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ANSWER TO KIRK-SHAHZAD'S QUESTION ON ZHANG-JIANG'S FIXED POINT THEOREM

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Abstract. In this note we give a positive answer to Kirk-Shahzad's question on Zhang-Jiang's fixed point theorem [8, Question on page 18].
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In 2014 Kirk and Shahzad [8, Chapter 2] surveyed the relation between well-known results that are Ekeland's variational principle [5], [6] and Caristi's fixed point theorem [3, Theorem (2.1)'], [4, Theorem 1]. Recall that a function $f : X \longrightarrow \mathbb{R}$ is called *lower semi-continuous* at x_0 if $\liminf_{x\to x_0} f(x) \geq f(x_0)$ and is called *upper semi-continuous* at x_0 if $\limsup_{x\to x_0} f(x) \leq f(x_0)$; f is called *lower semi-continuous* (*upper semi-continuous*) if f is lower semi-continuous (upper semi-continuous) at every $x \in X$, respectively.

Theorem 1.1. ([5]) *Let (X, d) be a complete metric space, $\varphi : X \longrightarrow [0,\infty)$ be lower semi-continuous and $\preceq$ be a partial order on X defined as follows: for all $x, y \in X$,*

$$x \preceq y \text{ if and only if } d(x, y) \leq \varphi(x) - \varphi(y).$$

Then $(X, \preceq)$ has a maximal element.

Theorem 1.2. ([4], Theorem 1) *Let (X, d) be a complete metric space, the function $\varphi : X \longrightarrow [0,\infty)$ be lower semi-continuous and the map $f : X \longrightarrow X$ be such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \text{ for all } x \in X.$$

Then f has a fixed point.

Many authors in nonlinear analysis claimed that Theorem 1.1 and Theorem 1.2 are equivalent, see [8, pages 7-8]. However to a logician these two results are not equivalent since the proof Theorem 1.2 implying Theorem 1.1 invokes the Axiom of Choice [8, page 8]. In fact, Brunner [2] has shown that any proof of Theorem 1.1 requires at least the basic axioms of Zermemo-Fraenkel plus a form of the Axiom of Choice called the Axiom of Dependent Choice, whereas Manka [10] has shown that Theorem 1.2 holds within Zermemo-Fraenkel. So from a purely logical point of view the two theorems are not equivalent. Recall that the Axiom of Dependent Choice is strictly weaker than the Axiom of Choice but strictly stronger than the Axiom of Countable Choice [8, page 8].

In 1976 Brézis and Browder [1, Theorem 1] derived Theorem 1.1 from an order principle which requires only Zermemo-Fraenkel and the Axiom of Choice. They then derived Theorem 1.2 from Theorem 1.1. However in 1990 Goebel and Kirk [7, Proof of Caristi's theorem on page 13] have shown that Theorem 1.2 can be derived directly from Brézis-Browder order principle without recourse to Theorem 1.1.

Recall that there were many generalizations of Theorem 1.2 in the literature. At the same time, many of them except for Zhang-Jiang's fixed point theorem [11, Theorem 2.1] turn out to be consequences of Theorem 1.2, see [8, Chapter 2].

Theorem 1.3. ([1], Theorem 1) *Let $(X, \preceq)$ be a partially ordered set, the function $\psi : X \longrightarrow \mathbb{R}$ and $S(x) = \{y \in X : x \preceq y\}$ for each $x \in X$ satisfying the following.*

(1) *$x \preceq y$ and $x \neq y$ implies $\psi(x) < \psi(y)$.*
(2) *For any increasing sequence $\{x_n\}$ in X such that $\psi(x_n) \leq C < \infty$ for all n there exists some $y \in X$ such that $x_n \preceq y$ for all n.*
(3) *For each $x \in X$, $\psi(S(x))$ is bounded above.*

Then for each $x \in X$ there exists $x^* \in S(x)$ such that x^* is a maximal element of $(X, \preceq)$, that is, $S(x^*) = \{x^*\}$.

Definition 1.4. ([11], page 524) Let Γ denote the collection of all functions $\gamma : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following.

(1) γ is sub-additive, that is, $\gamma(s + t) \leq \gamma(s) + \gamma(t)$ for all $s, t \in [0, \infty)$.
(2) γ is increasing and continuous.
(3) $\gamma^{-1}(0) = 0$.

Let $\mathcal{A}$ denote the collection of all functions $\eta : [0, \infty) \longrightarrow [0, \infty)$ satisfying there exist $\varepsilon_0 > 0$ and $\gamma \in \Gamma$ such that if $\eta(t) \leq \varepsilon_0$ then $\eta(t) \geq \gamma(t)$. Let $\mathcal{F}$ denote the collection of all functions $F : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following.

(1) F is increasing and upper semi-continuous.
(2) $F(0) = 0$ and $F^{-1}([0, \infty)) \subset [0, \infty)$.
(3) $F(t) + F(s) \leq F(t + s)$ for all $s, t \in [0, \infty)$.

Theorem 1.5. ([11], Theorem 2.1) *Let (X, d) be a complete metric space, $\varphi : X \longrightarrow \mathbb{R}$ be lower semi-continuous and bounded below, and $f : X \longrightarrow X$ be a map such that*

$$\eta(d(x, f(x))) \leq F(\varphi(x) - \varphi(f(x))) \tag{1.1}$$

for some $\eta \in \mathcal{A}$, some $F \in \mathcal{F}$ and all $x \in X$. Then f has a fixed point.

In 2001 Kirk and Saliga [9] weakened the notion of a lower semi-continuous function to the notion of a *lower semi-continuous from above* function and given a straightforward modification of Theorem 1.2 as follows.

Definition 1.6. ([9], page 2767) A function $\varphi : X \longrightarrow \mathbb{R}$ is called *lower semi-continuous from above* if for each $x \in X$, $\lim_{n\to\infty} x_n = x$, $\{\varphi(x_n)\}$ is decreasing, and $\lim_{n\to\infty} \varphi(x_n) = r$ implies $\varphi(x) \leq r$.

Theorem 1.7. ([9], Theorem 2.1) *Let (X, d) be a complete metric space, $\varphi : X \longrightarrow \mathbb{R}$ be lower semi-continuous from above and bounded below, and $f : X \longrightarrow X$ be a map* *such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \text{ for all } x \in X.$$

Then f has a fixed point.

To prove Theorem 1.7 the authors of [9] used Theorem 1.3.

The proof of Theorem 1.7 was represented by Kirk and Shahzad [8, Proof of Theorem 2.3 on page 9]. Kirk and Shahzad also posed the following question on the relation between Theorem 1.3 and Theorem 1.5.

Question 1.8. ([8], Question on page 18) *Is it possible to derive Theorem 1.5 from Theorem 1.3?*

In this paper we show that it is possible to derive Theorem 1.5 from Theorem 1.3. Then the answer to Question 8 is positive. The main result of the paper is as follows.

Theorem 1.9. *Theorem 1.5 is a consequence of Theorem 1.3.*

Proof. Let $\varphi_0 = \inf\{\varphi(x) : x \in X\}$. Since F is upper semi-continuous on $[0, \infty)$, we get

$$\limsup_{t \to 0^+} F(t) \leq F(0) = 0.$$

Then for ε_0 in the definition of $\mathcal{A}$ there exists $\delta > 0$ such that $F(t) < \varepsilon_0$ for $0 \leq t \leq \delta$. Denote

$$X_\delta = \{x \in X : \varphi(x) \leq \varphi_0 + \delta\}.$$

Since $\varphi_0 = \inf\{\varphi(x) : x \in X\}$, we get $X_\delta \neq \emptyset$, and since φ is lower semi-continuous, we get X_δ is closed. Therefore (X_δ, d) is a complete metric space.

We next define a relation $\preceq$ on X_δ as follows: for all $x, y \in X_\delta$,

$$x \preceq y \text{ if and only if } \gamma(d(x, y)) \leq F(\varphi(x) - \varphi(y)).$$

We will prove that $(X_\delta, \preceq)$ is a partially ordered set. Indeed, for all $x, y, z \in X_\delta$, we find that

$$\gamma(d(x, x)) = \gamma(0) = 0 = F(0) = F(\varphi(x) - \varphi(x)).$$

Therefore $\gamma(d(x, x)) = F(\varphi(x) - \varphi(x))$. This proves that $x \preceq x$.

If $x \preceq y$ and $y \preceq x$ then

$$\gamma(d(x, y)) \leq F(\varphi(x) - \varphi(y)) \text{ and } \gamma(d(y, x)) \leq F(\varphi(y) - \varphi(x)).$$

Since $d(x, y) \geq 0$ and γ is increasing, we get

$$\gamma(d(x, y)) \geq \gamma(0) = 0 \text{ and } \gamma(d(y, x)) \geq \gamma(0) = 0.$$

Then

$$F(\varphi(x) - \varphi(y)) \geq 0 \text{ and } F(\varphi(y) - \varphi(x)) \geq 0.$$

Since $F^{-1}[0, \infty) \subset [0, \infty)$, we have $\varphi(x) - \varphi(y) \geq 0$ and $\varphi(y) - \varphi(x) \geq 0$. Then $\varphi(x) = \varphi(y)$. So

$$0 \leq \gamma(d(x, y)) \leq F(\varphi(x) - \varphi(y)) = F(0) = 0.$$

It implies that $d(x, y) = 0$. Then $x = y$.

If $x \preceq y$ and $y \preceq z$ then

$$\gamma(d(x, y)) \leq F(\varphi(x) - \varphi(y)) \text{ and } \gamma(d(y, z)) \leq F(\varphi(y) - \varphi(z)).$$

Since $d(x, y) \geq 0$ and γ is increasing, we get

$$\gamma(d(x, y)) \geq \gamma(0) = 0 \text{ and } \gamma(d(y, z)) \geq \gamma(0) = 0.$$

Then

$$F(\varphi(x) - \varphi(y)) \geq 0 \text{ and } F(\varphi(y) - \varphi(z)) \geq 0.$$

Since $F^{-1}[0, \infty) \subset [0, \infty)$, we have $\varphi(x) - \varphi(y) \geq 0$ and $\varphi(y) - \varphi(z) \geq 0$. From properties of γ and F we find that

$$\begin{aligned}
\gamma(d(x, z)) &\leq \gamma(d(x, y) + d(y, z)) \\
&\leq \gamma(d(x, y)) + \gamma(d(y, z)) \\
&\leq F(\varphi(x) - \varphi(y)) + F(\varphi(y) - \varphi(z)) \\
&\leq F(\varphi(x) - \varphi(y) + \varphi(y) - \varphi(z)) \\
&= F(\varphi(x) - \varphi(z)).
\end{aligned}$$

Therefore $\gamma(d(x, z)) \leq F(\varphi(x) - \varphi(z))$. This proves that $x \preceq z$.

Now for each $x \in X_\delta$, let $S(x) = \{y \in X_\delta : x \preceq y\}$ and let $\psi(x) = -\varphi(x)$ for all $x \in X_\delta$. We will show that show that all assumptions of Theorem 1.3 are satisfied for $(X_\delta, \preceq)$ and ψ.

Indeed, if $x \preceq y$ then $\gamma(d(x, y)) \leq F(\varphi(x) - \varphi(y))$. As the above we find that $\varphi(x) - \varphi(y) \geq 0$. Then $\varphi(x) \geq \varphi(y)$ and thus $\psi(x) \leq \psi(y)$. Moreover, if $x \neq y$ then $\psi(x) < \psi(y)$. We also find that $\psi(S(x))$ is bounded above since φ is bounded below.

Let $\{x_n\}$ be an increasing sequence in $(X_\delta, \preceq)$ and $\psi(x_n) \leq C < \infty$ for all n. Then $\{\psi(x_n)\}$ is an increasing sequence in $\mathbb{R}$. It implies that $\{\varphi(x_n)\}$ is a decreasing sequence in $\mathbb{R}$. Since φ is bounded below, there exists $\lim\limits_{n\to\infty} \varphi(x_n) = r \in \mathbb{R}$. Since $\{x_n\}$ is increasing, for each $n < m$ we have $x_n \preceq x_m$. Then

$$\gamma(d(x_n, x_m)) \leq F(\varphi(x_n) - \varphi(x_m)).$$

Notice that γ is increasing continuous and F is upper semi-continuous. Moreover, if $n < m$ then $\varphi(x_n) - \varphi(x_m) \geq 0$ and $\lim\limits_{n\to\infty}(\varphi(x_n) - \varphi(x_m)) = r - r = 0$. Then for $n < m$ we have

$$\begin{aligned}
0 &\leq \gamma(\limsup_{n\to\infty} d(x_n, x_m)) \\
&= \limsup_{n\to\infty} \gamma(d(x_n, x_m)) \\
&\leq \limsup_{n\to\infty} F(\varphi(x_n) - \varphi(x_m)) \\
&\leq F(0) \\
&= 0.
\end{aligned}$$

Therefore $\limsup\limits_{n\to\infty} d(x_n, x_m) = 0$ for $n < m$. It implies that $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$ and hence $\{x_n\}$ is a Cauchy sequence in (X_δ, d). Since (X_δ, d) is complete, there exists $\lim\limits_{n\to\infty} x_n = y \in X_\delta$. Since φ is lower semi-continuous and $\lim\limits_{n\to\infty} \varphi(x_n) = r$, we have

$\varphi(y) \leq r$. So we have

$$\begin{aligned}\gamma(d(x_n, y)) &= \gamma(\lim_{m\to\infty} d(x_n, x_m)) \\ &= \lim_{m\to\infty} \gamma(d(x_n, x_m)) \\ &\leq \lim_{m\to\infty} F(\varphi(x_n) - \varphi(x_m)) \\ &\leq F(\varphi(x_n) - r) \\ &\leq F(\varphi(x_n) - \varphi(y)).\end{aligned}$$

Therefore $\gamma(d(x_n, y)) \leq F(\varphi(x_n) - \varphi(y))$. This proves that $x_n \preceq y$ for all n.

The above arguments show that all assumptions of Theorem 1.3 are satisfied for $(X_\delta, \preceq)$ and ψ. So $(X_\delta, \preceq)$ has a maximal element x^*.

We will prove that x^* is a fixed point of f. Indeed, by (1.1) we have

$$\eta(d(x^*, f(x^*))) \leq F(\varphi(x^*) - \varphi(f(x^*))). \tag{1.2}$$

Since $0 \leq \eta(d(x^*, f(x^*)))$, $0 \leq F(\varphi(x^*) - \varphi(f(x^*)))$. Since $F^{-1}([0, \infty)) \subset [0, \infty)$, we get $\varphi(x^*) - \varphi(f(x^*)) \geq 0$, that is,

$$\varphi(f(x^*)) \leq \varphi(x^*). \tag{1.3}$$

Since $x^* \in X_\delta$, we get

$$\varphi(x^*) \leq \varphi_0 + \delta. \tag{1.4}$$

By definition of φ_0 and (1.3), (1.4) we have $\varphi_0 \leq \varphi(f(x^*)) \leq \varphi(x^*) \leq \varphi_0 + \delta$. This implies that $f(x^*) \in X_\delta$ and $0 \leq \varphi(x^*) - \varphi(f(x^*)) \leq \delta$. By the selection of δ we get

$$F(\varphi(x^*) - \varphi(f(x^*))) \leq \varepsilon_0. \tag{1.5}$$

From (1.2) and (1.5) we obtain $\eta(d(x^*, f(x^*))) \leq \varepsilon_0$. Since $\eta \in \mathcal{A}$, we have

$$\gamma(d(x^*, f(x^*))) \leq \eta(d(x^*, f(x^*))). \tag{1.6}$$

It follows from (1.2) and (1.6) that

$$\gamma(d(x^*, f(x^*))) \leq F(\varphi(x^*) - \varphi(f(x^*))). \tag{1.7}$$

Note that we have shown $f(x^*) \in X_\delta$. So from (1.7) we conclude that $x^* \preceq f(x^*)$. However, x^* is the maximal element of $(X_\delta, \preceq)$, so $f(x^*) = x^*$. Then x^* is a fixed point of f.

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