of completeness weaker than compactness.

Beside this we will see some aspects of G-Cauchy sequences as we explain in the next paragraph.

Usually, concepts in classical metrics are extended to fuzzy context. In this paper, although it is not usual, we first extend in a natural way the concepts of G-Cauchyness and G-completeness to ordinary metrics. Moreover, we will introduce and study an appropriate weaker concept than convergence, called weaker G-convergence. Accordingly, we introduce the concept of weak G-completeness in metric spaces, and later in (*KM*-) fuzzy metric spaces, that fulfills in all cases the next nice diagram of implications.

$$\begin{array}{ccccc} G-completeness & \rightarrow & weak\ G-completeness & \rightarrow & completeness \\ & & \uparrow & & \\ & & compactness & & \end{array}$$

The above implications are not reversed, in general.

Later, inspired in a contractive condition due to D. Mihet [8] we give a more general contractive condition (Definition 6.1) than the one given by Grabiec (Definition 2.4). So, using Lemma 6.3 we generalize the Grabiec's fuzzy Banach Contraction Theorem for these new contractive mappings which, on the other hand, are now defined on weak G-complete *spaces* (Theorem 6.4). Example 6.7 shows that Theorem 6.4 is really a generalization of Grabiec's theorem, in both mentioned senses. Also, a Mihet's fixed point theorem in [8] (and consequently a Gregori and Sapena's fixed point theorem in [6]) stated for fuzzy *contractive* mappings defined on G-complete *spaces* (Corollary 6.9) is extended to weak G-complete *spaces* (Theorem 6.8). Imitating the proof of Theorem 6.4 with slight modifications, many fuzzy fixed point theorems, appeared in the literature stated on G-complete *spaces*, can be extended to weak G-complete *spaces*. (See for instance [11, 12, 13, 15]). Several appropriate examples along the paper illustrate our theory. This is an interesting aspect because when studying topics involving G-completeness in (*KM*-) fuzzy metric spaces one miss examples.

The structure of the paper is as follows. After the preliminary section, in Section 3 we study the concept of G-Cauchy sequence in metric spaces. In Section 4 and Section 5 we introduce and study the concept of weak G-completeness in metric and (*KM*-) fuzzy metric spaces, respectively. And finally, in Section 6 we give two fixed point theorems that generalize the corresponding ones due to Grabiec and Mihet, respectively.

2. Preliminaries

Definition 2.1. (George and Veeramani [1]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, \infty[\rightarrow]0, 1]$ is continuous.

It is also said that M is a fuzzy metric on X.

In the definition of fuzzy metric space of Kramosil and Michalek, [7], M is a fuzzy set on $X^2 \times [0, \infty[$ that satisfies (GV3) and (GV4), and (GV1), (GV2), (GV5) are replaced by (KM1), (KM2), (KM5), respectively, below:

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left continuous.

We will refer to these fuzzy metric spaces as KM-fuzzy metric spaces.

If M is a fuzzy metric on X then M can be considered a KM-fuzzy metric on X defining $M(x, y, 0) = 0$ for all $x, y \in X$.

The authors in [1] proved that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, \epsilon \in]0, 1[, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$, $t > 0$. A sequence $\{x_n\}$ in X converges to x if and only if $\lim_n M(x_n, x, t) = 1$ for all $t > 0$. The same is true in KM-fuzzy metric spaces.

Let (X, d) be a metric space and let M_d a function on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space [1] and M_d is called the *standard fuzzy metric* induced by d. The topology τ_{M_d} coincides with the topology $\tau(d)$ on X deduced from d.

There is not any problem in given the next definitions for fuzzy metrics and KM-fuzzy metrics.

Definition 2.2. (Gregori and Romaguera [5]) A (*KM*-) fuzzy metric M on X is called *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

Definition 2.3. (Grabiec [2]) A sequence $\{x_n\}$ in a (*KM*-) fuzzy metric space $(X, M, *)$ is called *G-Cauchy* if $\lim_n M(x_{n+p}, x_n, t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$. $(X, M, *)$, or simply X, is called *G-complete* if every G-Cauchy sequence in X is convergent in X.

Definition 2.4. (Grabiec [2], Sehgal and Bharucha-Reid [14]) A self-mapping in a (KM-)fuzzy metric space $(X, M, *)$ is called fuzzy *G-contractive* if there exists $k \in]0, 1[$ such that for all $x, y \in X$, $t > 0$

$$M(f(x), f(y), kt) \geq M(x, y, t).$$

Definition 2.5. (Gregori and Sapena [6]) A self-mapping in a (KM-)fuzzy metric space $(X, M, *)$ is called *fuzzy contractive* if there exists $k \in]0, 1[$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right)$$

for all $x, y \in X$ and $t > 0$.

Definition 2.6. (Mihet [8]) A self-mapping in a (KM-)fuzzy metric space $(X, M, *)$ is called *weak B-contraction* (for ψ) if it satisfies

$$M(x, y, t) > 0 \Rightarrow M(f(x), f(y), t) \geq \psi(M(x, y, t)),$$

where $\psi : [0, 1] \to [0, 1]$ is an increasing function and $\lim_n \psi^n(t) = 1$ for each $t \in]0, 1[$ (note that $\psi(t) \geq t$ for all $t \in [0, 1]$).

Although it is not usual we start extending the concept of G-Cauchy sequence to the classical case. So, in the next two sections (X, d) is a metric space.

3. G-Cauchy sequences in metric spaces

In a metric space (X, d) we will denote the open (closed) ball centered at $x_0 \in X$ and radius $r > 0$ by $B_d(x_0, r)$ ($B_d[x_0, r]$).

Definition 3.1. A sequence $\{x_n\}$ in X is called G-Cauchy if $\lim_n d(x_n, x_{n+1}) = 0$.

If $\{x_n\}$ is G-Cauchy, then obviously $\lim_n d(x_n, x_{n+p}) = 0$ for all $p \in \mathbb{N}$.

A sequence $\{x_n\}$ satisfying $\lim_n d(x_n, x_{n+p}) = 0$ for some $p \in \mathbb{N}$ (even for infinite values of p) is not necessarily G-Cauchy. In fact, we have the next proposition.

Proposition 3.2. *A sequence $\{x_n\}$ is G-Cauchy if and only if there exist positive integers $p_1, p_2, \ldots, p_m$ co-prime such that $\lim_n d(x_n, x_{n+p_i}) = 0$ for $i = 1, 2, \ldots, m$.*

Proof. The direct implication is obvious, since if $\{x_n\}$ is G-Cauchy then $\lim_n d(x_n, x_{n+p}) = 0$ for all $p \in \mathbb{N}$.

Conversely, let $p_1, p_2, \ldots, p_m$ co-prime and suppose that $\lim_n d(x_n, x_{n+p_i}) = 0$ for $i = 1, 2, \ldots, m$. By Bezout's identity there exist $t_1, t_2, \ldots, t_m \in \mathbb{Z}$ such that $t_1p_1 + t_2p_2 + \cdots + t_mp_m = 1$. By the triangle inequality, it is easy to observe that $\lim_n d(x_n, x_{n+t_ip_i}) = 0$ for $i = 1, 2, \ldots, m$.

We have that

$$d(x_n, x_{n+1}) = d(x_n, x_{n+t_1p_1+t_2p_2+\cdots+t_mp_m}) \leq d(x_n, x_{n+t_1p_1})$$
$$+d(x_{n+t_1p_1}, x_{n+t_1p_1+t_2p_2}) + \cdots + d(x_{n+t_1p_1+\cdots+t_{m-1}p_{m-1}}, x_{n+t_1p_1+\cdots+t_mp_m})$$

Taking limit in both sides of the inequality as n tends to ∞, by the above observation we have that $\lim_n d(x_n, x_{n+1}) = 0$ and so $\{x_n\}$ is G-Cauchy. □

The next proposition is obvious.

Proposition 3.3. *Every Cauchy sequence is G-Cauchy.*

The converse of this proposition is, in general, false, as shows the next example.

Example 3.4. (George and Veeramani [1]) Consider $\mathbb{R}$ endowed with its usual metric. Let $\{x_n\}$ be the sequence defined by $x_n = \sum_{i=1}^{n} \frac{1}{i}$ (i.e., x_n are the corresponding partial sums in the harmonic series). It is obvious that $\{x_n\}$ is G-Cauchy and it is well-known that $\{x_n\}$ is not Cauchy.

The concept of G-Cauchyness is so weak that interesting properties of Cauchy sequences are not preserved by G-Cauchy sequences. The next examples point out this fact.

Example 3.5. (A non-bounded G-Cauchy sequence.)
The sequence $\{x_n\}$ in Example 3.4 is G-Cauchy and it is not bounded.

Example 3.6. (A non-G-Cauchy subsequence of a G-Cauchy sequence.)
Consider $\mathbb{R}$ endowed with its usual metric. The sequence $\{x_n\}$ in $\mathbb{R}$ defined by $x_n = \sin\sqrt{n}$ is G-Cauchy (see [10]). Take $n_i = i^2$ for $i \in \mathbb{N}$. Then $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ and it is not G-Cauchy, since $x_{n_i} = \sin i$ and $\lim_i |x_{n_{i+1}} - x_{n_i}| = \lim_i |\sin(i+1) - \sin i|$ does not exist.

Example 3.7. (A G-Cauchy sequence with infinite cluster points.)
Consider $\mathbb{R}^2$ endowed with the metric d_∞. For each $n \in \mathbb{N}$ there exists a unique $m \in \mathbb{N}$ such that $2^m - 1 \le n \le 2^{m+1} - 2$. Since $2^m - 1 \le 3\cdot 2^{m-1} - 2 < 2^{m+1} - 2$, then, we can define the sequence $\{x_n\}$ in $\mathbb{R}^2$, given by

$$x_n = \begin{cases} \left(\frac{n-2^m+1}{2^{m-1}}, \frac{n-2^m+1}{2^{2(m-1)}}\right), & \text{if} \quad 2^m - 1 \le n \le 3\cdot 2^{m-1} - 2 \\ \left(\frac{2^{m+1}-1-n}{2^{m-1}}, \frac{2^{m+1}-1-n}{2^{2(m-1)}}\right), & \text{if} \quad 3\cdot 2^{m-1} - 1 \le n \le 2^{m+1} - 2 \end{cases}$$

for each $n \in \mathbb{N}$.

After an easy computation one can obtain in all cases that $d_\infty(x_n, x_{n+1}) = \frac{1}{2^{m-1}}$ for some $m \in \mathbb{N}$ satisfying the above relations with respect to n, and tacking into account that $n \to \infty$ if and only if $m \to \infty$, then $\lim_n d_\infty(x_n, x_{n+1}) = 0$.

Now, we will see that $(x, 0)$ is a cluster point of $\{x_n\}$ for all $x \in [0, 1]$. Let $x \in [0, 1]$ and take $\epsilon > 0$. Consider $B_{d_\infty}((x, 0), \epsilon)$. Given $s \in \mathbb{N}$ we can find $m \in \mathbb{N}$ such that $m > s$ and $\frac{1}{2^{m-1}} < \epsilon$. Then, we can take $p_s \in \mathbb{N}$, with $p_s \le 2^{m-1} - 1$ such that $\left|\frac{p_s}{2^{m-1}} - x\right| \le \frac{1}{2^{m-1}}$. For $n_s = p_s + 2^m - 1$, we have that $2^m - 1 \le n_s \le 3\cdot 2^{m-1} - 2$, and so we can choose $x_{n_s} = \left(\frac{n_s-2^m+1}{2^{m-1}}, \frac{n_s-2^m+1}{2^{2(m-1)}}\right) \in \{x_n\}$. Then

$$\begin{aligned} d_\infty(x_{n_s}, (x, 0)) &= \sup\left\{\left|\frac{n_s - 2^m + 1}{2^{m-1}} - x\right|, \left|\frac{n_s - 2^m + 1}{2^{2(m-1)}}\right|\right\} \\ &= \sup\left\{\left|\frac{p_s}{2^{m-1}} - x\right|, \left|\frac{p_s}{2^{2(m-1)}}\right|\right\} \le \frac{1}{2^{m-1}} < \epsilon. \end{aligned}$$

Therefore, $x_{n_s} \in B_{d_\infty}((x, 0), \epsilon)$. Then $\{x_n\}$ is frequently in $B_{d_\infty}((x, 0), \epsilon)$ and so $(x, 0)$ is a cluster point of $\{x_n\}$.

Example 3.8. (A G-Cauchy non-convergent sequence with a unique cluster point.)
Now, consider $X = \mathbb{R}\times\mathbb{R}^+ \cup \{(0, 0)\}$ endowed with the metric d_∞ on $\mathbb{R}^2$ restricted to X. The sequence $\{x_n\}$ of the last example is a G-Cauchy sequence in X with a unique cluster point $(0, 0) \in X$, and $\{x_n\}$ is not convergent.

Nevertheless in locally compact spaces a classical nice property of Cauchy sequences is restated as shows the next proposition.

Proposition 3.9. *Every G-Cauchy sequence with a unique cluster point in a locally compact metric space is convergent.*

Proof. Suppose (X, d) is locally compact and let $\{x_n\}$ be a G-Cauchy sequence in X with a unique cluster point $y \in X$.

Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging to y and that $\{x_n\}$ does not converge to y. Then we can find a closed compact ball centered at y, $B_d[y,\epsilon]$, such that for each $i \in \mathbb{N}$ there exists $m_i \geq i$ with $x_{m_i} \notin B_d[y,\epsilon]$. By induction we construct a subsequence $\{x_{m_i}\}$, with $m_i > m_j$ whenever $i > j$, of $\{x_n\}$ such that $x_{m_i} \notin B_d[y,\epsilon]$ for all $i \in \mathbb{N}$. On the other hand, since $\{x_{n_k}\}$ converges to y, for $\epsilon/2 > 0$ we can find $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have that $x_{n_k} \in B_d(y,\epsilon/2)$. Note that $d(x_{m_i}, x_{n_k}) \geq \epsilon/2$ for all $k \geq k_0$ and all $i \in \mathbb{N}$. Now, we will construct a subsequence $\{x_{l_j}\}$ of $\{x_n\}$ such that $\{x_{l_j}\} \subset A = B_d[y,\epsilon] \setminus B_d(y,\epsilon/2)$, as follows.

Take $\epsilon/4 > 0$. Since $\{x_n\}$ is a G-Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \epsilon/4$ for all $n \geq n_0$. Let $i_1 \geq n_0$ and consider $k_1 \geq k_0$ with $n_{k_1} > m_{i_1}$. We claim that we can find $m_{i_1} \leq l_1 \leq n_{k_1}$ such that $x_{l_1} \in A$. Indeed, suppose the contrary, i.e., for all $n \in \mathbb{N}$ with $m_{i_1} \leq n \leq n_{k_1}$ we have that $x_n \notin A$. Then $x_n \in B_d(y,\epsilon/2)$ or $x_n \notin B_d[y,\epsilon]$, and taking into account that $x_{m_{i_1}} \notin B_d[y,\epsilon]$ and that $x_{n_{k_1}} \in B_d(y,\epsilon/2)$, there exists $l \in \mathbb{N}$ with $m_{i_1} \leq l \leq n_{k_1}$ such that $x_l \notin B_d[y,\epsilon]$ and $x_{l+1} \in B_d(y,\epsilon/2)$, thus $d(x_l, x_{l+1}) > \epsilon/4$, a contradiction. Now, we take $i_2 \geq i_1$ such that $m_{i_2} > n_{k_1}$. Since $x_{n_{k_1}} \in B_d(y,\epsilon/2)$ and $x_{m_{i_2}} \notin B_d[y,\epsilon]$, in a similar way that before, we can find $n_{k_1} \leq l_2 \leq m_{i_2}$ such that $x_{l_2} \in A$. Iteratively, we construct a subsequence $\{x_{l_j}\} \subset A$. But A is, obviously, compact and so $\{x_{l_j}\}$ has a cluster point $z \in A$. Then z is a cluster point of $\{x_n\}$ and $z \neq y$, a contradiction. □

As usual, it is defined the following concept.

Definition 3.10. (X,d) is called G-complete if every G-Cauchy sequence in X converges in X.

Clearly a G-complete space is complete. The next proposition is obvious.

Proposition 3.11.
(i) *A G-complete subspace of a (G-) complete space is closed.*
(ii) *A closed subspace of a G-complete space is G-complete.*

4. Weak G-completeness

In order to obtain a weaker concept than G-completeness based on the concept of G-Cauchy sequence we introduce the next definition.

Definition 4.1. A sequence $\{x_n\}$ is called *weak G-convergent* if $\lim_n d(x_n, x_{n+1}) = 0$ and $\{x_n\}$ has (at least) a cluster point. X is called *weak G-complete* if every G-Cauchy sequence is weak G-convergent.

Notice that the concept of weak G-convergence involves, in some sense, convergence. Indeed, $\{x_n\}$ is weak G-convergence if and only if $\{x_n\}$ is G-Cauchy and it has a convergent subsequence. Obviously every convergent sequence is weak G-convergent.

The next result is obvious.

Proposition 4.2. *Every compact space is weak G-complete.*

The new situation can be summarized in the next Diagram of implications.

$$\begin{array}{ccccc} G-completeness & \rightarrow & weak\ G-completeness & \rightarrow & completeness \\ & & \uparrow & & \\ & & compactness & & \end{array}$$

The next examples show that the implications of the last Diagram are not reversed, in general.

Example 4.3. (A complete non weak G-complete metric space.)

The real line $\mathbb{R}$ endowed with the usual metric is complete. Now it is not weak G-complete because the sequence $\{x_n\}$ of Example 3.4 is G-Cauchy but $\{x_n\}$ has not any cluster point.

Example 4.4. (A weak G-complete non-G-complete metric space.)

Let $X = [-1, 1]$ and let d be the usual metric on $\mathbb{R}$ restricted to X. Then by Proposition 4.2 (X, d) is weak G-complete, since $[0, 1]$ is compact, and it is not G-complete. Indeed, for instance $\{sin\sqrt{n}\}$ is a G-Cauchy non-convergent sequence in X ([10]).

In Remark 5.9 we give an example of a weak G-complete space which is not compact.

In the next section we will extend the concepts here introduced for ordinary metrics to fuzzy metrics.

5. G-complete and weak G-complete fuzzy metric spaces

As it is observed in [9] we can characterize a G-Cauchy sequence as follows.

Proposition 5.1. *A sequence $\{x_n\}$ in X is G-Cauchy if and only if $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.*

With small changes on Proposition 3.2 we can obtain the next result.

Proposition 5.2. *A sequence $\{x_n\}$ is G-Cauchy if and only if there exist positive integers $p_1, p_2, \ldots, p_m$ co-prime such that $\lim_n M(x_n, x_{n+p_i}, t) = 1$ for $i = 1, \ldots, m$ and for all $t > 0$.*

The following concepts are now natural.

Definition 5.3. A sequence $\{x_n\}$ in X is called weak G-convergent if $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$ and it has (at least) a cluster point.

Definition 5.4. $(X, M, *)$, or simply X, is called weak G-complete if every G-Cauchy sequence in X is weak G-convergent in X.

The next proposition shows, in some sense, that Definitions 5.3 and 5.4 are appropriate.

Proposition 5.5. *Let $(X, M_d, \cdot)$ be the standard fuzzy metric space induced by a metric d on X, and let $\{x_n\}$ be a sequence in X. Then:*

(i) *$\{x_n\}$ is G-Cauchy in (X, d) if and only if $\{x_n\}$ is G-Cauchy in $(X, M_d, \cdot)$.*

(ii) $\{x_n\}$ *is weak* G*-convergent in* (X, d) *if and only if* $\{x_n\}$ *is weak* G*-convergent in* $(X, M_d, \cdot)$.

Consequently we have:

(iii) (X, d) *is* G*-complete if and only if* $(X, M_d, \cdot)$ *is* G*-complete.*
(iv) (X, d) *is weak* G*-complete if and only if* $(X, M_d, \cdot)$ *is weak* G*-complete.*

Proof. It is obvious from the previous definitions and because $\tau(d) = \tau_{M_d}$. □

Clearly the implications of the above Diagram are also satisfied in fuzzy setting. Also the implications of the mentioned Diagram cannot be reversed, in general. Indeed, Examples 4.3 and 4.4 can be stated for the standard fuzzy metric space, attending to the above proposition.

Next we will see an example of a compact (non-standard) fuzzy metric space which is not G-complete.

Example 5.6. (A compact non-G-complete fuzzy metric space.)

Let $(X, M, \cdot)$ be the fuzzy metric space, where $X = [0, 1]$ and M is given by $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$, for all $x, y \in X$ and for all $t > 0$. This fuzzy metric space is compact, since τ_M is the usual topology of $\mathbb{R}$ restricted to $[0, 1]$ (see [3]). Consider the sequence $\{y_n\}$ in X, where y_n is the projection of x_n onto x axis of the sequence of Example 3.7, i.e.,

$$y_n = \begin{cases} \frac{n-2^m+1}{2^{m-1}}, & \text{if} \quad 2^m - 1 \le n \le 3 \cdot 2^{m-1} - 2 \\ \frac{2^{m+1}-1-n}{2^{m-1}}, & \text{if} \quad 3 \cdot 2^{m-1} - 1 \le n \le 2^{m+1} - 2 \end{cases}$$

Let $t > 0$. For proving that $\{y_n\}$ is G-Cauchy in $(X, M, \cdot)$, we distinguish four cases, but before starting we observe that for all $b > a \ge 0$ it is satisfied that $\frac{a+t}{b+t} \ge \frac{t}{b-a+t}$:

(1) If $2^m - 1 \le n < 3 \cdot 2^{m-1} - 2$, tacking into account the above observation, we have that

$$M(y_n, y_{n+1}, t) = \frac{\frac{n-2^m+1}{2^{m-1}} + t}{\frac{n-2^m+2}{2^{m-1}} + t} \ge \frac{t}{\frac{1}{2^{m-1}} + t}$$

(2) If $n = 3 \cdot 2^{m-1} - 2$, then

$$M(y_n, y_{n+1}, t) = \frac{\frac{2^{m-1}-1}{2^{m-1}} + t}{1 + t}$$

(3) If $3 \cdot 2^{m-1} - 1 \le n < 2^{m+1} - 2$, tacking into account the above observation, we have that

$$M(y_n, y_{n+1}, t) = \frac{\frac{2^{m-1}-n-2}{2^{m-1}} + t}{\frac{2^{m-1}-1-n}{2^{m-1}} + t} \ge \frac{t}{\frac{1}{2^{m-1}} + t}$$

(4) If $n = 2^{m+1} - 2$, then

$$M(y_n, y_{n+1}, t) = \frac{t}{\frac{1}{2^{m-1}} + t}$$

Tacking into account that $n \to \infty$ if and only if $m \to \infty$, then in all cases we have that $\lim_n M(y_n, y_{n+1}, t) = 1$, and so $\{y_n\}$ is G-Cauchy.

Seen Example 3.7 it is clear that each $x \in [0,1]$ is a cluster point of $\{y_n\}$, and so $\{y_n\}$ is not convergent. Therefore, $(X, M, \cdot)$ is not G-complete.

Next, we give an example of a non-compact weak G-complete fuzzy metric space $(X, M, *)$ where M is not a standard fuzzy metric.

Example 5.7. (A weak G-complete fuzzy metric space which is not G-complete and not compact.)

Let $X = \left\{\frac{1}{2^n} : n \geq 2\right\} \cup \left[\frac{1}{2}, 1\right]$. Consider the stationary fuzzy metric space $(X, M, \cdot)$, where $M(x,y) = \frac{\min\{x,y\}}{\max\{x,y\}}$. It is well-known that τ_M is the usual topology of $\mathbb{R}$ restricted to X (see [3]). Since $\left\{\frac{1}{2^n}\right\}$ is open for each $n \geq 2$ then X is not compact.

We claim that every non-eventually constant sequence $\{a_i\}$ which only takes values on $\left\{\frac{1}{2^n} : n \geq 2\right\}$ is not G-Cauchy. Indeed, suppose $\{a_i\}$ only takes values on $\left\{\frac{1}{2^n} : n \geq 2\right\}$ and, without lost of generality, suppose that a_i and a_{i+1} are distinct for $i \in \mathbb{N}$. Then we can write $a_i = \frac{1}{2^{n_i}}$ where $n_i \geq 2$ and $n_i \neq n_{i+1}$. We have that $M(a_i, a_{i+1}) = \frac{1}{2^{|n_{i+1} - n_i|}} \leq \frac{1}{2}$. So, $\lim_i M(a_i, a_{i+1}) \leq \frac{1}{2}$ and $\{a_i\}$ is not G-Cauchy.

Suppose now that the sequence $\{a_i\}$ is frequently in the set $\left\{\frac{1}{2^n} : n \geq 2\right\}$ and also in $\left[\frac{1}{2}, 1\right]$. In such case for any $n_0 \in \mathbb{N}$ we can find $i \geq n_0$ such that $a_i = \frac{1}{2^{n_i}}$ with $n_i \geq 2$ and $a_{i+1} \in \left[\frac{1}{2}, 1\right]$. Then $M(a_i, a_{i+1}) \leq M(a_i, \frac{1}{2}) = \frac{1}{2^{n_i - 1}} \leq \frac{1}{2}$ and again $\{a_i\}$ cannot be G-Cauchy.

So if $\{x_n\}$ is a G-Cauchy sequence in X, after certain stage, x_n is in $\left[\frac{1}{2}, 1\right]$, and since $\left[\frac{1}{2}, 1\right]$ is compact then $\{x_n\}$ has a cluster point in $\left[\frac{1}{2}, 1\right]$ and hence $\{x_n\}$ is weak G-convergent. So, X is weak G-complete.

Now, X is not G-complete. Indeed, the sequence $\left\{\frac{|\sin\sqrt{n}|}{2}\right\}$ is in X and it is G-Cauchy since $\lim_n \left|\frac{\sin\sqrt{n}}{\sin\sqrt{n+1}}\right| = \left|\frac{\sin\sqrt{n+1}}{\sin\sqrt{n}}\right| = 1$ and, clearly, this sequence is not convergent.

Proposition 5.8. *Let $(X, M, *)$ be a stationary fuzzy metric space where $* \geq \mathfrak{L}$. Consider the metric d on X given by $d(x,y) = 1 - M(x,y)$ (see [4]). Then:*

(i) *$\{x_n\}$ is G-Cauchy in (X, d) if and only if $\{x_n\}$ is G-Cauchy in $(X, M, *)$.*
(ii) *$\{x_n\}$ is weak G-convergent in (X, d) if and only if $\{x_n\}$ is weak G-convergent in $(X, M, *)$.*

Consequently we have:

(iii) *(X, d) is G-complete if and only if $(X, M, *)$ is G-complete.*
(iv) *(X, d) is weak G-complete if and only if $(X, M, *)$ is weak G-complete.*

Proof. It is obvious from the previous definitions and because $\tau(d) = \tau_M$ (see [4]). □

Remark 5.9. If we consider the metric space (X, d), where $d(x,y) = 1 - M(x,y)$ and $(X, M, \cdot)$ is the stationary fuzzy metric space of Example 5.7, then by last proposition we have that (X, d) is a weak G-complete metric space which is not G-complete and not compact.

6. Fuzzy Banach contraction theorems

Inspired in the concept of weak B-contraction due to Mihet [8], we introduce the next more general concept of contractivity than the concept due to Grabiec.

Definition 6.1. Let Λ be the class of all mappings $\lambda :]0,\infty[\to]0,\infty[$ such that λ is increasing and $\lim_n \lambda^n(t) = \infty$ for each $t \in]0,\infty[$ (note that $\lambda(t) > t$ for all $t \in]0,\infty[$). Let $(X, M, *)$ be a (KM-)fuzzy metric space. A self-mapping f on X is called *fuzzy λ-contractive mapping* if there exists $\lambda \in \Lambda$ satisfying

$$M(x,y,t) > 0 \Rightarrow M(f(x), f(y), t) \geq M(x, y, \lambda(t)).$$

The next example shows that this concept is, really, more general than the concept of fuzzy G-contraction.

Example 6.2. Consider the fuzzy metric space $(X, M, \cdot)$ of Example 5.6 and consider the self-mapping f on X given by $f(x) = \frac{1}{1+x}$.

First, we will see that f is fuzzy λ-contractive for $\lambda(t) = t + 1 \in \Lambda$. Let $x, y \in X$ (suppose, without lost of generality, that $x \leq y$) and let $t > 0$.

Then $f(x) = \frac{1}{1+x} \geq \frac{1}{1+y} = f(y)$, and so

$$M(f(x), f(y), t) = \frac{\frac{1}{1+y} + t}{\frac{1}{1+x} + t} = \frac{(1+x)(1+t+yt)}{(1+y)(1+t+xt)} = \frac{x+t+1+(x+y+xy)t}{y+t+1+(x+y+xy)t}$$
$$\geq \frac{x+t+1}{y+t+1} = M(x, y, \lambda(t)).$$

Therefore, f is fuzzy λ-contractive.

Now, we will see that f is not fuzzy G-contractive.

Suppose the contrary, that is, f is fuzzy G-contractive. Then there exists $k \in]0,1[$ such that $M(f(x), f(y), kt) \geq M(x, y, t)$ for all $x, y \in X$ and all $t > 0$. Consider $x = 0$, $y \in]0,1]$ such that $y < \frac{1}{k} - 1$. Take $t > \frac{1}{1-k(1+y)}$. Note that $\frac{1}{1-k(1+y)} > 0$. Then $f(x) = f(0) = 1 > \frac{1}{1+y} = f(y)$, and so

$$M(f(x), f(y), kt) = \frac{\frac{1}{1+y} + kt}{1+kt} = \frac{1+kt+kty}{1+kt+y+kty} \geq M(x,y,t) = \frac{t}{y+t}$$

by our above assumption. Then

$$(y+t)(1+kt+kty) \geq (1+kt+y+kty)t.$$

Thus, $y + ykt + y^2kt \geq yt$ and therefore $\frac{1}{1-k(1+y)} \geq t$, a contradiction.

The following lemma is crucial for our purpose.

Lemma 6.3. *Let $(X, M, *)$ be a (KM-)fuzzy metric space and let $\{x_n\}$ be a G-convergent sequence. If $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging to $y \in X$, then $\{x_{n_k+1}\}$ converges to y.*

Proof. Let $t > 0$. For each $k \in \mathbb{N}$ we have that

$$M(x_{n_k+1}, y, t) \geq M(x_{n_k+1}, x_{n_k}, t/2) * M(x_{n_k}, y, t/2),$$

and so, since $\{x_n\}$ is G-Cauchy and $\{x_{n_k}\}$ converges to y we conclude that $\lim_k M(x_{n_k+1}, y, t) = 1$. $\square$

Theorem 6.4. *Let $(X, M, *)$ be a weak G-complete (KM-)fuzzy metric space such that* $\lim_{t\to\infty} M(x, y, t) = 1$ *for all* $x, y \in X$*. If* f *is a fuzzy* λ*-contractive mapping then* f *has a unique fixed point.*

Proof. Let $x \in X$. Construct by induction the sequence $\{x_n\}$ defined by $x_n = f^n(x)$. It is easy to verify that $M(x_n, x_{n+1}, t) \geq M(x, x_1, \lambda^n(t))$ for all $n \in \mathbb{N}$ and $t > 0$. Then $\lim_n M(x_n, x_{n+1}, t) \geq \lim_n M(x, x_1, \lambda^n(t)) = 1$, for all $t > 0$, and so $\{x_n\}$ is G-Cauchy. Since X is weak G-complete, then $\{x_n\}$ is weak G-convergent, i.e. there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to $y \in X$.

Now, we will see that y is a fixed point of f. Indeed, for each $t > 0$ we have that

$$M(y, f(y), t) \geq M(y, x_{n_k+1}, t/2) * M(f(x_{n_k}), f(y), t/2)$$
$$\geq M(y, x_{n_k+1}, t/2) * M(x_{n_k}, y, \lambda(t/2)) \geq M(y, x_{n_k+1}, t/2) * M(x_{n_k}, y, t/2)$$

for all $k \in \mathbb{N}$. Since $\{x_{n_k}\}$ converges to y then by Lemma 6.3 the sequence $\{x_{n_k+1}\}$ converges to y and hence if we take limit as $k \to \infty$ we have that $M(y, f(y), t) = 1$ and so $y = f(y)$.

As in [2] it is proved that y is the unique fixed point. □

Corollary 6.5. *([2], Grabiec's fuzzy Banach contraction theorem.) Let $(X, M, *)$ be a G-complete KM-fuzzy metric space such that*

(i) $\lim_{t\to\infty} M(x, y, t) = 1$ *for all* $x, y \in X$.

Let $f : X \to X$ *be a mapping satisfying*

(ii) $M(f(x), f(y), kt) \geq M(x, y, t)$

for all $x, y \in X$*, where* $k \in]0, 1[$*. Then* T *has a unique fixed point.*

Proof. It is easy to see that f is a fuzzy λ-contractive self-mapping on X for $\lambda(t) = \frac{t}{k}$. The conclusion follows by the last theorem, since X is weak G-complete. □

Remark 6.6. Notice that Theorem 6.4 is a generalization of Grabiec's theorem in two aspects. Indeed, the conditions of contractivity and completeness both have been extended (see the end of next example).

Example 6.7. Consider the fuzzy metric space $(X, M, \cdot)$ of Example 5.6. M satisfies the condition $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. In Example 6.2 we have just seen that $f(x) = \frac{1}{1+x}$ is fuzzy λ-contractive. Moreover $[0, 1]$ is compact, since τ_M is the usual topology of $\mathbb{R}$ restricted to $[0, 1]$, and consequently $[0, 1]$ is weak G-complete. Hence Theorem 6.4 can be applied to ensure the existence of unique fixed point of f in $[0, 1]$.

Notice that Grabiec's theorem cannot be applied because f is not fuzzy G-contractive (see Example 6.2) and also because $[0, 1]$ is not G-complete (see Example 5.6).

Next we generalize Theorem 3.1 of [8]. We omit its proof which can be obtained imitating the proof of the mentioned theorem and the proof of Theorem 6.4.

Theorem 6.8. *Let $(X, M, *)$ be a weak G-complete (KM-)fuzzy metric space and let* f *be a fuzzy weak B-contraction (for* ψ*). If there exists* $x \in X$ *such that* $M(x, f(x), t) > 0$ *for all* $t > 0$*, then* f *has a fixed point.*

Corollary 6.9. *([8], Theorem 3.15, Mihet's fixed point theorem) If $(X, M, *)$ is a G-complete KM-fuzzy metric space and f is a weak B-contraction on X such that for some $x \in X$ $M(x, f(x), t) > 0$ for all $t > 0$ then f has a fixed point.*

Corollary 6.10. *([6], Theorem 5.2, Gregori and Sapena's fixed point theorem) Let $(X, M, *)$ be a G-complete fuzzy metric space and let $f : X \to X$ be a fuzzy contractive mapping. Then f has a unique fixed point.*

Proof. In [9] the author shows that every fuzzy contractive mapping in a KM-fuzzy metric space is a weak B-contraction mapping for $\psi(t) = \frac{t}{t+k(1-t)}$, where $k \in]0, 1[$. Since $(X, M, *)$ is a fuzzy metric space then the condition $M(x, f(x), t) > 0$ for all $t > 0$ is fulfilled for all $x \in X$. Therefore, applying Theorem 6.8 f has a fixed point.

We will see that this fixed point is unique. Suppose that $y, z \in X$ are fixed points of f. Then for all $t > 0$ we have that

$$\begin{aligned} M(y, z, t) &= M(f(y), f(z), t) \geq \psi(M(y, z, t)) = \psi(M(f(y), f(z), t)) \\ &\geq \psi^2(M(y, z, t)) \geq \cdots \geq \psi^n(M(y, z, t)) \end{aligned}$$

for all $n \in \mathbb{N}$. Then $M(y, z, t) \geq \lim_n \psi^n(M(y, z, t)) = 1$ and so $z = y$. □

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