Fixed Point Theory, 19(2018), No. 1, 167-178
DOI 10.24193/fpt-ro.2018.1.12
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

PROJECTIONS ONTO CONES IN BANACH SPACES

A. DOMOKOS* AND M.M. MARSH**

*Department of Mathematics and Statistics
California State University at Sacramento
6000 J Street, Sacramento, CA, 95819, USA
E-mail: domokos@csus.edu

**Department of Mathematics and Statistics
California State University at Sacramento
6000 J Street, Sacramento, CA, 95819, USA
E-mail: mmarsh@csus.edu

Abstract. We propose to find algebraic characterizations of the metric projections onto closed, convex cones in reflexive, locally uniformly convex Banach spaces with locally uniformly convex dual.

Key Words and Phrases: Banach space, closed convex cone, metric projection.
2010 Mathematics Subject Classification: 41A65, 46C50, 52A27.

1. Introduction and preliminaries

As many optimization problems depend on properties related to closed, convex cones, characterizations of the metric projection mapping onto cones are important. Theorem 1.1 below gives necessary and sufficient algebraic conditions for a mapping to be the metric projection onto a closed, convex cone in a real Hilbert space.

Theorem 1.1 ([7]). *Let H be a Hilbert space, $P : H \to H$ be a continuous function, and $C = \{x \in H \mid P(x) = x\}$. Then C is a closed, convex cone and P is the metric projection onto C if and only if P satisfies the following properties:*

(1) $P^2(x) = P(x)\,,\ \forall\, x \in H.$
(2) $P(\alpha x) = \alpha P(x)\,,\ \forall\, \alpha > 0\,,\ x \in H.$
(3) $P(x + y) = P(x) + P(y)$ *if and only if*

$$\langle P(x), y\rangle = \langle P(x), P(y)\rangle = \langle x, P(y)\rangle\,. \tag{1.1}$$

Theorem 1.1 generalizes earlier results on the algebraic characterization of a metric projection onto a closed subspace as an idempotent, symmetric, and linear operator.

Theorem 1.2 ([10, Theorem 13.5.1]). *Let H be a Hilbert space, $P : H \to H$ be a continuous function, and $C = \{x \in H \mid P(x) = x\}$. Then C is a closed, linear subspace and P is the metric projection onto C if and only if P satisfies the following properties:*

(a) $P^2(x) = P(x)\,,\ \forall\, x \in H.$
(b) P *is linear.*
(c) $\langle P(x), y\rangle = \langle x, P(y)\rangle\,,\ \forall\, x, y \in H\,.$

We will be examining the possibilities of generalizing Theorem 1.1 to Banach spaces. It is clear that this is not an immediately available process, because the proofs in [7] use exclusive properties of real Hilbert spaces, such as bilinearity of the inner product, the Pythagorean theorem, orthogonal decomposition relative to cones, non-expansivity of the metric projection, and the convexity properties of polar cones. In the same way, it would be equally interesting to generalize, from Hilbert to Banach spaces, other results about the relation between metric projections and orders generated by cones, see [12, 13].

In our paper the geometry of Banach spaces plays a key role and we have to assume adequate smoothness and convexity properties of the norm. The existence of a metric projection, and the uniqueness of its point images onto closed convex sets in Banach spaces require at least reflexivity of the Banach space and strict convexity of the norm. The semi-inner product in non-Hilbert Banach spaces is not bilinear and the orthogonality relation is not symmetric, but under the assumptions of reflexivity, local uniform convexity and local uniform smoothness, we will find adequate properties to obtain results. Metric projections onto closed linear subspaces are not linear in general, except for maximal, closed, linear subspaces, so we have to impose some geometrical properties on the closed, convex cones to guarantee that the metric projection satisfies a condition similar to (3) in Theorem 1.1, which is called *face-linear* in [7].

One of the motivations mentioned in [7] for establishing such a characterization was to provide new methods towards the solution of the still open problem on the convexity of Chebyshev sets: Is each subset of a Hilbert space, which admits unique nearest point to each point of the space, necessarily convex? We also have this goal in mind, particularly as it relates to Chebyshev cones. Also, we wish to determine the additional geometric properties necessary for a similar characterization in the more general setting of a Banach space.

We continue with basic definitions and properties related to the geometry of Banach spaces, duality mappings, semi-inner products and metric projections. The results which are well-known will be listed without proofs, since they can be found in a variety of books and articles. See, for example [1, 4, 6, 11, 14, 15, 16].

Hereafter, we assume that X is a real, reflexive, locally uniformly convex Banach space, with locally uniformly convex dual X^*. There could be minor variations regarding the geometric properties of the Banach spaces, but we opted for those above for the following reasons. First, the simplicity of formulation and the fact that every reflexive Banach space can be renormed so that both X and X^* become locally uniformly convex, which is a familiar setting in the theory of perturbations of maximal monotone operators, see [9]. Second, these assumptions imply that the norm of X is also strictly convex and Fréchet differentiable at each $x \neq 0$. Third, they imply that the norm satisfies the Kadec-Klee property, which is essential in proving the continuity of the metric projection onto closed, convex sets [2]. The Kadec-Klee property states as follows: If a sequence $\{x_n\}$ converges weakly to x and $\{||x_n||\}$ converges to $||x||$, then $\{x_n\}$ converges in norm to x.

We will use the usual duality notation $\langle x^*, x\rangle$ for $x^*(x)$, when $x^* \in X^*$ and $x \in X$.

Definition 1.3. In a Banach space X the *duality mapping* $J : X \rightsquigarrow X^*$ is a (possibly set-valued) mapping, defined by

$$J(x) = \{x^* \in X^* \ : \ \langle x^*, x\rangle = ||x||^2 \text{ and } ||x^*|| = ||x||\} .$$

The duality mapping coincides with the subdifferential of the function $f(x) = \frac{1}{2}||x||^2$. If X^* is locally uniformly convex, then X has a Fréchet differentiable norm [3, page 32] and hence the duality mapping is single-valued. On the other hand, the norm of X is Fréchet differentiabile if and only if the duality mapping is norm-to-norm continuous [3, page 30]. We list the most important properties of the duality mapping in the following proposition.

Proposition 1.4 ([9, 17]). *In a reflexive Banach space X, with X and X^* locally uniformly convex, the duality mapping has the following properties:*

1. *Considering the norm topologies on both X and X^*, $J : X \to X^*$ is a homeomorphism.*
2. $\langle J(x), y\rangle = \lim_{t\to 0} \frac{||x+ty||^2 - ||x||^2}{2t}$, $\forall\, x, y \in X$.
3. $J(tx) = tJ(x)$, $\forall\, t \in \mathbb{R}$, $x \in X$.
4. *For every $R > 0$ and $x_0 \in X$, there exists a nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$ such that*
$$\langle J(x) - J(x_0), x - x_0\rangle \geq \phi(||x - x_0||)\, ||x - x_0|| ,$$
for all $x \in X$ with $||x - x_0|| \leq R$.

The following proposition is fundamental for the connection of the duality mapping with the metric projection. For consistency, we continue using our locally uniformly convex assumptions, even though Proposition 1.5 is true under slightly weaker conditions.

Proposition 1.5 ([14]). *Let X be a reflexive Banach space, with X and X^* locally uniformly convex. Let $K \subset X$ be nonempty, closed and convex, and $x_0 \in K$. Then*

$$||x_0|| = \min_{x\in K} ||x|| \text{ if and only if } \langle J(x_0), x - x_0\rangle \geq 0 ,\ \forall\, x \in K .$$

Definition 1.6. Let X be a reflexive Banach space, with X and X^* locally uniformly convex. The *semi-inner product* in X is defined by

$$\langle x, y\rangle_+ = \langle J(x), y\rangle .$$

The following properties of the semi-inner product are clearly implied by Propositions 1.4 and 1.5. We can also refer to [4, 6, 11, 14].

Proposition 1.7. *Let X be a reflexive Banach space, with X and X^* locally uniformly convex.*

1. *The semi-inner product coincides with the inner product in Hilbert spaces.*
2. $\langle tx, y\rangle_+ = t\langle x, y\rangle_+$, $\forall\, t \in \mathbb{R},\ x, y \in X$.
3. $\langle x, \cdot\rangle_+ \in X^*$, $\forall\, x \in X$.
4. *Considering the norm topology of X, $\langle\cdot,\cdot\rangle_+ : X \times X \to \mathbb{R}$ is continuous.*

(5) $|\langle x, y\rangle_+| \leq ||x|| \cdot ||y||, \ \forall\, x, y \in X$.
(6) $\langle x, x\rangle_+ = ||x||^2 , \ \forall\, x \in X$.
(7) $\langle x, x-y\rangle_+ > \langle y, x-y\rangle_+ , \ \forall\, x \neq y \in X$.
(8) $\langle x+y, y\rangle_+ > \langle x, y\rangle_+ , \ \forall\, x, y \in X , \ y \neq 0$.
(9) *If $K \subset X$ is nonempty, closed and convex, and $x_0 \in K$, then*

$$||x_0|| = \min_{x \in K} ||x|| \ \text{ if and only if } \ \langle x_0, x - x_0\rangle_+ \geq 0 , \ \forall\, x \in K .$$

Definition 1.8. Let X be a reflexive Banach space, with X and X^* locally uniformly convex. If $K \subset X$ is a nonempty, closed, convex set, then for all $x \in X$ we define the metric projection of x onto K by $P_K(x) \in K$ and

$$||P_K(x) - x|| = \min_{y \in K} ||y - x|| .$$

Under the assumptions of a reflexive and locally uniformly convex Banach space X, and K a nonempty, closed, convex subset of X, the metric projection $P_K : X \to X$ is a well defined mapping. The following two propositions list the most important properties of the metric projection mapping. For the continuity of the metric projection we refer to [2, Corollary 2.18], while the rest of the properties can be found in [16, Chapters 3-5].

For simplicity, when there is no risk of confusion, we will use P instead of P_K.

Proposition 1.9. *Let X be a reflexive Banach space, with X and X^* locally uniformly convex. Let K be a nonempty, closed, convex subset of X, and let P be the metric projection onto K. Then, we have the following properties.*

(a) *The projection P is continuous.*
(b) $P\Big(P(x) + t\big(x - P(x)\big)\Big) = P(x) , \ \forall\, t \geq 0 , \ x \in X$.
(c) *For any $x \in X$, $P(x) \in K$ is the metric projection of x onto K if and only if*

$$\langle x - P(x), k - P(x)\rangle_+ \leq 0 , \ \forall\, k \in K .$$

Proposition 1.10 ([15, 16]). *If K is a closed, linear subspace of X, the metric projection operator has the following additional properties:*

(1) *For all $x \in X$, we have $\langle x - P(x), k - P(x)\rangle_+ = 0 , \ \forall\, k \in K$.*
(2) $X = K \bigoplus P^{-1}(0)$.
Note that $P^{-1}(0)$ might not be a linear subspace, but we still have that each $x \in K$ can be uniquely written as $x = y + z$, where $x \in K$ and $z \in P^{-1}(0)$.
(3) $P(tx + k) = tP(x) + k \ \ \forall\, t \in \mathbb{R}, \ x \in X, \ k \in K$.
(4) *P is linear if and only if $P^{-1}(0)$ is a closed, linear subspace.*
(5) *If K is a closed, maximal linear subspace, then P is linear. If P is linear for all closed linear subspaces, then X is a Hilbert space.*

We can generalize Theorem 1.2 from Hilbert to Banach spaces in the following way.

Theorem 1.11. *Let X be a reflexive, locally uniformly convex Banach space, with a locally uniformly convex dual X^*.*

(a) *If K is a closed, maximal linear subspace, then the metric projection $P : X \to X$ has the following properties:*
  (i) *$K = P(X)$.*
  (ii) *$P^2(x) = P(x) , \ \forall\, x \in X$.*

(iii) *P is linear and continuous.*

(iv) $\langle x - P(x), P(y)\rangle_+ = 0\,,\ \forall\, x, y \in X$.

(b) *If $P : X \to X$ is a mapping such that $P(X) \neq X$ and P satisfies properties (i)-(iv) from part (a), then $K = P(X)$ is a closed linear subspace and P is the metric projection onto K.*

Proof. The proof is simple, we will just make some comments, which can be considered as a first glimpse of what lies ahead.
(a) Each of (i)-(iv) follows immediately from Proposition 1.9 (a) and Proposition 1.10 (1) and (5).
(b) From (i)-(iii) it follows that K is a closed subspace and then (iv) and Proposition 1.9 (c) imply that P coincides with the metric projection.

Comparing Theorems 1.2 and 1.11, we see that in non-Hilbert Banach spaces there is no one-to-one correspondence between closed linear subspaces, or even maximal, closed, linear subspaces, and linear orthogonal projections.
Note. Condition (iv) of Theorem 1.11 could be described by the James-orthogonality [8], which can be defined as x is orthogonal to y if $\langle x, y\rangle_+ = 0$, and which is equivalent to $||x|| \leq ||x + ty||$, for all $t \in \mathbb{R}$.
Proposition 1.12 ([15, 16]). *If K is a closed, convex cone, $x \in X\setminus K$, and $P(x) \neq 0$, then*

(a) *$P(x)$ is the metric projection of x onto K if and only if*

$$\langle x - P(x), P(x)\rangle_+ = 0 \text{ and } \langle x - P(x), k\rangle_+ \leq 0\,,\ \forall\, k \in K\,.$$

So, the kernel of $J(x - P(x))$, denoted by $\ker J(x - P(x))$, is a supporting hyperplane for K at $P(x)$.

(b) $P(tx) = tP(x)\,,\ \forall\, t \geq 0$.

Example 1.13. This example shows that, in non-Hilbert Banach spaces, Theorem 1.1 does not hold.

Let $X = \mathbb{R}^3$ with $||x|| = \sqrt[3]{|x_1|^3 + |x_2|^3 + |x_3|^3}\,.$

In this case (see, for example [1, 14]), for $x \neq 0$,

$$J(x) = \frac{1}{||x||}(x_1|x_1|, x_2|x_2|, x_3|x_3|)\,, \text{ and}$$

$$\langle x, y\rangle_+ = \frac{1}{||x||}(x_1|x_1|y_1 + x_2|x_2|y_2 + x_3|x_3|y_3)\,.$$

Note that, by the non-linear and non-symmetric properties of the semi-inner product, condition (1.1) in Theorem 1.1 has to be interpreted as

$$\langle x - P(x), P(y)\rangle_+ = 0 = \langle y - P(y), P(x)\rangle_+\,.$$

Consider the closed convex cone $K = \{\lambda(-1, -2, 1) \mid \lambda \geq 0\}$ and the points $x = (0, 0, 4)$ and $y = (\sqrt{2} - 1, -1, 3)$. If P is the metric projection onto K, then we have by Proposition 1.12 (a) that

$$P(x) = P(y) = (-1, -2, 1)\,,$$

since $x - P(x) = (1, 2, 3)$, $y - P(y) = (\sqrt{2}, 1, 2)$ and

$$\langle (1, 2, 3), (-1, -2, 1)\rangle_+ = 0 = \langle (\sqrt{2}, 1, 2), (-1, -2, 1)\rangle_+ .$$

Furthermore, $x + y = (\sqrt{2} - 1, -1, 7)$, but

$$P(x + y) \neq P(x) + P(y) = (-2, -4, 2),$$

because

$$\begin{aligned}&\langle (\sqrt{2} - 1, -1, 7) - (-2, -4, 2), (-2, -4, 2)\rangle_+ \\ &\quad = \langle (\sqrt{2} + 1, 3, 5), (-2, -4, 2)\rangle_+ \neq 0 .\end{aligned}$$

2. Necessary conditions for the metric projection onto smooth, round cones

Definition 2.1. We call $K \subset X$ a *pointed, smooth, round, solid cone* (or more simply, a *smooth round cone*) if the following properties are satisfied:

(1) K is a nonempty, closed convex cone.
(2) $K \cap (-K) = \{0\}$ and $\operatorname{int} K \neq \emptyset$.
(3) For all $0 \neq k \in \operatorname{bd} K$ there exists a unique hyperplane of support for K at k.
(4) For nonzero $k, l \in K$ we have $k + l \in \operatorname{bd} K$ if and only if $k, l \in \operatorname{bd} K$ and the hyperplanes of support at k and l coincide.

We make a few observations, which clarify this definition. Let K be a closed convex cone.

- If $K \cap (-K) = \{0\}$, then K is called a pointed cone.
- If $\operatorname{int} K \neq \emptyset$, then K is called a solid cone.
- If condition (3) is satisfied, we call K a smooth cone. This is related to the smoothness of the boundary of K at each non-zero point.
- The unique hyperplane at $0 \neq k \in \operatorname{bd} K$ is the same for any point along the non-closed half line $\{\lambda k \mid \lambda > 0\}$. If $x \notin K$ and $P(x) \neq 0$, then $H = \ker J(x - P(x))$ is the unique hyperplane of support for K at $P(x)$ and $P_H(x) = P_K(x)$.
- If condition (4) is satisfied, we call K a round cone. This condition is weaker than the strict convexity of cones, which means that if $0 \neq k, l \in \operatorname{bd} K$ and $k \notin \{\lambda l \mid \lambda > 0\}$, then the line segment connecting k and l, except the endpoints, lies entirely in the interior of K. If strict convexity is assumed, then a closed, convex cone cannot have line segments in the boundary, other than the generator half-lines $\{\lambda k \mid \lambda \geq 0\}$ where $0 \neq k \in \operatorname{bd} K$.
- By assuming conditions (3) and (4), we allow the boundary to contain besides the generator half-lines, parts of hyperplanes through the origin, but no other linear subspaces.

The following theorem gives necessary conditions for the metric projection onto smooth, round cones.

Theorem 2.2. *Suppose that X is a reflexive Banach space with X and X^* locally uniformly convex. If K is a smooth round cone, then the metric projection $P : X \to K$ has the following properties:*

(i) $P^2 = P$.

(ii) *$P(\lambda x) = \lambda P(x)$ for all $\lambda \geq 0$, $x \in X$.*

(iii) *If at least one of $P(x)$ or $P(y)$ is not zero, then*

$$P(x+y) = P(x) + P(y)$$

if and only if

$$\langle x - P(x), P(y)\rangle_+ = 0 \text{ and } \langle y - P(y), P(x)\rangle_+ = 0\,.$$

Proof. The first two properties are common properties of projections onto all closed, convex cones, so we will prove (iii).

Case 1. Suppose $P(x) \neq 0$ and $P(y) \neq 0$.

$\Longrightarrow$: We assume that $P(x+y) = P(x) + P(y)$.

If $x, y \in K$, then $x = P(x)$ and $y = P(y)$ and the conclusion is evident.

If $x \notin K$ and $y \in K$, then we must have $x + y \notin K$, since otherwise

$$P(x+y) = x + y = P(x) + y\,,$$

and hence $x = P(x)$, which contradicts $x \notin K$. Now, $x + y \notin K$ implies that

$$P(x+y) = P(x) + y \in \operatorname{bd} K\,.$$

Therefore, by Definition 2.1 (4), $P(x), y \in \operatorname{bd} K$ and the supporting hyperplanes at $P(x)$ and y coincide. So, $y \in \ker J(x - P(x))$ and hence

$$\langle x - P(x), P(y)\rangle_+ = 0 = \langle x - P(x), y\rangle_+\,.$$

Since $y = P(y)$, we clearly have that $\langle y - P(y), P(x)\rangle_+ = 0$.

If $x \notin K$, $y \notin K$, and $x + y \in K$, then

$$P(x+y) = x + y = P(x) + P(y)\,,$$

and hence

$$x - P(x) = P(y) - y\,.$$

Therefore,

$$\langle x - P(x), P(y)\rangle_+ = 0 = -\langle y - P(y), P(y)\rangle_+\,.$$

Similarly,

$$\langle y - P(y), P(x)\rangle_+ = 0\,.$$

If $x \notin K$, $y \notin K$, and $x + y \notin K$, then

$$P(x) + P(y) = P(x+y) \in \operatorname{bd} K\,.$$

So, by Definition 2.1 (4), the hyperplanes of support at $P(x)$ and $P(y)$ coincide, giving us a $\lambda > 0$ such that

$$J(x - P(x)) = \lambda J(y - P(y))\,.$$

Therefore, by the homogeneity of the duality mapping and of the metric projection,

$$\langle x - P(x), P(y)\rangle_+ = \lambda\langle y - P(y), P(y)\rangle_+ = 0\,.$$

Similarly,

$$\langle y - P(y), P(x)\rangle_+ = 0\,.$$

$\Longleftarrow$: For the reverse implication, assume that

$$\langle x - P(x), P(y)\rangle_+ = 0 = \langle y - P(y), P(x)\rangle_+\,.$$

If $x \in K$ and $y \in K$, then $x + y \in K$, and therefore

$$P(x + y) = x + y = P(x) + P(y)\,.$$

If $x \notin K$ and $y \in K$, then $\langle x - P(x), y\rangle_+ = 0 = \langle x - P(x), P(x)\rangle_+$, which shows that the common supporting hyperplane at $P(x)$ and y is $H = \ker J(x - P(x))$. Therefore, by Definition 2.1 (4), we have that $P(x) + y \in H \cap \operatorname{bd} K$ and

$$P(x) + y = P(P(x) + y + (x - P(x)) = P(x + y)\,.$$

Let $x \notin K$ and $y \notin K$. Then, by assumption, the supporting hyperplanes at $P(x)$ and $P(y)$ coincide, which means

$$\ker J(x - P(x)) = \ker J(y - P(y)) = H$$

and hence for some $\lambda > 0$,

$$x - P(x) = \lambda(y - P(y))\,.$$

By Definition 2.1 (4), $P(x) + P(y) \in \operatorname{bd} K \cap H$, and hence

$$P(x) + P(y) = P(P(x) + P(y) + x - P(x) + y - P(y)) = P(x + y)\,.$$

Case 2. Suppose $P(x) = 0$ and $P(y) \neq 0$. We want to prove that

$$P(x + y) = P(y) \iff \langle x, P(y)\rangle_+ = 0\,.$$

We can assume that $x \neq 0$, since the biconditional is trivial otherwise.

$\Longrightarrow$: Suppose $P(x + y) = P(y)$. This implies that $y \notin \operatorname{int} K$, because otherwise $x + y = y$ and this contradicts $x \neq 0$. By the uniqueness of the supporting hyperplane at $P(y)$, there exists $\lambda > 0$ such that

$$x + y = P(y) + \lambda(y - P(y))\,.$$

It follows that,

$$\langle x, P(y)\rangle_+ = (\lambda - 1)\langle y - P(y), P(y)\rangle_+ = 0\,.$$

$\Longleftarrow$: Suppose $\langle x, P(y)\rangle_+ = 0$. By Proposition 1.9 (c), since $P(x) = 0$, we have that $\langle x, k\rangle_+ \leq 0$ for all $k \in K$. So, $P(y) \in \operatorname{bd} K$.

If $y = P(y)$, then $P(x + y) = y$ since for each $k \in K$,

$$\langle x + y - P(x + y), k - P(x + y)\rangle_+ = \langle x, k - y\rangle_+ = \langle x, k\rangle_+ \leq 0\,.$$

If $y \notin K$, the conditions $\langle x, k\rangle_+ \leq 0$ for each $k \in K$, and $\langle x, P(y)\rangle_+ = 0$ imply that the kernel of $J(x)$ is the supporting hyperplane for K at $P(y)$. Therefore, there exists $\lambda > 0$ such that $\lambda J(x) = J(y - P(y)$, which, by Proposition 1.4 (1) and (3), gives

$$\lambda x = y - P(y)\,.$$

Noticing that

$$P(y) + \frac{1 + \lambda}{\lambda}(y - P(y)) = x + y\,,$$

the conclusion follows from Proposition 1.9 (b).

3. Sufficient conditions for the metric projection onto smooth, round cones

Let us assume that $N : X \to X$ is a mapping satisfying the following conditions:

(i) $\operatorname{int} N(X) \neq \emptyset$ and $N(X) \cap (-N(X)) = \{0\}$.
(ii) $N^2 = N$.
(iii) $N(\lambda x) = \lambda N(x)$ for all $x \in X$ and $\lambda \geq 0$.
(iv) If at least one of $N(x)$ or $N(y)$ is non-zero, then

$$N(x + y) = N(x) + N(y)$$

if and only if

$$\langle x - N(x), N(y)\rangle_+ = 0 = \langle y - N(y), N(x)\rangle_+ \ .$$

(v) N is continuous.

Let $C = \{x \in X \mid N(x) = x\}$ and $C^\circ = \{x \in X \mid \langle x, y\rangle_+ \leq 0\, , \forall\, y \in C\}$.

Proposition 3.1. *Suppose that X is a reflexive Banach space with X and X^* locally uniformly convex. If the mapping N satisfies conditions (i)-(v), then the set C has the following properties:*

(1) *C is closed and $0 \in \operatorname{bd} C$.*
(2) *$C = N(X)$.*
(3) *$\operatorname{int} C \neq \emptyset$.*
(4) *If $x \in C$ and $\lambda \geq 0$, then $\lambda x \in C$.*
(5) *If $x, y \in C$, then $x + y \in C$.*
(6) *If $x \in X$, then $\langle x - N(x), N(x)\rangle_+ = 0$.*
(7) *If $N(x) = N(y) \neq 0$, then for all $\alpha, \beta \geq 0$ we have*

$$N(\alpha x + \beta y) = \alpha N(x) + \beta N(y)\, .$$

(8) *For all $x \in X$ and $0 \leq \alpha \leq 1$, we have*

$$N(\alpha x + (1 - \alpha) N(x)) = N(x)\, .$$

(9) *If $x \notin C$ then $N(x) \in \operatorname{bd} C$.*
(10) *If $x \in C^\circ$, then $N(x) = 0$.*

Proof. Properties (1)-(4) are clear, so we begin with (5).

(5) Let $x, y \in C$. If $x = 0$ or $y = 0$, then clearly $x + y \in C$. Assume that $x \neq 0$ and $y \neq 0$. Then $x - N(x) = 0$ and $y - N(y) = 0$, so by condition (iv) of N, we have that

$$N(x + y) = N(x) + N(y) = x + y\, .$$

Hence, $x + y \in C$.

(6) If $x \in C$ or $N(x) = 0$, then $\langle x - N(x), N(x)\rangle_+ = 0$.
If $x \notin C$ and $N(x) \neq 0$, then

$$N(x + x) = N(2x) = 2N(x) = N(x) + N(x)\, ,$$

which, by condition (iv), gives

$$\langle x - N(x), N(x)\rangle_+ = 0\, .$$

(7) If $N(x) = N(y) \neq 0$, then $\langle x - N(x), N(y)\rangle_+ = 0$ and $\langle y - N(y), N(x)\rangle_+ = 0$. By conditions (iii) and (iv), we get that for each $\alpha, \beta \geq 0$,

$$N(\alpha x + \beta y) = \alpha N(x) + \beta N(y)\,.$$

(8) If $N(x) = 0$, then the result follows from condition (iii). If $N(x) \neq 0$, the result follows from property (7) for $x = x$, $y = N(x)$ and $\beta = 1 - \alpha$.

(9) If $N(x) = 0$, then by condition (i) we have that $N(x) \in \operatorname{bd} C$. Suppose, by way of contradiction, that $0 \neq N(x) \in \operatorname{int} N(X)$. Since $x \neq N(x)$, there exists a sufficiently small $0 < \alpha < 1$ such that $\alpha x + (1-\alpha)N(x) \in \operatorname{int} N(X)$. Hence,

$$N(\alpha x + (1-\alpha)N(x)) = \alpha x + (1-\alpha)N(x) \neq N(x)\,,$$

which contradicts (8).

(10) Let $x \in C^\circ$. If $x = 0$, then $N(x) = 0$. If $x \neq 0$, then $x \notin C$, because otherwise $\langle x, x\rangle_+ = ||x||^2 > 0$, which contradicts $x \in C^\circ$. Also, if $N(x) \neq 0$, then by Proposition 1.7 (8), we have that

$$0 = \langle x - N(x), N(x)\rangle_+ < \langle x, N(x)\rangle_+\,,$$

which contradicts $x \in C^\circ$.

The following theorem gives a characterization of the metric projection onto a pointed, closed, convex, solid cone in Banach spaces.

Theorem 3.2. *Suppose that X is a reflexive Banach space with X and X^* locally uniformly convex, and $N : X \to X$ is a mapping satisfying the conditions (i) - (v) at the beginning of this section. Let $C = \{x \in X \mid N(x) = x\}$. Then C is a pointed, closed, convex, solid cone and N is the metric projection onto C.*

Proof. That C is a pointed, closed, convex, solid cone, follows from Proposition 3.1, (1)-(5). Denote by P the metric projection onto C.

If $x \in C$, then $x = N(x) = P(x)$.

Suppose that $x \notin C$. Then $x - P(x) \in C^\circ$, and by Proposition 3.1 (10), we get that $N(x - P(x)) = 0$.

If $N(P(x)) = 0$, then using the fact that $P(x) \in C$, we get that $P(x) = 0$, and hence, $N(x) = 0$.

If $N(P(x)) \neq 0$, then $P(x) \neq 0$ and $N(P(x)) = P(x)$, so we use condition (iv) of N for $x - P(x)$ and $P(x)$, and get that

$$N(x) = N(x - P(x) + P(x)) = N(P(x)) = P(x)\,.$$

Remark 3.3. If we analyze the proof of Theorem 1.1, the smoothness and roundedness of the cone is not needed in a Hilbert space, because the metric projection onto subspaces, which may contribute to the boundary of the cone, is always linear. In order to obtain the face-linear property of the metric projection onto cones in Banach spaces, we have to assume extra smoothness and roundedness assumptions. However, as we analyze the sufficiency of these properties, we realize that under general assumptions on Banach spaces, we cannot exclude the possibility that $X = H \oplus Y$, where H is a Hilbert space, and some parts of the boundary of the cone lie in H, where we would have face-linearity without smoothness or roundedness. Therefore, under assumptions (i)-(v) of Section 3, the smoothness and roundedness of cones are not sufficient properties. This is in line with the result of [5], which says that for closed convex sets in Hilbert spaces we need at least a C^2-boundary in order to have a metric projection which is differentiable in the complement of the set C.

Acknowledgment. The authors would like to thank the referee for his/her comments, which helped improve the presentation of the results in the paper.

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Received: May 25, 2016; Accepted: August 12, 2016.