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EXISTENCE OF SOLUTIONS TO AN IMPULSIVE DIRICHLET BOUNDARY VALUE PROBLEM

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Abstract. In this paper, the authors establish the existence of nontrivial classical solutions for a class of Dirichlet boundary value problems with impulsive effects. The approach is based on variational methods and critical point theory.

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1. Introduction

Consider the nonlinear impulsive Dirichlet boundary value problem

$$\begin{cases} -u''(x) = \lambda f(x, u(x)) + g(u(x)), \quad \text{a.e. } x \in [0, T], \\ u(0) = u(T) = 0, \\ \Delta u'(x_j) = I_j(u(x_j)), \ j = 1, ..., p, \end{cases} \tag{IP}$$

where λ is a positive parameter, $T > 0$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $x_0 = 0 < x_1 < \ldots < x_p < x_{p+1} = T$, and $\Delta u'(x_j)$ is defined by

$$\Delta u'(x_j) = u'(x_j^+) - u'(x_j^-) = \lim_{x \to x_j^+} u'(x) - \lim_{x \to x_j^-} u'(x).$$

The following conditions will be assumed to hold throughout the remainder of this paper:

(H1) $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with a Lipschitz constant $L \in (0, 4/T^2)$, i.e., $|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$ for all $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$;

(H2) The impulsive functions $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, p$, are continuous and satisfy the condition $\sum_{j=1}^{p} (I_j(t_1) - I_j(t_2))(t_1 - t_2) \geq 0$ for all $t_1, t_2 \in \mathbb{R}$;

(H3) I_j, $j = 1, ..., p$, have sublinear growth, i.e., there exist constants $a_j > 0$, $b_j \geq 0$, and $\gamma_j \in [0, 1)$ such that $|I_j(t)| \leq a_j + b_j|t|^{\gamma_j}$ for every $t \in \mathbb{R}$ and $j = 1, ..., p$.

The theory and applications of impulsive functional differential equations are emerging as important areas of investigation, and in some senses they have proved to be far richer than those for non-impulsive equations. Various population models that are characterized by the fact that sudden changes of their state depends on their prior history can be expressed by impulsive differential equations with deviating arguments. These occur in such areas as population dynamics, ecology, epidemics, etc. In recent decades, impulsive differential equations have also become more important in mathematical models of spacecraft control, impact mechanics, physics, chemistry, biotechnology, economics, and inspection processes in operations research. It is now recognized that the theory of impulsive differential equations is a natural framework for a mathematical modeling of many natural phenomena.

The questions of the existence and multiplicity of solutions for such problems have been studied by several authors; we refer the reader to the monographs [6, 11] and the papers [1, 3, 4, 5, 8, 10, 15, 14, 16, 21, 23, 25, 26, 27] as examples of results of this type. Using a result of Bonanno (see Lemma 2.1 below), we establish some new results on the existence of nontrivial classical solutions to the problem (IP).

2. Preliminaries

For a given nonempty set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define

$$\varrho(r) = \sup_{v \in \Phi^{-1}(r,\infty)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty,r]} \Psi(u)}{\Phi(v) - r}$$

for all $r \in \mathbb{R}$. We also let X^* denote the dual space of X.

Lemma 2.1. ([7, Theorem 5.3]) *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that $\varrho(r) > 0$, and for each*

$$\lambda > \frac{1}{\varrho(r)},$$

the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then, for each $\lambda > \frac{1}{\varrho(r)}$, there exists $u_{0,\lambda} \in \Phi^{-1}(r, \infty)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(r, \infty)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Let $X = H_0^1(0, T)$ and $H^2(0, T) = \{u \in C^1[0, T] : u'' \in L^2[0, T]\}$. In the space X, consider the inner product

$$\prec u, v \succ = \int_0^T u'(x)v'(x)dx,$$

which induces the norm

$$\|u\| = \Big(\int_0^T |u'(x)|^2 dx \Big)^{\frac{1}{2}}.$$

It is not hard to see that

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)| \le \frac{\sqrt{T}}{2}\|u\| \quad \text{for } u \in X. \tag{2.1}$$

Next, we define what we mean by a solution to our problem.

Definition 2.1. By a classical solution of the problem (IP), we mean a function $u \in \{u(x) \in H^1(0,T):\ u(x) \in H^2(x_j, x_{j+1}),\ j = 0, 1, ..., p\}$ such that u satisfies (IP).

Definition 2.2. A function $u \in X$ is a weak solution of the problem (IP) if

$$\int_0^T u'(x)v'(x)dx + \sum_{j=1}^p I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx - \lambda \int_0^T f(x,u(x))v(x)dx = 0$$

for every $v \in X$.

Remark 2.1. Using an approach such as that in [3, Lemma 5], it is not difficult to show that a weak solution of (IP) is in fact a classical solution.

In what follows, we let

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in [0,T]\times\mathbb{R}$$

and

$$G(t) = -\int_0^t g(\xi)d\xi \quad \text{for all } t \in \mathbb{R}.$$

We will also need the following lemma.

Lemma 2.2. *Let $T : X \to X^*$ be the operator defined by*

$$T(u)v = \int_0^T u'(x)v'(x)dx + \sum_{j=1}^p I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx \tag{2.2}$$

for every u, $v \in X$. Then T has a continuous inverse on X^*.

Proof. Note that (H1) implies

$$|g(t)| \le L|t| \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.3}$$

Then from (H3), (2.1), and (2.3), it follows that

$$\begin{aligned}
&\lim_{\|u\|\to\infty} \frac{\langle T(u), u\rangle}{\|u\|} \\
= &\lim_{\|u\|\to\infty} \frac{\int_0^T (u'(x))^2dx + \sum_{j=1}^p I_j(u(x_j))u(x_j) - \int_0^T g(u(x))u(x)dx}{\|u\|} \\
\ge &\lim_{\|u\|\to\infty} \frac{\int_0^T (u'(x))^2dx + \sum_{j=1}^p I_j(u(x_j))u(x_j) - \frac{LT^2}{4}\|u\|^2}{\|u\|} \\
\ge &\lim_{\|u\|\to\infty} \frac{\left(1 - \frac{LT^2}{4}\right)\|u\|^2 - \sum_{j=1}^p \left(a_j \frac{\sqrt{T}}{2}\|u\| + b_j\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}\|u\|^{\gamma_j+1}\right)}{\|u\|} = \infty.
\end{aligned}$$

Hence, the map T is coercive.

By (H1) and (H2), we have

$$\langle T(u) - T(v), u - v\rangle \geq \left(1 - \frac{LT^2}{4}\right) \|u - v\|^2.$$

Hence, T is uniformly monotone. Note that T is also hemicontinuous on X. Then, by [24, Theorem 26.A (d)], T^{-1} exists and is continuous on X^*. $\square$

We will need to define the constants (see [3])

$$C_1 = \frac{1}{2} - \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1} \left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1},$$

$$C_2 = \frac{1}{2} + \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1} \left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1},$$

and

$$C_3 = \frac{\sqrt{T}}{2} \sum_{j=1}^{p} a_j + \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1} \left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1},$$

and the function $H : [0, \infty) \to \mathbb{R}$ by

$$H(t) = \left(C_1 - \frac{LT^2}{8}\right) t^2 - C_3 t \tag{2.4}$$

3. Main results

Our main result in this paper is contained in the following theorem.

Theorem 3.1. *Assume that $C_1 - \frac{LT^2}{8} > 0$ and there exist four positive constants ν, τ, η, and δ with η, $\delta < T/2$, and $\tau > \nu > \frac{\sqrt{T}C_3}{2\left(C_1 - \frac{LT^2}{8}\right)}$ such that*

- (C1) *$F(x, t) \geq 0$ for all $(x, t) \in ([0, \eta] \cup [T - \delta, T]) \times [0, \tau]$;*
- (C2) *$\int_0^T \sup_{t\in[-\nu,\nu]} F(x, t)dx < \int_\eta^{T-\delta} F(x, \tau)dx$;*
- (C3) *there exist $K \in \mathbb{R}$ and $\kappa \in (0, 2)$ such that $\limsup_{|\xi|\to\infty} \frac{F(x,\xi)}{|\xi|^\kappa} < K$ uniformly for all $x \in [0, T]$.*

Then, for each $\lambda > \overline{\lambda}$, where

$$\overline{\lambda} = \frac{\frac{\eta + \delta}{\eta\delta} \left(C_2 + \frac{LT^2}{8}\right) \tau^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3\tau - \frac{4}{T} \left(C_1 - \frac{LT^2}{8}\right) \nu^2 + \frac{2}{\sqrt{T}} C_3\nu}{\int_\eta^{T-\delta} F(x, \tau)dx - \int_0^T \sup_{t\in[-\nu,\nu]} F(x, t)dx},$$

the problem (IP) has at least one nontrivial classical solution $u \in X$ such that

$$\frac{4}{T} \left(C_1 - \frac{LT^2}{8}\right) \nu^2 - \frac{2}{\sqrt{T}} C_3\nu < \frac{1}{2} \int_0^T (u'(x))^2 dx + \sum_{j=1}^{p} \int_0^{u(x_j)} I_j(t)dt + \int_0^T G(u(x))dx. \tag{3.1}$$

Proof. Define the functionals Φ, $\Psi : X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_0^T (u'(x))^2 dx + \sum_{j=1}^p \int_0^{u(x_j)} I_j(t)dt + \int_0^T G(u(x))dx \tag{3.2}$$

and

$$\Psi(u) = \int_0^T F(x, u(x))dx. \tag{3.3}$$

It is well known that Ψ is a Gâteaux differentiable functional, is sequentially weakly lower semicontinuous, and its Gâteaux derivative at $u \in X$ is the functional $\Psi'(u) \in X^*$ defined by

$$\Psi'(u)(v) = \int_0^T f(x, u(x))v(x)dx \quad \text{for every } v \in X.$$

To show that $\Psi' : X \to X^*$ is a compact operator, let $u_n \to u \in X$ weakly in X as $n \to \infty$. Then, $u_n \to u$ strongly in $C([0, T])$. Since $f(x, \cdot)$ is continuous in $\mathbb{R}$ for every $x \in [0, T]$, we have $f(x, u_n) \to f(x, u)$ strongly as $n \to \infty$. By the Lebesgue dominated convergence theorem, we see that $\Psi'(u_n) \to \Psi'(u)$ strongly. Thus, Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by [24, Proposition 26.2].

We also know that Φ is Gâteaux differentiable with Gâteaux derivative at $u \in X$ being the functional $\Phi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_0^T u'(x)v'(x)dx + \sum_{j=1}^p I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx \text{ for every } v \in X.$$

Lemma 2.2 implies that Φ' has a continuous inverse on X^*. Since Φ' is monotonic, Φ is sequentially weakly lower semicontinuous (see [24, Proposition 25.20]).

Since $\|u\|^{\gamma_j+1} \le \|u\|^2$ for every $\|u\| \ge 1$ and $\|u\|^{\gamma_j+1} \le \|u\|$ for every $\|u\| < 1$, we have that $\|u\|^{\gamma_j+1} \le \|u\| + \|u\|^2$ for all $u \in X$. From (H3) and (2.1) we then obtain

$$\begin{aligned}\left|\sum_{j=1}^p \int_0^{u(x_j)} I_j(x)dx\right| &\le \sum_{j=1}^p \left|\int_0^{u(x_j)} I_j(x)dx\right| \\ &\le \sum_{j=1}^p \left(a_j|u(x_j)| + \frac{b_j}{\gamma_j+1}|u(x_j)|^{\gamma_j+1}\right) \\ &\le \sum_{j=1}^p \left(a_j\frac{\sqrt{T}}{2}||u|| + \frac{b_j}{\gamma_j+1}\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}||u||^{\gamma_j+1}\right). \end{aligned} \tag{3.4}$$

Thus, from (2.1), (3.4), and (2.3), for $u \in X$ we see that

$$\left(C_1 - \frac{LT^2}{8}\right)\|u\|^2 - C_3\|u\| \le \Phi(u) \le \left(C_2 + \frac{LT^2}{8}\right)\|u\|^2 + C_3\|u\|. \tag{3.5}$$

Now η, $\delta \in (0, T/2)$ implies $\frac{1}{\eta} + \frac{1}{\delta} > \frac{4}{T}$, so from (C2) and the fact that $C_2 \geq C_1$, we see that $\frac{1}{\lambda} > 0$. Choose $r = \frac{4}{T}\left(C_1 - \frac{LT^2}{8}\right)\nu^2 - \frac{2}{\sqrt{T}}C_3\nu$ and let w be defined by

$$w(x) = \begin{cases} \frac{\tau}{\eta}x, & x \in [0, \eta], \\ \tau, & x \in [\eta, T - \delta], \\ \frac{\tau}{\delta}(T - x), & x \in [T - \delta, T]. \end{cases} \tag{3.6}$$

It is clear that $w \in X$ and

$$\|w\| = \sqrt{\frac{\eta + \delta}{\eta\delta}}\tau. \tag{3.7}$$

Since $C_1 - \frac{LT^2}{8} > 0$ and $C_3 > 0$, we have $H(t) \leq 0$ for $t \in \left[0, \frac{C_3}{C_1 - \frac{LT^2}{8}}\right]$, $H(t)$ is strictly increasing on $\left[\frac{C_3}{2(C_1 - \frac{LT^2}{8})}, \infty\right)$, and $H(t) \geq 0$ on $\left[\frac{C_3}{C_1 - \frac{LT^2}{8}}, \infty\right)$. Since $\tau > \nu > \frac{\sqrt{T}C_3}{2\left(C_1 - \frac{LT^2}{8}\right)} > 0$, we have

$$\sqrt{\frac{\eta + \delta}{\eta\delta}}\tau > \sqrt{\frac{\eta + \delta}{\eta\delta}}\nu > \frac{2}{\sqrt{T}}\nu > \frac{C_3}{C_1 - \frac{LT^2}{8}}.$$

Hence, from (3.7), we see that $\|w\| > \frac{C_3}{C_1 - \frac{LT^2}{8}}$, which, together with the fact that $H(t)$ is strictly increasing on $\left[\frac{C_3}{C_1 - \frac{LT^2}{8}}, \infty\right)$, implies

$$H(\|w\|) > H\left(\frac{2}{\sqrt{T}}\nu\right) > H\left(\frac{C_3}{C_1 - \frac{LT^2}{8}}\right) = 0.$$

Thus, $r = H\left(\frac{2}{\sqrt{T}}\nu\right)$, and so

$$\left(C_1 - \frac{LT^2}{8}\right)\|w\|^2 - C_3\|w\| > r > 0. \tag{3.8}$$

Since $H(t)$ is strictly increasing on $\left[\frac{C_3}{C_1 - \frac{LT^2}{8}}, \infty\right)$, $\frac{2}{\sqrt{T}}\nu > \frac{C_3}{C_1 - \frac{LT^2}{8}}$, and $r = H\left(\frac{2}{\sqrt{T}}\nu\right)$, we see that

$$\left[\frac{C_3}{C_1 - \frac{LT^2}{8}},\ \frac{2}{\sqrt{T}}\nu\right] \subseteq \{t \in [0, \infty)\ :\ H(t) \leq r\} \tag{3.9}$$

and

$$\left(\frac{2}{\sqrt{T}}\nu,\ \infty\right) \cap \{t \in [0, \infty)\ :\ H(t) \leq r\} = \emptyset. \tag{3.10}$$

Recalling that $H(t) \le 0$ for $t \in \left[0, \frac{C_3}{C_1 - \frac{LT^2}{8}}\right]$, we have

$$\left[0, \frac{C_3}{C_1 - \frac{LT^2}{8}}\right] \subseteq \{t \in [0,\infty) \ : \ H(t) \le r\}. \tag{3.11}$$

From (3.9)–(3.11), we have

$$\{t \in [0,\infty) \ : \ H(t) \le r\} = \left[0, \ \frac{2}{\sqrt{T}}\nu\right]. \tag{3.12}$$

For any $u \in \Phi^{-1}(-\infty, r]$, from (3.5), we observe that

$$H(\|u\|) \le \Phi(u) \le r.$$

Hence, using (3.12), we have $\|u\| \le \frac{2\nu}{\sqrt{T}}$. Then, in view of (2.1), we see that

$$\Phi^{-1}(-\infty, r] \subseteq \{u \in X \ : \ ||u||_\infty \le \nu\}.$$

Thus,

$$\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,r]} \int_0^T F(x,u(x))dx \le \int_0^T \sup_{t\in[-\nu,\nu]} F(x,t)dx.$$

Since $0 \le w(x) \le \tau$ for each $x \in [0,T]$, condition (C1) implies

$$\int_0^\eta F(x,w(x))dx + \int_{T-\delta}^T F(x,w(x))dx \ge 0. \tag{3.13}$$

Therefore,

$$\begin{aligned} \varrho(r) &\ge \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{\Phi(w) - r} \\ &\ge \frac{\Psi(w) - \int_0^T \sup_{t\in[-\nu,\nu]} F(x,t)dx}{\Phi(w) - r} \\ &\ge \frac{\int_\eta^{T-\delta} F(x,\tau)dx - \int_0^T \sup_{t\in[-\nu,\nu]} F(x,t)dx}{\frac{\eta+\delta}{\eta\delta}(C_2 + \frac{LT^2}{8})\tau^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\tau - \frac{4}{T}(C_1 - \frac{LT^2}{8})\nu^2 + \frac{2}{\sqrt{T}}C_3\nu} \\ &= \frac{1}{\lambda} > 0. \end{aligned}$$

From (C3), there exists a constant $E > 0$ such that

$$F(x,t) \le Kt^\kappa + E \quad \text{for } (x,t) \in [0,T] \times \mathbb{R}.$$

Thus, for $u \in X$, we have

$$F(x,u(x)) \le K|u(x)|^\kappa + E \quad \text{for } x \in [0,T]. \tag{3.14}$$

For any fixed $\lambda > \overline{\lambda}$, from (2.1), (3.2), (3.3), and (3.14), it follows that

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\int_0^T (u'(x))^2 dx + \sum_{j=1}^{p}\int_0^{u(x_j)} I_j(t)dt + \int_0^T G(u(x))dx \\ &\quad - \lambda\int_0^T F(x,u(x))dx \\ &\geq \left(\frac{1}{2} - \frac{LT^2}{8}\right)\|u\|^2 - \sum_{j=1}^{p}\left(a_j\frac{\sqrt{T}}{2}\|u\| + b_j\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}\|u\|^{\gamma_j+1}\right) \\ &\quad - \lambda KT\left(\frac{\sqrt{T}}{2}\right)^{\kappa}\|u\|^{\kappa} - \lambda ET.\end{aligned}$$

Then, in view of the fact that $\gamma_j \in [0,1)$ and $\kappa \in (0,2)$, we see that

$$\lim_{||u||\to+\infty}(\Phi(u) - \lambda\Psi(u)) = \infty,$$

i.e., $\Phi - \lambda\Psi$ is coercive.

All the conditions of Lemma 2.1 hold, so for each $\lambda > \overline{\lambda}$, $\Phi - \lambda\Psi$ admits at least one local minimum u satisfying (3.1). Finally, taking into account Remark 2.1 and the fact that the weak solutions of the problem (IP) are exactly the critical points of the functional $\Phi - \lambda\Psi$, completes the proof of the theorem. □

Remark 3.1. The role of condition (C3) is to guarantee that $\Phi - \lambda\Psi$ is coercive. By examining the proof of Theorem 3.1, we see that $\Phi - \lambda\Psi$ is still coercive if we replace (C3) with the condition

(C4) $\limsup_{|\xi|\to\infty}\frac{F(x,\xi)}{|\xi|^2} \leq 0$ uniformly for all $x \in [0,T]$.

As an application of Theorem 3.1, we give an existence result for the case where the function f is separable. Let $\alpha \in L^1([0,T])$ be such that $\alpha(x) \geq 0$ a.e. $x \in [0,T]$, $\alpha \not\equiv 0$, and let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that $h(0) \neq 0$. Let

$$H(t) = \int_0^t h(\xi)d\xi \quad \text{for all } t \in \mathbb{R}.$$

Corollary 3.1. *Assume that $C_1 - \frac{LT^2}{8} > 0$ and there exist four positive constants ν, τ, η, and δ with η, $\delta < T/2$, and $\tau > \nu > \frac{\sqrt{T}C_3}{2\left(C_1 - \frac{LT^2}{8}\right)}$ such that:*

(C5) $\|\alpha\|_{L^1([0,T])}H(\nu) < \|\alpha\|_{L^1([\eta,T-\delta])}H(\tau)$;

(C6) *there exist $K \in \mathbb{R}$ and $\kappa \in (0,2)$ such that* $\limsup_{|\xi|\to\infty}\frac{H(\xi)}{|\xi|^{\kappa}} < K$.

Then, for each $\lambda > \overline{\lambda'}$, where

$$\overline{\lambda'} = \frac{\frac{\eta+\delta}{\eta\delta}\left(C_2 + \frac{LT^2}{8}\right)\tau^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\tau - \frac{4}{T}\left(C_1 - \frac{LT^2}{8}\right)\nu^2 + \frac{2}{\sqrt{T}}C_3\nu}{\|\alpha\|_{L^1([\eta,T-\delta])}H(\tau) - \|\alpha\|_{L^1([0,T])}H(\nu)},$$

the problem

$$\begin{cases} -u''(x) = \lambda\alpha(x)h(u(x)) + g(u(x)), & a.e.\ x \in [0,T], \\ \Delta u'(x_j) = I_j(u(x_j)), & j = 1, 2, ..., p, \\ u(0) = u(T) = 0, \end{cases} \tag{3.15}$$

has at least one nontrivial classical solution $u \in X$ *satisfying* (3.1).

Remark 3.2. As noted in Remark 3.1, Corollary 3.1 is still true if we replace (C6) with the condition

(C7) $\limsup_{|\xi|\to\infty} \frac{H(\xi)}{|\xi|^2} = 0$.

We conclude with an example satisfying the hypotheses of Corollary 3.1.

Example 3.1. Consider the problem

$$\begin{cases} -u''(x) = 2\lambda x \left(\frac{5}{3}u^{2/3} + 1\right) + \frac{1}{8}u, & \text{a.e. } x \in [0,4], \\ u(0) = u(4) = 0, \\ \Delta u'(x_1) = \arctan u(x_1),\ \Delta u'(x_2) = \arctan u(x_2), & 0 < x_1 < x_2 < 4. \end{cases} \tag{3.16}$$

We claim that there exists $\lambda^* > 0$ such that, for each $\lambda > \lambda^*$, the problem (3.16) has at least one nontrivial classical solution.

In fact, with $T = 4$, $p = 2$, $\alpha(x) = 2x$, $h(t) = \frac{5}{3}t^{2/3} + 1$, $g(t) = \frac{1}{8}t$, and $I_1(t) = I_2(t) = \arctan t$, we see that problem (3.16) is of the form of the problem (3.15) and the covering assumptions (H1)–(H3) are satisfied. In particular, in (H1), we can take $L = \frac{1}{8}$, and in (H3), we can choose $a_1 = a_2 = \frac{\pi}{2}$ and $b_1 = b_2 = \gamma_1 = \gamma_2 = 0$. Clearly, we have $C_1 - \frac{LT^2}{8} = \frac{1}{4} > 0$.

Moreover, if we let $\nu = 27$, $\tau = 64$, $\eta = \delta = 1$, $K = 1$, and $\kappa = \frac{11}{6}$, then it is easy to check that the conditions (C5) and (C6) hold. Thus, all the assumptions of Corollary 3.1 are satisfied and so our claim holds.

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