Fixed Point Theory, 19(2018), No. 1, 185-192
DOI 10.24193/fpt-ro.2018.1.14
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SOLUTION OF A FUNCTIONAL INTEGRAL INCLUSION IN BANACH SPACE

AHMED M.A. EL-SAYED* AND NESREEN F.M. EL-HADDAD**

*Faculty of Science, Alexandria University, Alexandria, Egypt
E-mail: amasayed@alexu.edu.eg

**Faculty of Science, Damanhour University, Behera, Egypt
E-mail: nesreen_fawzy20@yahoo.com

Abstract. In this paper we study the existence of weak solutions $x \in C[I, E]$ for the nonlinear functional integral inclusion

$$x(t) \in a(t) + \int_0^t F(t, s, g(s, x(m(s))))ds, \;\; t \in I = [0, T]$$

where E is reflexive Banach space and the set-valued function F satisfy Caratheodory condition.
Key Words and Phrases: Set-valued functions, weak solutions, functional integral inclusion, fixed point, Caratheodory condition.
2010 Mathematics Subject Classification: 46Txx, 47Hxx, 39Bxx, 47H30, 47H10.

1. Introduction

Let $I = [0, T]$ and let $L^1(I)$ be the class of all Lebesgue integrable functions defined on the interval I. Let E be a reflexive Banach space with norm $\|.\|$ and dual E^*. Denote $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.

Consider the functional integral inclusion

$$x(t) \in a(t) + \int_0^t F(t, s, g(s, x(m(s))))ds \tag{1.1}$$

where $F : I \times I \times E \to P(E)$ is a nonlinear set-valued mapping, and $P(E)$ denote the family of nonempty subsets of the Banach space E.

Here we study the existence of weak solutions $x \in C[I, E]$ of the functional integral inclusion (1.1) in the reflexive Banach space E.

In the past few years, several papers have been devoted to the study of integral equations by different authors under different conditions on the kernel (see for instance [2]-[4] and [8]-[9] and their references). However very few results are available for integral inclusions (see [1], [6] and [10]).

The fundamental tools used in the proofs of all above mentioned works are essentially fixed point arguments or iterative methods.

In this paper, using a fixed point theorem, we shall be concerned with the existence of weak solutions $x \in C[I, E]$ of the functional integral inclusion (1.1), under the assumption that the set-valued function F satisfy Caratheodory condition.

2. Preliminaries

In this section we introduce definitions and some auxiliary results which are used throughout this paper.
let E be a Banach space and let $x : I \to E$. Then
(1) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0 .
(2) A function $h : E \to E$ is said to be sequentially continuous if h maps weakly convergent sequence in E to weakly convergent sequence in E .

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see [5] and [7]). Note that in reflexive Banach spaces weakly measurable functions are Pettis integrable (see [7] and [12]) if and only if $\phi(x(.))$ is Lebesgue integrable on I for every $\phi \in E^*$.

Now we state a fixed point theorem and some propositions which will be used in the sequal (see [11]).
Theorem 2.1. (O'Regan fixed point theorem) *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space $(C[0,T], E)$ and let $A : Q \to Q$ be a weakly sequentially continuous and assume that $AQ(t)$ is relatively weakly compact in E for each $t \in [0,T]$. Then A has a fixed point in the set Q.*
Proposition 2.2. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*
Proposition 2.3. *Let E be a normed space with $y \neq 0$. Then there exists a $\phi \in E^*$ with $\|\phi\| = 1$ and $\|y\| = \phi(y)$.*
Definition 2.4. (L^1-Caratheodory) The multivalued map $F : I \times E \to P(E)$ is L^1-Caratheodory if the following conditions hold:
(i) $t \to F(t, y)$ is measurable for each $y \in E$,
(ii) $y \to F(t, y)$ is upper semicontinuous for almost all $t \in I$.
(iii) For each $r > 0$, there exists $h_r \in L^1(I, E)$ such that
$\|F(t, y)\| = sup\{\|f\| : \ f \in F(t, y)\} \leq h_r(t)$, for all $\|y\| \leq r$, and for almost all $t \in I$.
Definition 2.5. Let $S^1_{F(.,x(.))} = \{f \in L^1(I, E) : f(t) \in F(t, x(t)) \ for \ a.e. \ t \in I\}$ denote the set of selections of F that belongs to $L^1[I, E]$.

3. Main results

Consider now the functional integral inclusion (1.1) under the following assumptions
(1) $t \to F(t, s, y)$ is measurable for each $y \in E$.
(2) $y \to F(t, s, y)$ is upper semicontinuous for almost all $(t, s) \in I \times I$.
(3) For each $r_1 > 0$, there exists an integrable function $k(t, s)$, $(t, s) \in I \times I$ such that

$$\|F(t, s, y)\| = sup\{\|f\| : \ f \in F(t, s, y)\} \leq |k(t, s)|$$

for all $\|y\| \leq r_1$ and for almost all $(t,s) \in I \times I$.
(4) The set of all Caratheodory selections $S^1_{F(t,.,x(.))}$ is nonempty.
(5) $g(.,x)$ is weakly measurable on I for every $x \in E$.
(6) g satisfies the weakly Lipschitz condition

$$\phi(g(t,x) - g(t,y)) \leq L\phi(x-y)$$

for every $(t,x), (t,y) \in I \times E$.
(7) $m : [0,T] \to [0,T],\ m(t) \leq t$ is continuous function.
(8) $a : [0,T] \to [0,T]$ is continuous function.
(9) $\int_0^t |k(t,s)|ds \leq K,\ \forall t \in I$.

Example 3.1. Let $\overline{S} = \{x \in E : \|x\| \leq 1\}$ and $I = [0,T]$. Consider a multifunction $F : I \times I \times \overline{S} \to P(E)$ defined by

$$F(t,s,x) = k(t,s)\overline{S},\ t,s \in I = [0,T].$$

Then F is L^1-Caratheodory. In fact, for the norm in the Banach space we have

$$\|F(t,s,x)\| = \sup\{\|f\| :\ f \in F(t,s,x)\} \leq |k(t,s)|$$

for almost all $(t,s) \in I \times I$.

Remark 3.2. From the assumptions (1)-(4), there exists $f \in S_F$ such that

$$\|f(t,s,y)\| \leq |k(t,s)|$$

with

$$x(t) = a(t) + \int_0^t f(t,s,g(s,x(m(s)))ds,\ t \in [0,T] \tag{3.1}$$

and then the solution of the functional integral equation (3.1), if it exists, is a solution of the functional integral inclusion (1.1).

Now, let

$$y(t) = g(t,x(m(t))),\ t \in [0,T], \tag{3.2}$$

then from (3.1) we have

$$x(t) = a(t) + \int_0^t f(t,s,y(s))ds,\ t \in [0,T] \tag{3.3}$$

and the functional integral equation (3.1) is equivalent to the coupled system (3.2) and (3.3).
Consider now the coupled system (3.2) and (3.3).

Definition 3.3. By a weak solution of the coupled system (3.2) and (3.3) we mean the ordered pair of functions $(x,y),\ x,y \in C[I,E]$ such that

$$\phi(x(t)) = a(t) + \int_0^t \phi(f(t,s,y(s)))ds,\ t \in [0,T]$$
$$\phi(y(t)) = \phi(g(s,x(m(s)))),\ t \in [0,T]$$

for all $\phi \in E^*$.

For the existence of weak solutions $x,y \in C[I,E]$ of the coupled system (3.2) and (3.3) we have the following theorem

Theorem 3.4. *Let the assumptions (1)-(9) be satisfied. Then the coupled system (3.2) and (3.3) has a weak solution $x,y \in C[I,E]$.*

Proof. Let X be the class of all ordered pair $U = (x, y),\ x, y \in C[I, E]$, with norm

$$\|(x, y)\| = \|x\| + \|y\|$$

Let

$$\begin{aligned} U(t) &= (x(t), y(t)) \\ &= (a(t) + \int_0^t f(t, s, y(s))ds,\ g(t, x(m(t)))),\ t \in [0, T]. \end{aligned}$$

Let A be any operator defined by

$$AU(t) = A(x(t), y(t)) = (A_1 y(t), A_2 x(t))$$

where

$$A_1 y(t) = a(t) + \int_0^t f(t, s, y(s))ds,\ t \in [0, T]$$

and

$$A_2 x(t) = g(t, x(m(t))),\ t \in [0, T].$$

Define the set Q_r by

$$Q_r = \{U = (x, y) \in X : x, y \in C[I, E],\ \|y\| \le r_1,\ \|x\| \le r_2,\ r = r_1 + r_2\}.$$

Let $U = (x, y) \in Q_r$ be an arbitrary ordered pair, then from proposition 2.3 we have

$$\begin{aligned} \|A_1 y(t)\| &= \phi(A_1 y(t)) \\ &= \phi(a(t)) + \int_0^t \phi(f(t, s, y(s)))ds \\ &= \|a\| + \int_0^t \|f(t, s, y(s))\|ds \\ &\le \|a\| + \int_0^t |k(t, s)|ds \\ &\le \|a\| + K. \end{aligned}$$

Therefore

$$\|A_1 y(t)\| \le \|a\| + K = r_1$$

and

$$\begin{aligned} \|A_2 x(t)\| &= \phi(A_2 x(t)) \\ &= \phi(g(t, x(m(t)))) \\ &= L\phi(x) + sup\ \phi(g(t, 0)) \\ &\le L\|x\| + M, \end{aligned}$$

where $M = sup\ \phi(g(t, 0))$.
Then

$$\|A_2 x(t)\| \le L r_2 + M = r_2,\ where\ r_2 = \frac{M}{1 - L}.$$

Now

$$\begin{aligned} \|AU(t)\| &= \|A_1y(t)\| + \|A_2x(t)\| \\ &\leq \|a\| + K + Lr_2 + M \\ &= r. \end{aligned}$$

Then

$$\|AU\| \leq r.$$

Hence, $AU \in Q_r$, which proves that $AQ_r \subset Q_r$, i.e. $A : Q_r \to Q_r$, and the class of functions $\{AQ_r\}$ is uniformly bounded.
Now Q_r is nonempty, closed, convex and uniformly bounded.
As a consequence of proposition 2.2, then $\{AQ_r\}$ is relatively weakly compact.

Now, we shall prove that $A : X \to X$.
Let $t_1, t_2 \in I,\ t_1 < t_2$ (without loss of generality assume that $\|AU(t_2) - AU(t_1)\| \neq 0$), then

$$\begin{aligned} A_1y(t_2) - A_1y(t_1) &= (a(t_2) - a(t_1)) + \int_0^{t_2} f(t_2, s, y(s))ds - \int_0^{t_1} f(t_1, s, y(s))ds \\ &= (a(t_2) - a(t_1)) + \int_0^{t_1} f(t_2, s, y(s))ds + \int_{t_1}^{t_2} f(t_2, s, y(s))ds \\ &- \int_0^{t_1} f(t_1, s, y(s))ds \\ &= (a(t_2) - a(t_1)) + \int_0^{t_1} (f(t_2, s, y(s)) - f(t_1, s, y(s)))ds \\ &+ \int_{t_1}^{t_2} f(t_2, s, y(s))ds. \end{aligned}$$

Therefore as a consequence of proposition 2.3, we obtain

$$\begin{aligned} \|A_1y(t_2) - A_1y(t_1)\| &= \phi(A_1y(t_2) - A_1y(t_1)) \\ &= \phi(a(t_2) - a(t_1)) + \int_0^{t_1} \phi(f(t_2, s, y(s)) - f(t_1, s, y(s)))ds \\ &+ \int_{t_1}^{t_2} \phi(f(t_2, s, y(s)))ds \\ &= \|a(t_2) - a(t_1)\| + \int_0^{t_1} \|f(t_2, s, y(s)) - f(t_1, s, y(s))\|ds \\ &+ \int_{t_1}^{t_2} \|f(t_2, s, y(s))\|ds \\ &\leq \|a(t_2) - a(t_1)\| + \int_0^{t_1} \|f(t_2, s, y(s)) - f(t_1, s, y(s))\|ds \\ &+ \int_{t_1}^{t_2} |k(t, s)|ds \end{aligned}$$

and

$$\begin{aligned}
\|A_2x(t_2) - A_2x(t_1)\| &= \phi(A_2x(t_2) - A_2x(t_1)) \\
&= \phi(g(t_2, x(m(t_2))) - g(t_1, x(m(t_1)))) \\
&\leq \phi(g(t_2, x(m(t_2))) - g(t_2, x(m(t_1)))) \\
&+ \phi(g(t_2, x(m(t_1))) - g(t_1, x(m(t_1)))) \\
&\leq L\phi(x(m(t_2)) - x(m(t_1))) \\
&+ \phi(g(t_2, x(m(t_1))) - g(t_1, x(m(t_1)))).
\end{aligned}$$

Then

$$\begin{aligned}
\|AU(t_2) - AU(t_1)\| &= \|(A_1y(t_2), A_2x(t_2)) - (A_1y(t_1), A_2x(t_1))\| \\
&= \|((A_1y(t_2) - A_1y(t_1)), (A_2x(t_2) - A_2x(t_1)))\| \\
&= \|A_1y(t_2) - A_1y(t_1)\| + \|A_2x(t_2) - A_2x(t_1)\|.
\end{aligned}$$

This proves that $A : X \to X$.

Finally, we prove that A is weakly sequentially continuous.
Let $\{U_n\}$ be a sequence in Q_r converges weakly to U $\forall t \in I$, then we have the two sequences $\{y_n\}$, $\{x_n\}$, such that $\{y_n\}$ converges strongly to y and $\{x_n\}$ converges weakly to x, i.e. $y_n \to y$, $x_n \rightharpoonup x$, $\forall t \in I$.
Since $g(t, x(m(t)))$ is weakly Lipschitz in x, i.e. weakly continuous in x, then $g(t, x_n(m(t)))$ converges weakly to $g(t, x(m(t)))$.
Thus $\phi(g(t, x_n(m(t))))$ converges strongly to $\phi(g(t, x(m(t))))$, and $\phi(f(t, s, y_n(s)))$ converges strongly to $\phi(f(t, s, y(s)))$.
Also,

$$\|f(t, s, y)\| \leq |k(t, s)|.$$

Applying Lebesgue dominated convergence theorem for Pettis integral, then we obtain

$$\begin{aligned}
\phi(A_1y_n(t)) &= \phi(a(t) + \int_0^t f(t, s, y_n(s))) \\
&= \phi(a(t)) + \int_0^t \phi(f(t, s, y_n(s))) \\
&\to \phi(a(t)) + \int_0^t \phi(f(t, s, y(s))) \\
&\to \|a(t)\| + \int_0^t \|f(t, s, y(s))\|, \ \forall\ \phi \in E^*, \ t \in I
\end{aligned}$$

i.e. $\phi(A_1y_n(t)) \to \phi(A_1y(t))$, and then

$$\|A_1y_n(t)\| \to \|A_1y(t)\|.$$

Also

$$\begin{aligned}
\phi(A_2x_n(t)) &= \phi(g(t, x_n(m(t)))) \\
&\to \phi(g(t, x(m(t)))) \\
&\to \|g(t, x(m(t)))\|, \ \forall\ \phi \in E^*, \ t \in I
\end{aligned}$$

i.e. $\phi(A_2x_n(t)) \to \phi(A_2x(t))$, and then

$$\|A_2x_n(t)\| \to \|A_2x(t)\|.$$

Therefore,

$$\begin{aligned}\|AU_n(t)\| &= \|A(x_n(t), y_n(t))\| \\ &= \|(A_1y_n(t), A_2x_n(t))\| \\ &= \|A_1y_n(t)\| + \|A_2x_n(t)\| \\ &\to \|A_1y(t)\| + \|A_2x(t)\| \\ &\to \|(A_1y(t), A_2x(t))\| \\ &\to \|AU(t)\|\end{aligned}$$

Hence, A is weakly sequentially continuous (i.e. $AU_n(t) \to AU(t)$, $\forall t \in I$ weakly). Since all conditions of O'Regan theorem are satisfied, then the operator A has at least one fixed point $U \in Q_r$ and then the coupled system (3.2) and (3.3) has at least one weak solution $(x, y) \in X$, then there exists at least one weak solution $x \in C[I, E]$ of the functional integral equation (3.2).

Consequently, there exists at least one weak solution $x \in C[I, E]$ of the functional integral inclusion (1.1).

Acknowledgement. The author is grateful to the editor in chief and the referee for there valuable comments and suggestions which improve this article.

References

[1] A.I. Bulgakov, L.N. Lyapin, *Some properties of the set of solutions for the Volterre-Hammerstein integral inclusion*, Differential Eq. (transl.), **14**(1978), 1043-1048.

[2] A. Constantin, *Monotone iterative technique for a nonlinear integral equation*, J. Math. Anal. Appl., **205**(1997), 280-283.

[3] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, New York, 1991.

[4] C. Corduneanu, *Perturbations of linear abstract volterra equations*, J. Internat. Eq. Appl., **2**(1990), 393-401.

[5] N. Dunford, J.T. Schwartz, *Linear Operators*, Interscience, Wiley, New York, 1958.

[6] G. Gripenberg, S.O. Londen, O. Stiffans, *Volterra Integral and Functional Equations*, Cambridge Univ. Press, New York, 1990.

[7] E. Hille, R.S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. 31, Amer. Math. Soc., Providence, R.I., 1957.

[8] D. O'Regan, *Existence results for nonlinear integral equations*, J. Math. Anal. Appl., **192**(1995), 705-726.

[9] D. O'Regan, *Existence theory for nonlinear Volterra and Hammerstein integral equations*, Dynamical Systems and Applications (R.P. Agarwal, ed.), World Scientific Series in Applicable Analysis, **4**(1995), 601-615.

[10] D. O'Regan, *Integral inclusions of upper semicontinuous or lower semicontionuous type*, Proceedings of the Amer. Math. Soc., **8**(1996), 2391-2399.

[11] D. O'Regan, *Fixed point theory for wealy sequentially continuous mapping*, Math. Comput. Modeling, **27**(1998), 1-14.

[12] B.J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc., **44**(1938), 277-304.

Received: November 19, 2015 ; Accepted: April 10, 2017.