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GENERAL ALGORITHM FOR EQUILIBRIUM PROBLEMS AND SET-VALUED OPERATORS

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Abstract. In this paper we introduce and study a general algorithm to approximate a common element of the set of solutions of a system of equilibrium problems and the set of common fixed points of an infinite family of quasi-nonexpansive set-valued mappings. We prove strong convergence of such algorithm in a real Hilbert space. This common solution is the unique solution of a variational inequality problem and is the optimality condition for a minimization problem. Our results improve and extend many related results in the literature.
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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. *Let* Υ *be a bifunction from* $C \times C$ *into* $\mathbb{R}$, *such that* $\Upsilon(x,x) = 0$ *for all* $x \in C$. *The Equilibrium problem for* $\Upsilon : C \times C \to \mathbb{R}$ *is to find* $x \in C$ *such that*

$$\Upsilon(x,y) \geq 0, \qquad \forall y \in C.$$

The set of solutions is denoted by $EP(\Upsilon)$. This problem is also often called the Ky Fan inequality due to his contribution to this field. Such problems arise frequently in mathematics, physics, engineering, game theory, transportation, electricity market, economics and network. In the literature, many techniques and algorithms have been proposed to analyze the existence and approximation of a solution to equilibrium problem; see [1-3].

If $\Upsilon(x,y) = \langle Fx, y - x\rangle$ for every $x, y \in C$, where F is a mapping from C into $\mathcal{H}$, then the equilibrium problem becomes the classical variational inequality problem which is formulated as finding a point $x^* \in C$ such that

$$\langle Fx^*, y - x^*\rangle \geq 0, \qquad \forall y \in C.$$

The set of solutions of this problem is denoted by $VI(F,C)$. It is well known that variational inequalities covers many branches of mathematics; such as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance, see [23, 19].

A subset $C \subset \mathcal{H}$ is called proximal if for each $x \in \mathcal{H}$, there exists an element $y \in C$ such that

$$\| x - y \| = dist(x, C) = inf\{\| x - z \| : z \in C\}.$$

We denote by $CB(C), K(C)$ and $P(C)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of C, respectively. The Hausdorff metric $\mathfrak{h}$ on $CB(\mathcal{H})$ is defined by

$$\mathfrak{h}(A, B) := \max\{\sup_{x \in A} dist(x, B), \sup_{y \in B} dist(y, A)\},$$

for all $A, B \in CB(\mathcal{H})$.
Let $T : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping. An element $x \in \mathcal{H}$ is said to be a fixed point of T, if $x \in Tx$. We use $Fix(T)$ to denote the set of all fixed points of T. An element $x \in \mathcal{H}$ is said to be an endpoint of a set-valued mapping T if x is a fixed point of T and $T(x) = \{x\}$. We say that T satisfies the endpoint condition if each fixed point of T is an endpoint of T. We also say that a family of set- valued mappings $T_i, (i \in \mathbb{N})$ satisfies the common endpoint condition if $T_i(x) = \{x\}$ for all $x \in \bigcap_{i=1}^{\infty} Fix(T_i)$.

Definition 1.1. A set-valued mapping $T : \mathcal{H} \to CB(\mathcal{H})$ is called

(i) nonexpansive if

$$\mathfrak{h}(Tx, Ty) \leq \|x - y\|, \quad x, y \in \mathcal{H}.$$

(ii) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $\mathfrak{h}(Tx, Tp) \leq \|x - p\|$ for all $x \in \mathcal{H}$ and all $p \in Fix(T)$.

The theory of set-valued mappings has applications in control theory, convex optimization, differential equations and economics. Fixed point theory for set-valued mappings has been studied by many authors, see [6-8] and the references therein.

In the recent years iterative algorithms for finding a common element of the set of solutions of equilibrium problem and the set of fixed points of nonlinear mappings in a real Hilbert space have been studied by many authors (see, e.g., [9-24]). The motivation for studying such a problem is in its possible application to mathematical models whose constraints can be expressed as fixed-point problems and/or equilibrium problem. This happens, in particular, in the practical problems as signal processing, network resource allocation, image recovery; see, for instance, [25-28]. In 2007, Takahashi and Takahashi [38], introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping in the setting of Hilbert spaces. They also studied the strong convergence of the sequences generated by their algorithm for a solution of the equilibrium problem which is also a fixed point of a nonexpansive mapping defined on a closed convex subset of a Hilbert space. Motivated by fixed point techniques of Takahashi and Takahashi in [38] and an improvement set of extragradient-type iteration methods in [24], Anh [3, 4], introduce some new iteration algorithms for finding a common element of the solution set of equilibrium problems with a monotone and Lipschitz-type continuous bifunction and the set of fixed points of a single valued nonexpansive mapping.

The purpose of this paper is to propose a general algorithm for finding a common element of the set of solutions of a system of equilibrium problems and the set of common fixed points of an infinite family of quasi-nonexpansive set-valued mappings. We prove the strong convergence theorem of such algorithm in a real Hilbert space. This common solution is the unique solution of a variational inequality problem and is the optimality condition for a minimization problem. Our results generalize and improve the results of Anh, Kim and Muu [5], Anh [3, 4], Plubtieg and Punpaeng [33], Petrusel and Yao [32] and many others.

2. Preliminaries

Throughout the paper, we denote by $\mathcal{H}$ a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ and $x \in \mathcal{H}$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$, and strong convergence by $x_n \to x$. Let C be a nonempty closed convex subset of $\mathcal{H}$. The nearest point projection from $\mathcal{H}$ to C, denoted P_C, assigns, to each $x \in \mathcal{H}$, the unique point $P_C x \in C$ with the property

$$\|x - P_C x\| := inf\{\|x - y\|, \quad \forall y \in C\}.$$

It is known that P_C is a nonexpansive mapping and for each $x \in \mathcal{H}$

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \qquad \forall y \in C.$$

Recently, J.Garcia-Falset, E. Llorens-Fuster and T. Suzuki [18], introduced a new generalization of the concept of a nonexpansive single valued mapping which called condition (E). Very recently, Abkar and Eslamian [1, 2], modify the condition (E), for set-valued mappings as follows:

Definition 2.1. A set-valued mapping $T : \mathcal{H} \to CB(\mathcal{H})$ is said to satisfy condition (E) provided that

$$\mathfrak{h}(Tx, Ty) \leq \mu \, dist(x, Tx) + \|x - y\|, \quad x, y \in \mathcal{H},$$

for some $\mu > 0$.

It is obvious that every set- valued nonexpansive mapping $T : \mathcal{H} \to CB(\mathcal{H})$ satisfies the condition (E), and every mapping $T : \mathcal{H} \to CB(\mathcal{H})$ which satisfies the condition (E) with nonempty fixed point set $Fix(T)$ is quasi-nonexpansive, see [2].

Lemma 2.2. ([2]) *Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$. Let $T : C \to CB(C)$ be a quasi-nonexpansive set-valued mapping satisfies the endpoint condition. Then $Fix(T)$ is closed and convex.*

Definition 2.3. Let C be a nonempty subset of a real Hilbert space $\mathcal{H}$ and let T :C$\to$ $CB(C)$ be a set-valued mapping. The mapping $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x^*$ and $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$, imply $x^* \in Fix(T)$.

Lemma 2.4. ([2]) *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $T : C \to K(C)$ be a set-valued mapping satisfying the condition (E). Then $I - T$ is demiclosed in zero.*

Definition 2.5. A bounded linear operator A on $\mathcal{H}$ is called strongly positive if there exists $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \qquad (x \in \mathcal{H}).$$

For a nonexpansive mapping T from a nonempty subset C of $\mathcal{H}$ into itself a typical problem is to minimize the quadratic function

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

over the set of all fixed points $Fix(T)$ of T (see [28]).

Lemma 2.6. ([28]) *Let A be a strongly positive linear bounded self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

For solving the equilibrium problem, we assume that the bifunction $\Phi : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $\Phi(x, x) = 0$ for all $x \in C$,
(A2) Φ is monotone, i.e., $\Phi(x, y) + \Phi(y, x) \leq 0$, for any $x, y \in C$,
(A3) Φ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} \Phi(tz + (1-t)x, y) \leq \Phi(x, y),$$

(A4) $\Phi(x, .)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 2.7. ([6]) *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let Φ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in \mathcal{H}$. Then, there exists $z \in C$ such that*

$$\Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \qquad \forall y \in C.$$

Lemma 2.8. ([17]) *Assume that $\Phi : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$ and $x \in \mathcal{H}$, define a mapping $S_r : \mathcal{H} \rightrightarrows C$ as follows:*

$$S_r x = \{z \in C : \Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}.$$

Then, the following holds:

(i) *S_r is single valued;*
(ii) *S_r is firmly nonexpansive, i.e., for any $x, y \in \mathcal{H}$,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle;$$

(iii) $Fix(S_r) = EP(\Phi)$;
(iv) *$EP(\Phi)$ is closed and convex.*

We also consider the following conditions for the bifunction $\Psi : C \times C \to \mathbb{R}$:

(B1) $\Psi(x, x) = 0$ for all $x \in C$,
(B2) Ψ is pseudomonotone, i.e., $\Psi(x, y) \geq 0 \Rightarrow \Psi(y, x) \leq 0,\ \forall x, y \in C$
(B3) Ψ is Lipschitz-type continuous, i.e., there exist constants $c_1 > 0$ and $c_2 > 0$ such that $\Psi(x, y)+\Psi(y, z) \geq \Psi(x, z)-c_1\|x-y\|^2-c_2\|y-z\|^2$, for all $x, y, z \in C$,
(B4) $\Psi(x, .)$ be convex and subdifferentiable on C.

Lemma 2.9. ([4]) *Let C be a nonempty closed convex subset of a real Hilbert spaces $\mathcal{H}$ and let Ψ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(B1)-(B4)$. Let $\{x_n\}, \{z_n\}$, and $\{w_n\}$ be sequences generated by $x_0 \in C$ and by*

$$\begin{cases} w_n = argmin\{\lambda_n \Psi(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : \quad w \in C\}, \\ z_n = argmin\{\lambda_n \Psi(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : \quad z \in C\}. \end{cases}$$

Let $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, where $L = \max\{2c_1, 2c_2\}$, then for each $x^\star \in EP(\Psi)$,

$$\|z_n - x^\star\|^2 \leq \|x_n - x^\star\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2, \quad \forall n \geq 0.$$

Lemma 2.10. ([40]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \eta_n)a_n + \eta_n\delta_n, \qquad n \geq 0,$$

where $\{\eta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\displaystyle\sum_{n=1}^{\infty} \eta_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\eta_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11. ([26]) *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$:*

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \qquad t_n \leq t_{\tau(n)+1}.$$

In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

Lemma 2.12. ([10]) *Let $\mathcal{H}$ be a Hilbert space and $\{x_n\}$ be a bounded sequence in $\mathcal{H}$. Then for any given $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ with $\displaystyle\sum_{n=1}^{\infty} \lambda_n = 1$ and for any positive integer i, j with $i < j$,*

$$\|\sum_{n=1}^{\infty} \lambda_n x_n\|^2 \leq \sum_{n=1}^{\infty} \lambda_n\|x_n\|^2 - \lambda_i\lambda_j\|x_i - x_j\|^2.$$

3. Algorithm and its convergence analysis

In this section we combine viscosity approximation method with subgradient algorithm to present a general algorithm for approximating the common element of the set of solutions of a system of Ky Fan inequalities and the set of common fixed points of an infinite family of quasi-nonexpansive set-valued mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and let $\Psi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(B1) - (B4)$. Let $T_i : C \to CB(C)$, $(i \in \mathbb{N})$ be a sequence of* *quasi-nonexpansive set-valued mappings such that $I - T_i$ are demiclosed at 0, and T_i satisfies the common endpoint condition. Assume that*

$$\mathcal{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \cap EP(\Phi) \cap EP(\Psi) \neq \emptyset.$$

Suppose f is a contraction of C into itself with constant $b \in (0,1)$ and A is a strongly positive bounded linear self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma} < 1$ and $0 < \gamma < \frac{\overline{\gamma}}{b}$. Let $\{x_n\}$ be sequence generated by $x_0 \in C$ and

$$\begin{cases} u_n = S_{r_n} x_n, \\ w_n = argmin\{\lambda_n \Psi(u_n, w) + \frac{1}{2}\|w - u_n\|^2 : \quad w \in C\}, \\ \nu_n = argmin\{\lambda_n \Psi(w_n, u) + \frac{1}{2}\|u - u_n\|^2 : \quad u \in C\}, \\ y_n = \gamma_{n,0}\, \nu_n + \displaystyle\sum_{i=1}^{\infty} \gamma_{n,i}\, z_{n,i}, \\ x_{n+1} = a_n \gamma f(y_n) + (I - a_n A) y_n, \qquad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $z_{n,i} \in T_i \nu_n$. Let the sequences $\{a_n\}, \{r_n\}, \{\lambda_n\}$ and $\{\gamma_{n,i}\}$ satisfy the following conditions:

(i) $\{a_n\} \subset (0,1)$, $\lim_{n\to\infty} a_n = 0$, $\displaystyle\sum_{n=1}^{\infty} a_n = \infty$,

(ii) $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$,

(iii) $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iv) $\displaystyle\sum_{i=0}^{\infty} \gamma_{n,i} = 1$, $\liminf_{n\to\infty} \gamma_{n,0}\gamma_{n,i} > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \mathcal{F}$ which solves the variational inequality:

$$\langle (A - \gamma f)x^\star, x - x^\star \rangle \geq 0, \qquad \forall x \in \mathcal{F}. \tag{3.2}$$

Proof. First we note that $P_{\mathcal{F}}(I - A + \gamma f)$ is a contraction of C into itself. By the Banach contraction principle there exists a unique element $x^\star \in C$ such that $x^\star = P_{\mathcal{F}}(I - A + \gamma f)x^\star$. Next we show that $\{x_n\}$ is bounded. From firmly nonexpansivity of the mapping S_{r_n} (Lemma 2.8) and using $x^\star = S_{r_n} x^\star$, we obtain that

$$\|u_n - x^\star\|^2 = \|S_{r_n} x_n - S_{r_n} x^\star\| \leq \|x_n - x^\star\|^2 - \|u_n - x_n\|^2. \tag{3.3}$$

Utilizing Lemma 2.9, we have

$$\|\nu_n - x^\star\|^2 \leq \|u_n - x^\star\|^2 - (1 - 2\lambda_n c_1)\|u_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - \nu_n\|^2. \tag{3.4}$$

Since T_i are quasi-nonexpansive and satisfies the common endpoint condition, from the convexity of $\|.\|^2$ we have

$$\begin{aligned}
\|y_n - x^\star\|^2 \quad &= \|\gamma_{n,0}\, \nu_n + \sum_{i=1}^{\infty} \gamma_{n,i}\, z_{n,i} - x^\star\|^2 \\
&\leq \gamma_{n,0}\|\nu_n - x^\star\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i}\|z_{n,i} - x^\star\|^2 \\
&= \gamma_{n,0}\|\nu_n - x^\star\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i}\, dist(z_{n,i}, T_i x^\star)^2 \\
&\leq \gamma_{n,0}\|\nu_n - x^\star\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i}\, \mathfrak{h}(T_i\nu_n, T_i x^\star)^2 \\
&\leq \gamma_{n,0}\|\nu_n - x^\star\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i}\|\nu_n - x^\star\|^2 \\
&= \|\nu_n - x^\star\|^2
\end{aligned} \tag{3.5}$$

Using inequalities (3.3),(3.4) and (3.5) we obtain that

$$\|y_n - x^\star\| \leq \|x_n - x^\star\|.$$

Since $\lim_{n\to\infty} a_n = 0$, we can assume that $a_n \in (0, \|A\|^{-1})$ for all $n \geq 0$. By Lemma 2.6 we have $\|I - a_n A\| \leq 1 - a_n\overline{\gamma}$. This implies that

$$\begin{aligned}
\|x_{n+1} - x^\star\| &= \|a_n(\gamma f y_n - Ax^\star) + (I - a_n A)(y_n - x^\star)\| \\
&\leq a_n\|\gamma f y_n - Ax^\star\| + \|I - a_n A\|\|y_n - x^\star\| \\
&\leq a_n\gamma\|f y_n - f x^\star\| + a_n\|\gamma f x^\star - Ax^\star\| + (1 - a_n\overline{\gamma})\|y_n - x^\star\| \\
&\leq a_n\gamma b\|y_n - x^\star\| + a_n\|\gamma f x^\star - Ax^\star\| + (1 - a_n\overline{\gamma})\|y_n - x^\star\| \\
&\leq (1 - a_n(\overline{\gamma} - \gamma b))\|x_n - x^\star\| + a_n\|\gamma f x^\star - Ax^\star\|.
\end{aligned}$$

By induction, we have

$$\|x_n - x^\star\| \leq \max\{\|x_0 - x^\star\|, \frac{1}{\overline{\gamma} - \gamma b}\|\gamma f x^\star - Ax^\star\|\},$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and we also obtain that $\{u_n\}, \{y_n\}$ and $\{f(y_n)\}$ are bounded. Next, we show that

$$\lim_{n\to\infty} dist(\nu_n, T_i\nu_n) = 0.$$

Indeed, by using Lemma 2.12 and inequalities (3.3) and (3.4) we have

$$\begin{aligned}
\|y_n - x^\star\|^2 &= \|\gamma_{n,0}\, \nu_n + \sum_{i=1}^{\infty} \gamma_{n,i}\, z_{n,i} - x^\star\|^2 \\
&\le \gamma_{n,0}\|\nu_n - x^\star\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i}\|z_{n,i} - x^\star\|^2 \\
&- \sum_{i=1}^{\infty} \gamma_{n,0}\gamma_{n,i}\|\nu_n - z_{n,i}\|^2 \\
&= \gamma_{n,0}\|\nu_n - x^\star\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i}\, dist(z_{n,i}, T_i x^\star)^2 \\
&- \sum_{i=1}^{\infty} \gamma_{n,0}\gamma_{n,i}\|\nu_n - z_{n,i}\|^2 \\
&\le \gamma_{n,0}\|\nu_n - x^\star\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i}\, \mathfrak{h}(T_i\nu_n, T_i x^\star)^2 \\
&- \sum_{i=1}^{\infty} \gamma_{n,0}\gamma_{n,i}\|\nu_n - z_{n,i}\|^2 \\
&\le \gamma_{n,0}\|\nu_n - x^\star\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i}\|\nu_n - x^\star\|^2 \\
&- \sum_{i=1}^{\infty} \gamma_{n,0}\gamma_{n,i}\|\nu_n - z_{n,i}\|^2 \\
&= \|\nu_n - x^\star\|^2 - \sum_{i=1}^{\infty} \gamma_{n,0}\gamma_{n,i}\|\nu_n - z_{n,i}\|^2 \\
&\le \|x_n - x^\star\|^2 - \|u_n - x_n\|^2 - \sum_{i=1}^{\infty} \gamma_{n,0}\gamma_{n,i}\|\nu_n - z_{n,i}\|^2 \\
&- (1 - 2\lambda_n\, c_1)\|u_n - w_n\|^2 - (1 - 2\lambda_n\, c_2)\|w_n - \nu_n\|^2.
\end{aligned} \tag{3.6}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|a_n(\gamma f y_n - Ax^\star) + (I - a_n A)(y_n - x^\star)\|^2 \\
&\le a_n^2\|\gamma f y_n - Ax^\star\|^2 + (1 - a_n\overline{\gamma})^2\|y_n - x^\star\|^2 \\
&+ 2a_n(1 - a_n\overline{\gamma})\|\gamma f y_n - Ax^\star\|\|y_n - x^\star\| \\
&\le a_n^2\|\gamma f y_n - Ax^\star\|^2 + (1 - a_n\overline{\gamma})^2\|x_n - x^\star\|^2 \\
&+ 2a_n(1 - a_n\overline{\gamma})\|\gamma f y_n - Ax^\star\|\|x_n - x^\star\| \\
&- (1 - a_n\overline{\gamma})^2 \sum_{i=1}^{\infty} \gamma_{n,0}\gamma_{n,i}\|\nu_n - z_{n,i}\|^2 - (1 - a_n\overline{\gamma})^2\|u_n - x_n\|^2 \\
&- (1 - a_n\overline{\gamma})^2(1 - 2\lambda_n\, c_1)\|u_n - w_n\|^2 \\
&- (1 - a_n\overline{\gamma})^2(1 - 2\lambda_n\, c_2)\|w_n - \nu_n\|^2.
\end{aligned} \tag{3.7}$$

So, we have

$$\begin{aligned}\sum_{i=1}^{\infty}(1-a_n\overline{\gamma})^2\gamma_{n,0}\gamma_{n,i}\|\nu_n-z_{n,i}\|^2 \quad &\leq \|x_n-x^\star\|^2-\|x_{n+1}-x^\star\|^2\\ &+2a_n(1-a_n\overline{\gamma})\|\gamma f y_n-Ax^\star\|\|x_n-x^\star\|\\ &+a_n^2\|\gamma f y_n-Ax^\star\|^2.\end{aligned} \tag{3.8}$$

From inequality (3.7) we have also the following inequality:

$$\begin{aligned}(1-a_n\overline{\gamma})^2\|x_n-u_n\|^2 \quad &\leq \|x_n-x^\star\|^2-\|x_{n+1}-x^\star\|^2\\ &+2a_n(1-a_n\overline{\gamma})\|\gamma f y_n-Ax^\star\|\|x_n-x^\star\|\\ &+a_n^2\|\gamma f y_n-Ax^\star\|^2.\end{aligned} \tag{3.9}$$

In order to prove that $x_n \to x^\star$ as $n \to \infty$, we consider two possible cases.
Case 1. Assume that $\{\|x_n - x^\star\|\}$ is a monotone sequence. Since $\|x_n - x^\star\|$ is bounded we have $\|x_n - x^\star\|$ is convergent. Since $\lim_{n\to\infty} a_n = 0$ and $\{f(y_n)\}$ and $\{x_n\}$ are bounded, we have

$$\lim_{n\to\infty}(1-a_n\overline{\gamma})^2\gamma_{n,0}\gamma_{n,i}\|\nu_n-z_{n,i}\|^2=0.$$

From $\lim_{n\to\infty} a_n = 0$, we can assume that for some $c \in (0,1)$, $0 < c < (1-a_n\overline{\gamma})^2$. By assumption that $\liminf_n \gamma_{n,0}\gamma_{n,i} > 0$, we get that

$$\lim_{n\to\infty}\|\nu_n-z_{n,i}\|=0. \tag{3.10}$$

Using similar argument, from inequalities (3.7) and (3.9) we obtain that

$$\lim_{n\to\infty}\|u_n-w_n\|=\lim_{n\to\infty}\|w_n-\nu_n\|=\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.11}$$

This implies that

$$\lim_{n\to\infty}\|u_n-\nu_n\|=0. \tag{3.12}$$

By our assumption that $z_{n,i} \in T_i\nu_n$, we have

$$\lim_{n\to\infty} dist(\nu_n,T_i\nu_n)\leq \lim_{n\to\infty}\|\nu_n-z_{n,i}\|=0. \tag{3.13}$$

Next, we show that $\lim sup_{n\to\infty}\langle (A-\gamma f)x^\star, x^\star-x_n\rangle \leq 0$, where $x^\star = P_{\mathcal{F}}(I-A+\gamma f)x^\star$ is a unique solution of the variational inequality (3.2). We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}(\langle A-\gamma f)x^\star,x^\star-x_{n_i}\rangle=\limsup_{n\to\infty}(\langle A-\gamma f)x^\star,x^\star-x_n\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to x^*. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup x^*$. Since $\lim_{n\to\infty}\|x_n - u_n\| = 0$, we have $u_{n_i} \rightharpoonup x^*$. We show that $x^* \in \mathcal{F}$. For proving this, first we show that $x^* \in EP(\Psi)$. Since $\Psi(x,.)$ is convex on C for each $x \in C$, we see that

$$w_n = argmin\{\lambda_n\,\Psi(u_n,y)+\frac{1}{2}\|y-u_n\|^2: \quad y\in C\}$$

if and only if

$$0 \in \partial_2(\lambda_n \Psi(u_n, y) + \frac{1}{2}\|y - u_n\|^2)(w_n) + N_C(w_n),$$

where $N_C(w_n)$ is the (outward) normal cone of C at $w_n \in C$. This follows that

$$0 = \lambda_n v + w_n - u_n + z_n$$

where $v \in \partial_2\Psi(u_n, w_n)$ and $z_n \in N_C(w_n)$. By the definition of the normal cone N_C we have

$$\langle w_n - u_n, y - w_n\rangle \geq \lambda_n\langle v, w_n - y\rangle, \qquad \forall y \in C. \tag{3.14}$$

Since $\Psi(u_n, .)$ is subdifferentiable on C, by the well-known Moreau-Rockafellar theorem [29], there exists $v \in \partial_2\Psi(u_n, w_n)$ such that

$$\Psi(u_n, y) - \Psi(u_n, w_n) \geq \langle v, y - w_n\rangle, \quad \forall y \in C.$$

Combining this with (3.14), we have

$$\lambda_n(\Psi(u_n, y) - \Psi(u_n, w_n)) \geq \langle w_n - u_n, w_n - y\rangle, \quad \forall y \in C.$$

Hence

$$\Psi(u_{n_i}, y) - \Psi(u_{n_i}, w_{n_i}) \geq \frac{1}{\lambda_{n_i}}\langle w_{n_i} - u_{n_i}, w_{n_i} - y\rangle, \quad \forall y \in C.$$

Since $\lim_{n\to\infty} \|u_n - w_n\| = 0$, we have that $w_{n_i} \rightharpoonup x^*$. Now by continuity of Ψ and assumption that $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$ we have

$$\Psi(x^*, y) \geq 0, \qquad \forall y \in C.$$

This implies that $x^* \in EP(\Psi)$. Let us show $x^* \in EP(\Phi)$. Since $u_n = S_{r_n}x_n$ we have

$$\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \qquad \forall y \in C.$$

From (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \Phi(y, u_n),$$

therefore

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq \Phi(y, u_{n_i}).$$

Since

$$\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$$

and $u_{n_i} \rightharpoonup x^*$, from (A4) we have

$$\Phi(y, x^*) \leq 0, \qquad \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1 - t)x^*$. Since $y, x^* \in C$, and C is convex we have $y_t \in C$ and hence $\Phi(y_t, x^*) \leq 0$. So, from (A1) and (A4) we have

$$0 = \Phi(y_t, y_t) \leq t\Phi(y_t, y) + (1 - t)\Phi(y_t, x^*) \leq t\Phi(y_t, y),$$

which gives $\Phi(y_t, y) \geq 0$. From (A3) we have $0 \leq \Phi(x^*, y), \forall y \in C$ and hence $x^* \in EP(\Phi)$. From the demiclosedness of $T_i - I$ and using inequality (3.13) we get that $x^* \in \bigcap_{i=1}^{\infty} Fix(T_i)$. Thus $x^* \in \mathcal{F}$. Since $x^\star = P_{\mathcal{F}}(I - A + \gamma f)x^\star$ and $x^* \in \mathcal{F}$, it follows that

$$\begin{aligned}\lim sup_{n\to\infty}\langle (A-\gamma f)x^\star, x^\star - x_n\rangle &= \lim_{i\to\infty}\langle (A-\gamma f)x^\star, x^\star - x_{n_i}\rangle \\ &= \langle (A-\gamma f)x^\star, x^\star - x^*\rangle \le 0.\end{aligned}$$

We note that in every Hilbert space $\mathcal{H}$

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \qquad \forall x, y \in \mathcal{H}.$$

From this and (3.1) we have

$$\begin{aligned}\|x_{n+1}-x^\star\|^2 &\le \|(I-a_nA)(y_n-x^\star)\|^2 + 2a_n\langle \gamma f y_n - Ax^\star, x_{n+1}-x^\star\rangle \\ &\le (1-a_n\overline{\gamma})^2\|y_n-x^\star\|^2 + 2a_n\gamma\langle fy_n - fx^\star, x_{n+1}-x^\star\rangle \\ &+ 2a_n\langle \gamma f x^\star - Ax^\star, x_{n+1}-x^\star\rangle \\ &\le (1-a_n\overline{\gamma})^2\|x_n-x^\star\|^2 + 2a_nb\gamma\|x_n-x^\star\|\|x_{n+1}-x^\star\| \\ &+ 2a_n\langle \gamma f x^\star - Ax^\star, x_{n+1}-x^\star\rangle \\ &\le (1-a_n\overline{\gamma})^2\|x_n-x^\star\|^2 + a_nb\gamma(\|x_n-x^\star\|^2 + \|x_{n+1}-x^\star\|^2) \\ &+ 2a_n\langle \gamma f x^\star - Ax^\star, x_{n+1}-x^\star\rangle \\ &= ((1-a_n\overline{\gamma})^2 + a_nb\gamma)\|x_n-x^\star\|^2 + a_n\gamma b\|x_{n+1}-x^\star\|^2 \\ &+ 2a_n\langle \gamma f x^\star - Ax^\star, x_{n+1}-x^\star\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-x^\star\|^2 &\le \tfrac{1-2a_n\overline{\gamma}+(a_n\overline{\gamma})^2+a_n\gamma b}{1-a_n\gamma b}\|x_n-x^\star\|^2 + \tfrac{2a_n}{1-a_n\gamma b}\langle \gamma f x^\star - Ax^\star, x_{n+1}-x^\star\rangle \\ &= (1-\tfrac{2(\overline{\gamma}-\gamma b)a_n}{1-a_n\gamma b})\|x_n-x^\star\|^2 + \tfrac{(a_n\overline{\gamma})^2}{1-a_n\gamma b}\|x_n-x^\star\|^2 \\ &+ \tfrac{2a_n}{1-a_n\gamma b}\langle \gamma f x^\star - Ax^\star, x_{n+1}-x^\star\rangle \\ &\le (1-\tfrac{2(\overline{\gamma}-\gamma b)a_n}{1-a_n\gamma b})\|x_n-x^\star\|^2 \\ &+ \tfrac{2(\overline{\gamma}-\gamma b)a_n}{1-a_n\gamma b}(\tfrac{(a_n\overline{\gamma}^2)M}{2(\overline{\gamma}-\gamma b)} + \tfrac{1}{\overline{\gamma}-\gamma b})\langle \gamma f x^\star - Ax^\star, x_{n+1}-x^\star\rangle \\ &= (1-\sigma_n)\|x_n-x^\star\|^2 + \sigma_n\eta_n,\end{aligned}$$

where $M = sup\{\|x_n-x^\star\|^2 : n \ge 0\}$, $\sigma_n = \frac{2(\overline{\gamma}-\gamma b)a_n}{1-a_n\gamma b}$ and

$$\eta_n = \frac{(a_n\overline{\gamma}^2)M}{2(\overline{\gamma}-\gamma b)} + \frac{1}{\overline{\gamma}-\gamma b}\langle \gamma f x^\star - Ax^\star, x_{n+1}-x^\star\rangle.$$

It is easy to see that $\sigma_n \to 0, \displaystyle\sum_{n=1}^{\infty}\sigma_n = \infty$ and $\limsup_{n\to\infty}\eta_n \le 0$. Hence, by Lemma 2.10 the sequence $\{x_n\}$ converges strongly to $x^\star$.

Case 2. Assume that $\{\|x_n - x^\star\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N}; k \le n : \|x_k - x^\star\| < \|x_{k+1} - x^\star\|\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$,

$$\|x_{\tau(n)} - x^\star\| < \|x_{\tau(n)+1} - x^\star\|.$$

From (7) we obtain $\lim_{n\to\infty} dist(\nu_{\tau(n)}, T_i\nu_{\tau(n)}) = 0$, and $\lim_{n\to\infty} \|u_{\tau(n)} - x_{\tau(n)}\| = 0$. Following an argument similar to that in Case 1 we have

$$\|x_{\tau(n)+1} - x^\star\|^2 \leq (1 - \sigma_{\tau(n)})\|x_{\tau(n)} - x^\star\|^2 + \sigma_{\tau(n)}\eta_{\tau(n)},$$

where $\sigma_{\tau(n)} \to 0, \sum_{n=1}^{\infty} \sigma_{\tau(n)} = \infty$ and $\limsup_{n\to\infty} \eta_{\tau(n)} \leq 0$. Hence, by Lemma 2.10, we obtain $\lim_{n\to\infty} \|x_{\tau(n)} - x^\star\| = 0$ and $\lim_{n\to\infty} \|x_{\tau(n)+1} - x^\star\| = 0$. Thus by Lemma 2.11 we have

$$0 \leq \|x_n - x^\star\| \leq \max\{\|x_{\tau(n)} - x^\star\|, \|x_n - x^\star\|\} \leq \|x_{\tau(n)+1} - x^\star\|.$$

Therefore $\{x_n\}$ converges strongly to $x^\star = P_{\mathcal{F}}(I - A + \gamma f)x^\star$, which complete the proof.

Now, we intent to remove the common endpoint condition. For this work, let $T : C \to P(C)$ be a set- valued mapping, we consider

$$P_T(x) = \{y \in Tx : \|x - y\| = dist(x, Tx)\}, \quad x \in C.$$

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $T_i : C \to CB(C)$, $(i \in \mathbb{N})$ be a sequence of set- valued mappings such that P_{T_i} are quasi-nonexpansive and $I - P_{T_i}$ are demiclosed in 0. Let Φ, Ψ, f, A and $\mathcal{F}$ be as in Theorem 3.1. Let $\{x_n\}$ be sequence generated by $x_0 \in C$ and by*

$$\begin{cases} u_n = S_{r_n} x_n, \\ w_n = argmin\{\lambda_n \Psi(u_n, w) + \frac{1}{2}\|w - u_n\|^2 : \quad w \in C\}, \\ \nu_n = argmin\{\lambda_n \Psi(w_n, u) + \frac{1}{2}\|u - u_n\|^2 : \quad u \in C\}, \\ y_n = \gamma_{n,0}\, \nu_n + \sum_{i=1}^{\infty} \gamma_{n,i}\, z_{n,i}, \\ x_{n+1} = a_n \gamma f(y_n) + (I - a_n A) y_n, \ \forall n \geq 0, \end{cases} \tag{3.15}$$

where $z_{n,i} \in P_{T_i}\nu_n$. Let the sequences $\{a_n\}, \{r_n\}, \{\lambda_n\}$ and $\{\gamma_{n,i}\}$ satisfy the conditions of Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \mathcal{F}$ which solves the variational inequality (3.2).

Proof. Let $p \in \mathcal{F}$, then $P_{T_i}(p) = \{p\}, (i \in \mathbb{N})$. Now by substituting P_{T_i} instead of T_i, and similar argument as in the proof of Theorem 3.1, the desired result holds.

Remark 3.3. Theorem 3.1 and Theorem 3.2 generalize the result of Anh [3] from a single valued nonexpansive mapping to an infinite family of quasi-nonexpansive set-valued mappings. We also weaken or remove some control conditions on parameters.

4. Application

In this section, we consider the particular equilibrium problem corresponding to the function Ψ defined, for every $x, y \in C$ by $\Psi(x, y) = \langle F(x), y-x\rangle$, with $F : C \to \mathcal{H}$. Doing so, we obtain the classical variational inequality:

$$Find\, z \in C\, such\, that \quad \langle F(z), y-z\rangle \geq 0, \qquad \forall y \in C.$$

The set of solutions of this problem is denoted by $VI(F, C)$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and let $F : C \to \mathcal{H}$ be a monotone and L- Lipschitz continuous operator on C. Let $T_i : C \to K(C)$, $(i \in \mathbb{N})$ be a sequence of set-valued mappings satisfying the condition (E) and the common endpoint condition. Assume that $\mathcal{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \cap EP(\Phi) \cap VI(F, C) \neq \emptyset$. Suppose f is a contraction of C into itself with constant $b \in (0, 1)$ and A is a strongly positive bounded linear self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma} < 1$ and $0 < \gamma < \frac{\overline{\gamma}}{b}$. Let $\{x_n\}$ be sequence generated by $x_0 \in C$ and*

$$\begin{cases} u_n = S_{r_n} x_n, \\ w_n = P_C(u_n - \lambda_n F(u_n)), \\ \nu_n = P_C(u_n - \lambda_n F(w_n)), \\ y_n = \gamma_{n,0}\, \nu_n + \displaystyle\sum_{i=1}^{\infty} \gamma_{n,i}\, z_{n,i}, \\ x_{n+1} = a_n \gamma f(y_n) + (I - a_n A) y_n, \qquad \forall n \geq 0, \end{cases} \tag{4.1}$$

where $z_{n,i} \in T_i \nu_n$. Let the sequences $\{a_n\}, \{r_n\}, \{\lambda_n\}$ and $\{\gamma_{n,i}\}$ satisfy the conditions of Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \mathcal{F}$ which solves the variational inequality (3.2).

Proof. Putting $\Psi(x, y) = \langle F(x), y - x\rangle$, we have that

$$w_n = argmin\{\lambda_n\, \Psi(u_n, w) + \frac{1}{2}\|w - u_n\|^2 : \quad w \in C\} = P_C(u_n - \lambda_n F(u_n)).$$

Since F is a L-Lipshchitz continuous on C, we have

$$\Psi(x, y) + \Psi(y, z) - \Psi(x, z) = \langle F(x) - F(y), y - z\rangle, \qquad x, y, z \in C.$$

Therefore

$$|\langle F(x) - F(y), y - z\rangle| \leq L\|x - y\|\|y - z\| \leq \frac{L}{2}(\|x - y\|^2 + \|y - z\|^2),$$

hence Ψ satisfies Lischiptz-type continuous condition with $c_1 = c_2 = \frac{L}{2}$. Since T_i satisfying the condition (E), we have T_i are quasi-nonexpansive. From Lemma 2.5 we have that $T_i - I$ are demiclosed at 0. Now, applying Theorem 3.1, we obtain the desired result.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let F be a function from C to $\mathcal{H}$ such that F is monotone and L-Lipschitz continuous on C. Let $T_i : C \to K(C)$, $(i \in \mathbb{N})$ be a sequence of set-valued mappings satisfying* *the condition* (E) *and the common endpoint condition. Assume that*

$$\mathcal{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \cap VI(F,C) \neq \emptyset.$$

Suppose f *is a contraction of* C *into itself with constant* $b \in (0,1)$. *Let* $\{x_n\}$ *be sequence generated by* $x_0 \in C$ *and*

$$\begin{cases} u_n = P_C(x_n - \lambda_n F(x_n)), \\ \nu_n = P_C(x_n - \lambda_n F(u_n)), \\ y_n = \gamma_{n,0}\, \nu_n + \displaystyle\sum_{i=1}^{\infty} \gamma_{n,i}\, z_{n,i}, \\ x_{n+1} = a_n f(y_n) + (1-a_n) y_n, \qquad \forall n \geq 0, \end{cases} \tag{4.2}$$

where $z_{n,i} \in T_i \nu_n$. *Let the sequences* $\{a_n\}, \{\lambda_n\}$ *and* $\{\gamma_{n,i}\}$ *satisfy the conditions of Theorem 3.1. Then, the sequence* $\{x_n\}$ *converges strongly to* $x^\star \in \mathcal{F}$ *which solves the variational inequality:*

$$\langle (I-f)x^\star, x - x^\star \rangle \geq 0, \qquad \forall x \in \mathcal{F}.$$

Proof. Putting $\Phi = 0$, $A = I$ and $\gamma = 1$ in Theorem 3.1 we obtain the desired result.

As an application of our main result we have the following strong convergence theorem for an infinite family of set valued mappings, which is new, even in the case of single valued mappings.

Theorem 4.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$ *and Let* $T_i : C \to CB(C)$, $(i \in \mathbb{N})$ *be a sequence of quasi-nonexpansive set-valued mappings such that* $I - T_i$ *are demiclosed at* 0, *and* T_i *satisfies the common endpoint condition. Assume that* $\mathcal{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. *Suppose* f *is a contraction of* C *into itself with constant* $b \in (0,1)$ *and* A *is a strongly positive bounded linear self-adjoint operator on* $\mathcal{H}$ *with coefficient* $\overline{\gamma} < 1$ *and* $0 < \gamma < \frac{\overline{\gamma}}{b}$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} y_n = \gamma_{n,0}\, x_n + \displaystyle\sum_{i=1}^{\infty} \gamma_{n,i}\, z_{n,i}, \\ x_{n+1} = P_C(a_n \gamma f(y_n) + (I - a_n A) y_n), \qquad \forall n \geq 0, \end{cases} \tag{4.3}$$

where $z_{n,i} \in T_i x_n$. *Let the sequences* $\{a_n\}$ *and* $\{\gamma_{n,i}\}$ *satisfy the conditions of Theorem 3.1. Then, the sequence* $\{x_n\}$ *converges strongly to* $x^\star \in \mathcal{F}$ *which solves the variational inequality (3.2).*

Remark 4.4. Theorem 4.1 and Theorem 4.2 generalize the results of Anh [3], Anh, Kim and Muu [5] and Petrusel and Yao [32] from a single valued nonexpansive mapping to an infinite family of set-valued mappings satisfying the condition (E). We also weaken or remove some control conditions on parameters.

Remark 4.5. Theorem 4.3 generalize and improve the result of Dhompongsa, Inthakon and Takahashi, [12]. Indeed, in [12], the authors presented an iterative process to obtain a weak convergence theorem for a generalized nonexpansive single valued mapping and a nonspreading mapping in Hilbert spaces (we note these class of mappings are quasi-nonexpansive). But in this paper we obtain a strong convergence theorem for an infinite family of set-valued quasi-nonexpansive mappings.

4.1. **Numerical example.** Now, we supply an example to illustrate the main result of this paper.

Example 4.6. We consider the nonempty closed convex subset $C = [0, 2]$ of the Hilbert space $\mathbb{R}$. Define a family of mappings T_i as follows:

$$T_1(x) = [\frac{x}{3}, x], \qquad T_2(x) = \begin{cases} [0, \frac{x}{5}], & x \neq 2 \\ [1, \frac{3}{2}] & x = 2, \end{cases} \qquad T_i(x) = [0, \frac{i-2}{i-1}x], \quad i = 3, 4, 5, \ldots$$

It is easy to see that T_1 is nonexpansive, T_2 satisfy the condition (E) and T_i, $i = 3, 4, 5, \ldots$ are nonexpansive. We define a bifunction Φ as follows:

$$\begin{cases} \Phi : C \times C \to \mathbb{R} \\ \Phi(x, y) = y^2 + xy - 2x^2. \end{cases}$$

It is easy to see that Φ satisfies the conditions $(A1) - (A4)$. If we put $r_n = 1$, then $u_n = S_{r_n} x_n = \frac{x_n}{3r_n+1} = \frac{x_n}{4}$, (for details, see [36]).

Put $\gamma_{n,0} = \gamma_{n,1} = \gamma_{n,2} = \frac{1}{5}, \gamma_{n,i} = \frac{12}{5\pi^2}\frac{1}{(i-2)^2}, (i = 3, 4, 5, \ldots)$ and $a_n = \frac{1}{n}$. (We note that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$). Then these sequences satisfy conditions of Theorem 3.1. We put $f(x) = \frac{x}{2}, \gamma = 1, \mathcal{A} = I$ and $F = 0$. Taking $x_0 = \frac{3}{2}$ and $z_{n,1} = x, z_{n,2} = \frac{x}{5}$ and $z_{n,i} = \frac{i-2}{i-1}x, (i = 3, 4, 5, \ldots)$, we have the following algorithm:

$$\begin{cases} u_n = S_{r_n} x_n = \frac{x_n}{3r_n+1} = \frac{x_n}{4}, \\ y_n = \frac{u_n}{5} + \frac{u_n}{5} + \frac{u_n}{25} + \frac{12}{5\pi^2} u_n = \frac{11\pi^2+60}{100\pi^2} x_n, \\ x_{n+1} = \frac{y_n}{2n} + \frac{n-1}{n} y_n = (\frac{2n-1}{2n})(\frac{11\pi^2+60}{100\pi^2}) x_n \end{cases}$$

We observe that x_n is convergent to zero. We note that

$$\mathcal{F} = \bigcap_{i=1}^{\infty} Fix(T_i) \cap EP(\Phi) = \{0\}.$$

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