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EXISTENCE OF BEST PROXIMITY POINTS FOR SET-VALUED CYCLIC MEIR–KEELER CONTRACTIONS

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Abstract. In this paper the concept of set-valued cyclic Meir–Keeler contraction map is introduced. The existence of best proximity point for such maps on a metric space with the UC property is presented.
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1. Introduction and preliminaries

Banach contraction fixed point theorem is a powerful tool in nonlinear analysis, differential equation and many others. It say that every contraction map on a complete metric space has a unique fixed point. Recently, Kirk et al. [10] introduced cyclic maps and obtained a fixed point theorem for such maps on a complete metric space.

By motivation of the Kirk et al. result, Eldred and Veeramani [6] introduced the notion of cyclic contraction as follows:

Definition 1.1. Let A and B be nonempty subsets of a metric space (M, d). Suppose that $f : A \cup B \to A \cup B$ is cyclic map (that is $f(A) \subset B$ and $f(B) \subset A$). Then f is said to be cyclic contraction if there exists $k \in (0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y) + (1 - k)D(A, B),\ \forall x \in A\ \ \forall y \in B,$$

where

$$D(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Since for a cyclic map f on $A \cup B$, we have $d(x, f(x)) \geq D(A, B)$ for all $x \in A \cup B$, the existence of best proximity points for such maps have been considered by many authors; see [1, 2, 3, 4, 5, 7, 8, 9, 12, 14, 15] and references therein. However, we say that $x \in A \cup B$ is a best proximity point for cyclic map f, if $d(x, f(x)) = D(A, B)$. Notice that in the case where $A \cap B \neq \emptyset$, best proximity points are fixed points.

Eldred and Veeramani [6] presented best proximity point results for cyclic contraction maps in uniformly convex Banach spaces and later these results have been generalized to cyclic Meir–Keeler contractions by Di Bari et al. [4]. Suzuki et al. [14] introduced a notion of the property UC in metric spaces and obtained best proximity points for cyclic Meir–Keeler contractions in metric spaces with the property UC. Here, we extend the concept of cyclic Meir–Keeler contraction for single valued maps to set-valued maps. Then, an existence result of best proximity point for such maps in metric spaces with the property UC is given. Our result extends the corresponding ones in Di Bari et al. [4] and Suzuki et al. [14] to set-valued maps.

2. Main result

In this section, we prove the existence of a best proximity point for set-valued Meir–Keeler contraction maps. We begin with the notions of cyclic Meir–Keeler contraction and the UC property which was introduced by Suzuki et al. [14] as follows.

Definition 2.1. [14] Let (M, d) be a metric space and A and B be nonempty subsets of M. Then a map $f : A \cup B \to A \cup B$ is said to be a cyclic Meir–Keeler contraction if the following conditions are satisfied:

(i) $f(A) \subset B$ and $f(B) \subset A$.

(ii) For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < D(A, B) + \varepsilon + \delta \text{ implies } d(f(x), f(y)) < D(A, B) + \varepsilon$$

for all $x \in A$ and $y \in B$.

Definition 2.2. [14] Let A and B be nonempty subsets of a metric space (M, d). Then the pair (A, B) is said to satisfy the property UC if the following holds:
If $\{x_n\}$ and $\{x'_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_n d(x_n, y_n) = D(A, B)$ and $\lim_n d(x'_n, y_n) = D(A, B)$, then $\lim_n d(x_n, x'_n) = 0$.

We need the following result for establish the main result.

Theorem 2.3. *(Theorem 3 of [14]) Let (M, d) be a metric space and let A and B be nonempty subsets of M such that (A, B) satisfies the property UC. Assume that A is complete and $f : A \cup B \to A \cup B$ is a cyclic Meir–Keeler contraction. Then, there exists a unique best proximity point z in A and z is a unique fixed point of f^2 in A. Also, $\{f^{2n}(x)\}$ converges to z for every $x \in A$.*

Let (M, d) be a metric space, $\mathcal{CB}(M)$ and $\mathcal{K}(M)$ denote the family of all nonempty closed and bounded subsets of M and the family of all nonempty compact subsets of M, respectively. Then, the Pompeiu-Hausdorff metric on $\mathcal{CB}(M)$ is given by

$$H(X, Y) = \max\{e(X, Y), e(Y, X)\},$$

where $e(X,Y) = \sup_{a\in X} d(a,Y)$ and $d(a,Y) = \inf_{b\in Y} d(a,b)$. It is well known that if (M,d) is a complete metric space, then $(\mathcal{K}(M),H)$ is a complete metric space.
In the first version of this paper the following proposition was proved by authors but recently we found this result in [13, Proposition 2.3]. Although the paper [13] has been submitted after this work but since the proof of Proposition 2.3 of [13] is similar to the our proof, therefore, we omit its proof in the the final version.

Proposition 2.4. *Let (M,d) be a metric space and A and B be nonempty subsets of M such that the pair (A,B) satisfies the property UC. Then the pair $(\mathcal{K}(A),\mathcal{K}(B))$ also satisfies the property UC in $(\mathcal{CB}(M),H)$.*

Now, we introduce the notion of set-valued cyclic Meir–Keeler contraction mappings.

Definition 2.5. Let (M,d) be a metric space and let A and B be nonempty subsets of M. Then a set-valued map $T : A\cup B \multimap A\cup B$ is called a set-valued cyclic Meir–Keeler contraction if the following are satisfied:

(i) $T(A) \subset B$ and $T(B) \subset A$.
(ii) For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x,y) < D(A,B) + \varepsilon + \delta \quad \text{implies} \quad H(T(x),T(y)) < D(A,B) + \varepsilon$$

for all $x \in A$ and $y \in B$.

In the sequel, Lim's characterization for Meir–Keeler contractions in [11] is extended to set-valued cyclic Meir–Keeler contractions.

Definition 2.6. [11] A function φ from $[0,\infty)$ into itself is called an L-function if $\varphi(0) = 0, \varphi(s) > 0$ for $s \in (0,\infty)$, and for every $s \in (0,\infty)$, there exists $\delta > 0$ such that $\varphi(t) \le s$ for all $t \in [s, s+\delta]$.

Lemma 2.7. [4, 11] *Let Y be a nonempty set, and let h and g be functions from Y into $[0,\infty)$. Then the following are equivalent.*
(i) For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in Y, \quad h(x) < \varepsilon + \delta \quad \textit{implies} \quad g(x) < \varepsilon.$$

(ii) There exists a (nondecreasing, continuous) L-function φ, such that

$$x \in Y, \quad h(x) > 0 \quad \textit{implies} \quad g(x) < \varphi(f(x))$$

and

$$x \in Y, \quad f(x) = 0 \quad \textit{implies} \quad g(x) = 0.$$

Lemma 2.8. [4] *Let φ be an L-function. Let $\{s_n\}$ be a nonincreasing sequence of nonnegative real numbers. Suppose $s_{n+1} < \varphi(s_n)$ for all $n \in \mathbb{N}$ with $s_n > 0$. Then $\lim_n s_n = 0$.*

According to Lemma 2.7 we deduce the following characterization of the set-valued cyclic Meir–Keeler contraction.

Proposition 2.9. *Let (M, d) be a metric space and let A and B be nonempty subsets of M. Assume that $T : A \cup B \multimap A \cup B$ is a set-valued map. Then T is set-valued cyclic Meir–Keeler contraction if and only if there exists a (nondecreasing, continuous) L-function φ such that*

$$d(x, y) > D(A, B) \text{ implies } H(T(x), T(y)) < \varphi(d(x, y) - D(A, B)) + D(A, B)$$

and

$$d(x, y) = D(A, B) \text{ implies } H(T(x), T(y)) = D(A, B)$$

for all $x \in A$ and $y \in B$.

As a consequence of the above proposition, we have the following set-valued version of Lemma 3 in [4].

Lemma 2.10. *Let (M, d) be a metric space and let A and B be nonempty subsets of M. Suppose that $T : A \cup B \multimap A \cup B$ is a set-valued cyclic Meir–Keeler contraction and φ is an L-function as in Proposition 2.9. Then,*

(i) $H(T(x), T(y)) \le d(x, y), \ \forall x \in A$ *and* $y \in B$.
(ii) $H(T(x), T(y)) \le \varphi(d(x, y) - D(A, B)) + D(A, B), \ \forall x \in A$ *and* $y \in B$.

Now, we are ready to state our main result.

Theorem 2.11. *Let (M, d) be a metric space and let A and B be nonempty subsets of M such that (A, B) satisfies the property UC. Assume that A is complete and $T : A \cup B \multimap A \cup B$ is a set-valued cyclic Meir–Keeler contraction such that $T(X)$ is compact for any $X \in \mathcal{K}(A) \cup \mathcal{K}(B)$. Then T has a best proximity point x in A. Furthermore, if $y \in T(x)$ and $d(x, y) = D(A, B)$, then y is a best proximity point in B and x is a fixed point of T^2.*

Proof. Let $F : \mathcal{K}(A) \cup \mathcal{K}(B) \to \mathcal{K}(A) \cup \mathcal{K}(B)$ be defined by $F(X) = T(X)$ for all $X \in \mathcal{K}(A) \cup \mathcal{K}(B)$. By our assumption F is well defined. Since T is cyclic, F is cyclic. From Proposition 2.4, the pair $(\mathcal{K}(A), \mathcal{K}(B))$ has the property UC. Now, we show that F is a Meir–Keeler contraction. Since T is set-valued Meir–Keeler contraction, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \varepsilon + \delta + D(A, B) \Rightarrow H(T(x), T(y)) < \varepsilon + D(A, B), x \in A, \ y \in B. \tag{2.1}$$

Let $X \in \mathcal{K}(A)$ and $Y \in \mathcal{K}(B)$ be such that $H(X, Y) < \delta + \varepsilon + D(A, B)$. We show that

$$H(F(X), F(Y)) < \varepsilon + D(A, B).$$

If $H(X, Y) < \delta + \varepsilon + D(A, B)$, then $e(X, Y) < \delta + \varepsilon + D(A, B)$. Therefore, if z is an arbitrary point in X, then $d(z, Y) < \delta + \varepsilon + D(A, B)$. Since Y is compact, there exists $w \in Y$ such that $d(z, w) = d(z, Y)$ and so $d(z, w) < \delta + \varepsilon + D(A, B)$. Hence, by (2.1) we have $H(T(z), T(w)) < \varepsilon + D(A, B)$. Thus, $e(T(z), T(w)) < \varepsilon + D(A, B)$. It follows that $e(T(z), T(X)) < \varepsilon + D(A, B)$. Since $z \in X$ is an arbitrary point, then $e(T(Y), T(X)) < \varepsilon + D(A, B)$. Also, $e(Y, X) \le H(X, Y) < \delta + \varepsilon + D(A, B)$. Therefore, by the same argument as the above we deduce that $e(T(X), T(Y)) < \varepsilon + D(A, B)$. Hence, $H(T(Y), T(X)) < \varepsilon + D(A, B)$. Thus, $H(X, Y) < \varepsilon + \delta + D(A, B)$ implies

$$H(F(X), F(Y)) < \varepsilon + D(A, B), \ X \in \mathcal{K}(A), \ Y \in \mathcal{K}(B).$$

Hence, F is Meir–Keeler contraction with respect to Pompeiu-Hausdorff metric. Therefore, by Theorem 2.3, there exists a unique point $E \in \mathcal{K}(A)$ such that $H(E, F(E)) = D(A,B)$ and $F^2(E) = E$. But T is set-valued cyclic Meir–Keeler contraction. So by Lemma 2.10, there exists a (nondecreasing, continuous) L-function φ such that

(i) $H(T(x), T(y)) \leq d(x,y), \ \forall x \in A$ and $y \in B$.
(ii) $H(T(x), T(y)) \leq \varphi(d(x,y) - D(A,B)) + D(A,B), \ \forall x \in A$ and $y \in B$.

Let $x_0 \in E$. If $d(x_0, y) = D(A,B)$ for some $y \in T(x_0)$, then $d(x_0, T(x_0)) = D(A,B)$. Therefore, x_0 is a best proximity point in A. Suppose that $x_1 \in T(x_0)$ and $d(x_0, x_1) > D(A,B)$, then by Proposition 2.9

$$H(T(x_1), T(x_0)) < \varphi(d(x_1, x_0) - D(A,B)) + D(A,B).$$

Since $d(x_1, T(x_1) \leq H(T(x_0), T(x_1))$ and $T(x_1)$ is compact, there exists $x_2 \in T(x_1)$ such that

$$d(x_2, x_1) \leq H(T(x_1), T(x_0)) < \varphi(d(x_1, x_0) - D(A,B)) + D(A,B).$$

If $d(x_2, x_1) = D(A,B)$, then

$$D(A,B) \leq d(x_2, T(x_2)) \leq H(T(x_1), T(x_2)) \leq d(x_1, x_2) = D(A,B).$$

Hence, x_2 is a best proximity point in A. Otherwise, by the same argument as the above, there exists $x_3 \in T(x_2)$ such that

$$d(x_3, x_2) \leq H(T(x_2), T(x_1)) < \varphi(d(x_2, x_1) - D(A,B)) + D(A,B).$$

By continuing in this way, either T has a best proximity point in A or there is a sequence $\{x_n\}$ in $E \cup T(E)$ such that $x_{n+1} \in T(x_n)$, $x_{2n} \in E$, $x_{2n+1} \in T(E)$ and

$$d(x_{n+1}, x_n) \leq H(T(x_n), T(x_{n-1})) < \varphi(d(x_n, x_{n-1}) - D(A,B)) + D(A,B), \tag{2.2}$$

for each $n \in \mathbb{N}$. Define a sequence $\{s_n\}$ in $(0,\infty)$ by $s_n = d(x_n, x_{n+1}) - D(A,B)$. Then, by inequality (2.2), $\{s_n\}$ is nonincreasing sequence and $s_{n+1} < \varphi(s_n)$. Therefore, from Lemma 2.8, we have $\lim_n s_n = 0$. Hence,

$$\lim_n d(x_n, x_{n+1}) = D(A,B). \tag{2.3}$$

On the other hand E is compact and $x_{2n} \in E$, then there exists a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ such that

$$\lim_k x_{2n_k} = x \in E. \tag{2.4}$$

Since

$$D(A,B) \leq d(x, x_{2n_k-1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}),$$

then from (2.3) and (2.4) we deduce

$$\lim_k d(x, x_{2n_k-1}) = D(A,B). \tag{2.5}$$

Since

$$D(A,B) \leq d(x_{2n_k}, T(x)) \leq H(T(x_{2n_k-1}), T(x)) < d(x_{2n_k-1}, x) \ \ \forall k \in \mathbb{N},$$

then from (2.4) and (2.5), we have $d(x,T(x)) = D(A,B)$. Therefore, T has a best proximity point in A. Let $y \in T(x)$ and $d(x,y) = D(A,B)$, then

$$D(A,B) \leq d(y,T(y)) \leq H(T(x),T(y)) \leq d(x,y) = D(A,B).$$

Therefore, $d(y,T(y)) = D(A,B)$. Hence, y is a best proximity point of T in B. Since $T(y) \subset T^2(x)$, then $D(A,B) \leq d(y,T^2(x)) \leq d(y,T(y)) = D(A,B)$ and but $d(y,T^2(x)) = D(A,B)$. Since $T^2(x)$ is compact, there exists $z \in T^2(x)$ such that $d(z,y) = D(A,B)$. As the pair (A,B) satisfies the property UC, then $d(x,z) = 0$. Hence, $x = z \in T^2(x)$, i.e., x is a fixed point of T^2. □

As a consequence of the above theorem, the following fixed point result can be obtained.

Corollary 2.12. *Let (M,d) be a metric space and let A and B be nonempty subsets of M such that A is complete and $A \cap B \neq \emptyset$. Assume that $T : A \cup B \multimap A \cup B$ is a set-valued cyclic Meir–Keeler contraction such that $T(X)$ is compact for any $X \in \mathcal{K}(A) \cup \mathcal{K}(B)$. Then, T has a fixed point in $A \cap B$*

Proof. Since $A \cap B \neq \emptyset$, then $D(A,B) = 0$ and so the pair (A,B) satisfies the property UC. Therefore, by the above theorem T has a best proximity point x in A. Hence, $d(x,T(x)) = D(A,B) = 0$. Therefore, x is a fixed point of T. Thus, $x \in T(x) \subset B$ and so $x \in A \cap B$. □

Notice that the hypothesis of Theorem 2.11 does not guarantee the uniqueness of the best proximity point. The following example justifies our claim.

Example 2.13. Let $M = \mathbb{R}$ with Euclidean metric, $A = [0,1]$ and $B = [\frac{1}{3}, 2]$. Assume that $T(x) = \{\frac{x+1}{3}, \frac{1}{3}\}$, for each $x \in A \cup B$. Then

$$H(T(x),T(y)) = \frac{1}{3}|x-y| < \frac{1}{2}|x-y|$$

for each $x, y \in A \cup B$. Therefore, T is a cyclic continuous map so it is cyclic Meir–Keeler contraction. Moreover, $x = \frac{1}{3}$ and $x = \frac{1}{2}$ are best proximity points of T in A.

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