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R_δ-STRUCTURE OF SOLUTIONS SET FOR A VECTOR EVOLUTION INCLUSIONS DEFINED ON RIGHT HALF-LINE

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Abstract. In this paper, we deal with the topological structure of a first order vector differential inclusion defined on right half-line. Under some general growth conditions, the R_δ structure of continue solution set for Cauchy problem on compact interval is investigated. Then by the inverse limit method, the R_δ structure is also obtained on noncompact interval. Further, using the related results of structure, we obtain the existence and topological structure of solution set for some nonlocal problems. Subsequently a optimal dual control problem is considered and an R_δ structure of attainable set based on the proven results is obtained.

Key Words and Phrases: Vector differential inclusion, topological structure, nonlocal condition, inverse limit, growth condition, R_δ set.
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1. Introduction

It is well known that the topological structure for differential inclusions is associated with processes of controlled heat transfer, describing hybrid systems with dry friction, obstacle problems that others (see, e.g., [7], [1], [22] and references therein), and has been studied intensively on the compact intervals, see DeBlasi and Myjak [9], Deimling [11], Hu and Papageorgiou [18], Staicu [28], Ke et al. [20] and Zhu [30] and references therein.

There exist various techniques for topological structure of solution sets for differential equations or differential inclusions on non-compact intervals(including infinite

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intervals), such as Andres et al. [2] for boundary value problems of differential equations and inclusions, Bakowska and Gabor [4] for differential equations and inclusions in Fréchet spaces, and Sěda and Belohorec [27] for initial value problem of second-order ODE with time delay. It is worthwhile to note that Gabor and Grudzka [15] treated recently an impulsive abstract Cauchy problem governed by a semi-linear differential inclusion involving a family of time-dependent linear operators in the linear part. An important aspect for the study of topological structure of solution sets is the R_δ-property on a non-compact interval. For more details on this topic,we refer the reader to, e.g., Gabor [14] and O' Regan [23] and references therein. In [3], the R_δ-structure is firstly investigated for second-order vector asymptotic boundary value problems, by means of the inverse limit method, on noncompact intervals. For nonlocal conditions which including periodic, anti-periodic, mean value condition and multi-point discrete mean condition, the situation is much more delicate and the related results are still very rare, see Chen et al. [8]. In [8], they dealt with a nonlinear delay differential inclusion of evolution type involving m-dissipative operator and source term of multi-valued type in a Banach space, and obtained the R_δ-property on non-compact intervals under the linear growth condition.

It is worth mentioning that the characterizations of solution sets including compactness, acyclicity and R_δ are important in the study of the qualitative theory for deterministic problems; please see [8], [3] and the references therein for more comments and citations. The contribution of this paper is to study the R_δ-property of solution sets for the nonlocal problems of a vector differential inclusion with time delay on a noncompact intervals under more general growth condition. Moreover, As far as we know, not much work has been done for this general nonlocal problem involving vector differential inclusion where the nonlocal function is multivalued. Following their lead, in this paper, motivated by applications of the topological structure, we establish the R_δ-property of continuous solution sets to nonlocal problem with the nonlocal multivalued functions. Further, as the applications of the information about the topological structure, we get the R_δ-property of solution sets of a class of neural networks with the discontinuous activations functions. Subsequently a dual optimal control problem defined on the half-line is considered and the existence theorems and R_δ-attainable set for control systems are obtained based on the information of the topological structure.

In this paper, we firstly prove that the solution set for Cauchy problem of a vector evolution inclusion with a closed and convex valued orient field is compact R_δ on a compact intervals. Secondly, the R_δ-structure of solution sets on compact intervals is, by means of the inverse limit method, extended to non-compact intervals cases. Then we also get the similar results on the topological structure for cases of the nonlocal multivalued function with convex or nonconvex value. Finally, some examples are also given to illustrate the effectiveness of our results.

2. Preliminaries

In this section, we recall some geometric notions of subsets of metric spaces, in particular, of retracts; see [16], [19] for more details.

Let X be a Hausdorff topological space. For a nonempty set $\mathcal{A} \subset X$, we say that $\mathcal{A}$ is a retract of X, if there exists a continuous map (retraction) $r : X \to \mathcal{A}$ such that $r\mid_{\mathcal{A}}$ is the identity map. It is easy to see that a retract $\mathcal{A} \subset X$ is closed.

Definition 2.1. Let X is a metric space, a closed subset $\mathcal{A}$ of X is said to be an absolute retract(AR-space), if for every metric space Y and a closed set $C \subset Y$, each continuous map $f : C \to \mathcal{A}$ has a continuous extension, $\widehat{f} : Y \to \mathcal{A}$.

Equivalently we say that the closed set $\mathcal{A} \subseteq X$ is an absolute retract, if every homeomorphic image of $\mathcal{A}$ in any metric space Y is a retract of Y.

Definition 2.2. Y is called an absolute neighborhood retract (ANR-space) if for any metric space H, closed subset $D \subset H$ and continuous function $\alpha : D \to Y$, there exists an neighborhood $D \subset U$ and a continuous extension $\bar{\alpha} : U \to Y$ of α.

Proposition 2.1. (see [16]) *If Y is an AR-space then it is an ANR-space. Moreover, if X is a retract of a convex set in a Fréchet space, then it is an AR-space.*

So, in particular, the spaces $C(I, R^N)$, $C^1(I, R^N)$, are AR-spaces as well as their convex subsets, where $I \subset \mathbb{R}$ is an arbitrary interval. Here, let $C([-\tau, m]; R^N)$ be the Banach space of all continuous functions from $[-\tau, m]$ to $\mathbb{R}^N$ equipped with the sup-norm

$$\|\cdot\|_m = \max_{t\in[-\tau,m]} \|\cdot\|,$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^N$. Denote by $\|\cdot\|_0$ the norm of $C([-\tau, 0], R^N)$.

Definition 2.3. A subset $\mathcal{A}$ of a metric space X is said to be contractible, if there exist a continuous function $\eta : [0, 1] \times \mathcal{A} \to X$ (homotopy) and a point $a \in \mathcal{A}$ such that for all $x \in \mathcal{A}$ we have $\eta(0, x) = a$ and $\eta(1, x) = x$.

Definition 2.4. A subset D of a metric space X is said to be R_δ, if it is homeomorphic to the intersection of a decreasing sequence $\{D_n\}_{n\geq 1}$ of absolute retracts. Moreover, if each D_n is also compact, then we say that D is a compact R_δ.

A subset $D \subseteq X$ is compact R_δ if and only if it is the intersection of a decreasing sequence of contractible compact metric spaces. Note that a compact R_δ set D is nonempty, compact and connected. The following hierarchy holds for nonempty subsets of a metric space:

$$\text{compact} + \text{convex} \ \subset \text{compact AR} - \text{space} \ \subset \text{compact} + \text{contractible} \ \subset R_\delta - \text{set},$$

and all the above inclusions are proper.

For $A, B \in X$, known in the literature as the Hausdorff metric, by

$$d_H(A, B) = \max[\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)].$$

Let X and Y be arbitrary metric spaces. We say that G is a multivalued map from X to Y $(G : X \to 2^Y)$ if, for every $x \in X$, a nonempty subset $G(x)$ of Y is prescribed. A multivalued map $G : X \to 2^Y$ is said to be measurable if, for all $y \in Y$, the $\mathbb{R}_+$-valued function $x \to d(y, G(x))$ is measurable.

Definition 2.5. A multivalued map $G(\cdot)$ is called upper semicontinuous(for short u.s.c.) provided for every open $U \subseteq Y$, the set $G^{-1}(U) = \{x \in X : G(x) \subseteq U\}$ is open in X.

Proposition 2.2. *A multivalued map $G : X \to Y$ is u.s.c. if and only if for every closed set $A \subset Y$ the set $G^{-1}(A)$ is a closed subset of X.*

We associate with G its graph Γ_G, the subset of $X \times Y$, defined by

$$\Gamma_G := \{(x, y) \in X \times Y \mid y \in G(x)\}.$$

Proposition 2.3. *If $G : X \to 2^Y$ is u.s.c. then the graph Γ_G is a closed subset of $X \times Y$.*

The reverse relation between upper semicontinuous mappings and those with closed graphs is expressed in the following proposition.

Proposition 2.4. *Let X, Y be metric spaces and $G : X \to 2^Y$ be a multivalued map with the closed graph such that $G(X) \subset K$, where K is a compact set. Then G is u.s.c.*

Definition 2.6. A multivalued map $G(\cdot) : X \to Y$ is called a lower semicontinuous (l.s.c.) provided for every open $U \subseteq Y$, the set $G_+^{-1}(U) = \{x \in X : G(x) \cap U \neq \emptyset\}$ is open in X.

Definition 2.7. A multivalued map $F : X \to 2^Y$ with bounded values is called Lipschitzian if there exists a constant $L > 0$ such that

$$d_H(F(x), F(y)) \leq Ld(x, y),$$

and called a contraction if the constant $L \in [0, 1)$.

A multivalued map $F : I \times R^m \to 2^{R^n}$ is called an upper-Carathéodory map if the map $F(\cdot, x) : I \to 2^{R^n}$ is measurable on every compact subinterval of I, for all $x \in R^m$, the map $F(t, \cdot) : R^m \to 2^{R^n}$ is u.s.c., for almost all (a.a.) $t \in I$, and the set $F(t, x)$ is closed and convex, for all $(t, x) \in I \times R^m$. Contrary to the singlevalued case, **Fix**(F) of a contraction F is more complex. So it is an interesting problem to study topological property. In this framework, the following two results worthy to recall is critical to our results.

Proposition 2.5. (see [26]) *Let X be a closed, convex subset of a Banach space E and let $\phi : X \to 2^X$ be a contraction with compact, convex values. Then* **Fix**(ϕ) *is a nonempty, compact AR-space.*

Definition 2.8. A multivalued map $\alpha : Y \to 2^Z$ is called an R_δ-map where Y and Z are both metric spaces if α is u.s.c. and $\alpha(y)$ is an R_δ-set for each $y \in Y$.

It is clear that every u.s.c. multivalued map with contractible values can be seen as an R_δ-mapping. In particular, every single-valued continuous map is an R_δ-mapping. Let $\varphi : X \to 2^Y$ and $\psi : Y \to 2^Z$ be two multivalued maps, then the composition $\psi \circ \varphi : X \to 2^Z$ of φ and ψ is defined by

$$(\psi \circ \varphi)(x) = \cup\{\psi(y) : y \in \varphi(x)\} \text{ for every } x \in X.$$

Proposition 2.6. (see [16]) *Let $\varphi : X \to 2^Y$ and $\psi : Y \to 2^Z$ are two u.s.c map with compact values, then the composition $\psi \circ \varphi : X \to 2^Z$ is an u.s.c map with compact values.*

The following is a fixed point theorem due to Górniewicz and Lassonde [17, Corollary 4.3].

Theorem 2.1. *Let Y be an ANR-space. Assume that $\alpha : Y \to 2^Y$ can be factorized as $\alpha = \alpha_N \circ \alpha_{N-1} \circ \ldots \circ \alpha_1$, where $\alpha_i : Y^i \to 2^{Y^i}$, $i = 1, ..., N$, are R_δ-mappings,*

$Y^i, i = 1, ..., N-1$, are ANR-spaces, and $Y^0 = Y^N = Y$ are AR-spaces. If there exists a compact subset $K \subset Y$ satisfying $\alpha(Y) \subset K$, then α admits a fixed point.

Proposition 2.7. (see [16]) *Let $\varphi : X \to 2^Y$ be an u.s.c maping with compact values and A be a compact subset of X. Then $\varphi(A)$ is compact.*

3. Main results

3.1. Topological structure on compact intervals

In this section, we first study a topological structure of solution set governed by a delay evolution inclusion on compact intervals. Let us consider the following problem:

$$\begin{aligned} &\dot{x} + D(t)x(t) \in H(t, x, x_t),\ for\ a.a.\ [0, m] \\ &x(t) = \psi(t)\ for\ t \in [-\tau, 0], \end{aligned} \tag{3.1}$$

where

(H1) $D : [0, m] \to \mathbb{R}^{N \times N}$ is an integrable matrix function satisfying $\langle D(t)u, u\rangle \geq 0$ for a.a. $t \in [0, m]$; $x_t(\omega) = x(t + \omega)$, for $\omega \in [-\tau, 0]$, and $\psi(t) \in C([-\tau, 0]; R^N)$;

(H2) $H : [0, m] \times \mathbb{R}^N \times C([-\tau, 0]; R^N) \to 2^{\mathbb{R}^N}$ is an upper carathéodory function with compact and convex value satisfying

$$d_H(H(t, x_1, y_1), H(t, x_2, y_2)) \leq \mu(t)(\|x_1 - x_2\| + \|y_1 - y_2\|_0),\ a.a.\ [0, m]$$

for all $x_1, x_2 \in \mathbb{R}^N$, and $y_1, y_2 \in C([-\tau, 0]; R^N)$ where $\mu(t)$ satisfies $0 < \int_0^m \mu(t)dt < \frac{1}{3}$;

(H3) there exists an integrable function $\alpha : [0, m] \to [0, +\infty)$ with $\int_0^m \alpha(s)ds$ sufficiently small, and a continuous function $g : \mathbb{R}_+ \to \mathbb{R}$ with $|g(\xi)| \leq g(\eta)$ for all $|\xi| \leq \eta$ such that

$$|H(t, x, y)| \leq \alpha(t)(1 + g(\|x\|) + g(\|y\|_0))$$

for all $(t, x, y) \in I \times \mathbb{R}^N \times C([-\tau, 0]; R^N)$;

(H4) the equation $\dot{x} = \alpha(t)g(x) + \alpha(t)$, for $t \in [0, m]$ where $x(0) = \eta \in \mathbb{R}^N$ has an uniformly solution for the Cauchy problem.

Remark 3.1. The assumption (H3) and (H4) on guaranteeing the property of solution set of this problem (3.1) for a wide class of superlinear sources, for example, functions $g(x) = e^x$ and $g(x) = x^p (p \geq 1)$, is formulated. For sublinear case (for example $\ln(1 + |x|)$) this condition is automatically fulfilled.

We denote the solution set of (3.1) by S^m_ψ, and shall show that S^m_ψ is an R_δ set in $C([-\tau, m]; R^N)$. For every $f \in L^1([0, m]; R^N)$, consider the following differential equation:

$$\begin{aligned} &\dot{x} + D(t)x(t) = f(t),\ a.a.\ for\ t \in [0, m], \\ &x(t) = \psi(t)\ for\ t \in [-\tau, 0]. \end{aligned} \tag{3.2}$$

If hypothesis (H1) hold, for every $f \in L^1([0, m]; R^N)$, it is easy to check that the differential equation (3.2) has a unique solution $x \in C([-\tau, m]; R^N)$. So, we can define a solution map $P_m : L^1([0, m]; R^N) \to C([-\tau, m]; R^N)$ such that $x = P_m(f)$ for each $f \in L^1([0, m]; R^N)$. If $f, g \in L^1([0, m]; R^N)$, and x_f, x_g are two solutions to the differential equation (3.2) corresponding to f and g, respectively, then one has that

$$\int_0^t \langle \dot{x}_f(s) - \dot{x}_g(s), x_f(s) - x_g(s)\rangle ds + \int_0^t \langle D(s)(x_f(s) - x_g(s)), x_f(s) - x_g(s)\rangle ds$$
$$= \int_0^t \langle f(s) - g(s), x_f(s) - x_g(s)\rangle ds,\ t \in [0, m]. \tag{3.3}$$

From (H1), we have

$$\|x_f(t) - x_g(t)\|^2 \leq \|x_f(0) - x_g(0)\|^2 + 2\int_0^t \|f(s) - g(s)\|\|x_f(s) - x_g(s)\| ds,\ t \in [0, m]. \tag{3.4}$$

By Brezis [6, p. 157], we have

$$\|x_f(t) - x_g(t)\| \leq \|x_f(0) - x_g(0)\| + \int_0^t \|f(s) - g(s)\| ds,\ t \in [0, m]. \tag{3.5}$$

In order to study the topological structure of solution set for problem (3.1), we first establish the following existence result on compact intervals.

Theorem 3.1. *If hypotheses (H1)-(H4) hold, the solution set of problem (3.1) is a nonempty, compact R_δ-set.*

Proof. Let us define the Nemytskii operator $N : C([-\tau, m]; R^N) \to L^1([0, m]; R^N)$ corresponding to H as follows

$$N(x) = \{f \in L^1([0, m]; R^N), f(t) \in H(t, x, x_\tau),\ for\ a.a.\ t \in [0, m]\}.$$

The closedness and decomposability of the values of $N(\cdot)$ are easy to check. For the non-emptiness, note that if $x \in C([-\tau, m]; R^N)$, by hypothesis (H3), $H(t, x, x_t)$ possesses a measurable selection. Thus, $N(x) \neq \emptyset$, for all $x \in C([-\tau, m]; R^N)$. For given $\psi(t) \in C([-\tau, 0]; R^N)$, the set Q_ψ^m is defined by

$$Q_\psi^m = \{x \in C([-\tau, m]; R^N) : x(t) = \psi(t)\ for\ t \in [-\tau, 0], and$$
$$\|x(t)\| \leq u_\psi(t)\ for\ all\ t \in [0, m]\},$$

where $u_\psi(t) \in C([0, m], R_+)$ is the unique continuous solution of the integral equation in the form

$$u_\psi(t) = \|\psi\|_0 + \int_0^t \alpha(s)(1 + 2g(u_\psi(s)))ds,\ t \in [0, m]. \tag{3.6}$$

Next, we will seek for solutions in Q_ψ^m. For this, let us define a multivalued mapping F_ψ^m on Q_ψ^m by setting

$$F_\psi^m(x) := P_m \circ (N(x)),\ x \in Q_\psi^m.$$

What follow to check that for every $x \in Q_\psi^m$, $F_\psi^m(x) \neq \emptyset$. To this end, we assume that $x \in C([-\tau, m]; R^N)$ and (x_n, y_n) is a step functions from $[0, m]$ to $\mathbb{R}^N \times C([-\tau, 0]; R^N)$ such that

$$x_n \to x \text{ in } \mathbb{R}^N \text{ with } \|x_n\| \leq \|x\|,$$

and

$$y_n \to x_t \text{ in } C([-\tau, 0]; R^N) \text{ with } \|y_n\|_0 \leq \|x_t\|_0$$

for every $t \in [0, m]$. By (H2), we see readily that for each n, $H(\cdot, x_n(\cdot), y_n(\cdot))$ admits a measurable selection $f_n(\cdot)$. Furthermore, from (H2), it follows that $\{f_n\}$ is integrably bounded in $L^1([0, m]; R^N)$. So by Dunford-Pettis theorem, and by passing to a subsequence if necessary, we may assume that $f_n \to f$ weakly in $L^1([0, m]; R^N)$. Similar to Theorem 3.1.2 in [29], we obtain $f \in N(x)$, so $N(x) \neq \emptyset$ which means $F_\psi^m(x) \neq \emptyset$. For each $x \in C([-\tau, m]; R^N)$, therefore, $F_\psi^m(x) \subset C([-\tau, m]; R^N)$ is nonempty. Also, it is noted that $\{u\mid_{-\tau}^0, u \in F_\psi^m(x)\} = \psi$ where $u\mid_{-\tau}^0$ is the restriction of u on $[-\tau, 0]$. Moreover, taking $f \in N(x)$ with $x \in Q_\psi^m$, from (H2) and (3.5), it follows that for each $t \in [0, m]$, we have

$$\begin{aligned}
\|P_m(f)\| &\leq \|\psi(0)\| + \int_0^t \|f\| ds \\
&\leq \|\psi\|_0 + \int_0^t \alpha(s)(1 + g(\|x(s)\|) + +g(\|x_t\|_0)) ds \\
&\leq \|\psi\|_0 + \int_0^t \alpha(s)(1 + 2g(u_\psi(s))) ds \\
&= u_\psi(t).
\end{aligned}$$

Hence, we know that $\|P_m(f)\| \leq u_\psi(t)$ for each $t \in [-\tau, m]$, and then $P_m(f) \in Q_\psi^m$. Therefore, we obtain that $F_\psi^m(x) \subset Q_\psi^m$ for every $x \in Q_\psi^m$.

Obviously, the solutions set of problem (3.1) is equal to the set of fixed-points of the operator F_ψ^m. Next, we shall show that the $\mathbf{Fix}(F_\psi^m(x))$ is, by means of Proposition 2.5, a nonempty, convex and compact. The following two claims are given to complete the proof.

Claim 1. *For each $x \in Q_\psi^m$, the map $F_\psi^m(x)$ has convex value.*

If $v_1, v_2 \in F_\psi^m(x)$, then there exists integrable selections $f_1(\cdot), f_2(\cdot)$ of $H(\cdot, x(\cdot), x_t(\cdot))$ such that, $v_1 = P_m(f_1), v_2 = P_m(f_2)$, i.e.

$$\dot{v}_1 + D(t)v_1 = f_1(t),\ \dot{v}_2 + D(t)v_2 = f_2(t),\ for\ a.a.\ t \in [0, m].$$

For any $\lambda \in [0, 1]$, then we have

$$\lambda\dot{v}_1 + (1-\lambda)\dot{v}_2 + D(t)(\lambda v_1 + (1-\lambda)v_2) = \lambda f_1(t) + (1-\lambda)f_2(t).$$

Since the multi-valued map $N(x)$ has convex value, then

$$\lambda f_1(t) + (1-\lambda)f_2(t) \in N(x),\ for\ a.a.\ t \in [0, m].$$

Therefore,

$$\lambda v_1 + (1-\lambda)v_2 = P_m(\lambda f_1(t) + (1-\lambda)f_2(t)) \in P_m \circ N(x),$$

i.e.

$$\lambda v_1 + (1-\lambda)v_2 \in F_\psi^m(x),$$

which implies the map $F_\psi^m(x)$ has convex value, as claimed.

Claim 2. *For each $x \in Q_\psi^m$, the map $F_\psi^m(x)$ has compact value.*

Let $x \in C([-\tau, m]; R^N)$ be arbitrary and from Claim 1, we see that $F_\psi^m(x) \subseteq Q_\psi^m \subset C([-\tau, m]; R^N)$. Then the set $F_\psi^m(x)$ is bounded and equicontinuous. Therefore, the set $F_\psi^m(x)$ is relatively compact due to the well-known Arzelá-Ascoli Lemma.

The closedness of $F^m_\psi(x)$ follows from the fact that, according to [29], F^m_ψ can be expressed as the closed graph composition of operators $P_m \circ N(x)$, where $P_m : L^1([0,m];R^N) \to C([-\tau,m];R^N)$, and $N : C([-\tau,m];R^N) \to L^1([0,m];R^N)$. Hence, $F^m_\psi(x)$ has compact value.

What follow to show that the operator F^m_ψ is a contraction. In fact, for any $x, y \in C([-\tau,m];R^N)$, there exist $v_x \in F^m_\psi(x), v_y \in F^m_\psi(y)$ and integrable selections $f_x(\cdot) \in H(t,x,x_t)$ and $f_y(\cdot) \in H(t,y,y_t)$ such that

$$\begin{aligned} d_H(F^m_\psi(x), F^m_\psi(y)) &= \|v_x - v_y\|_C \\ &\leq \|v_x(0) - v_y(0)\| + \int_0^m \|f_x(s) - f_y(s)\| ds \\ &\leq 2\int_0^m \mu(s) ds \|x - y\|_C. \end{aligned} \tag{3.7}$$

Since $\omega := 2\int_0^m \mu(s)ds < 1$, then the operator F^m_ψ is a desired contraction with a Lipschitz constant $\omega \in [0,1)$. Finally, since F^m_ψ is a contraction with compact and convex values, due to Proposition 2.5, the set $\mathbf{Fix}(F^m_\psi)$ is a nonempty, compact AR-space, i.e. the solution set of problem (3.1) is a compact R_δ set, which completes the proof.

Remark 3.1.
$(\mathrm{H3})_1$ there exist $(x_0, y_0) \in R^N \times C([-\tau,0];R^N)$ and a constant $C_0 \geq 0$ such that

$$|H(t,x_0,y_0)| \leq C_0\gamma(t) \ for \ a.a. \ t \in [0,m],$$

where r satisfies $\int_0^m \gamma(s)ds < \frac{1}{2}$.

From hypotheses (H2),$(\mathrm{H3})_1$, we can infer that there exists a constant

$$M = \|x_0\| + \|y_0\|_0$$

such that

$$|H(t,x,y)| \leq \gamma(t)(M + C_0 + \|x\| + \|y\|_0)$$

for each $(x,y) \in R^N \times C([-\tau,0];R^N)$. So let $g(s) = s$, then the assumption (H2) and $(\mathrm{H3})_1$ satisfies (H3). Thus, if hypotheses (H1),(H2) and $(\mathrm{H3})_1$ holds, the solution set of the problem (3.1) is a compact R_δ-set.

Remark 3.2. If H is a lower carathéodory function with closed value, by the Bressan-Colombo continuous selection theorem (see [5]), the differential inclusion (3.1) is reduced to the single value differential equation, so the existence results of solutions can be established for this case from the information of topological structure.

As an application of the previous results, we present some examples.

Example 3.1. Let us consider the following second order differential inclusion of the form

$$\begin{gathered} \ddot{x}(t) + A(t)\dot{x} \in F(t,\dot{x},\dot{x}_t) \ for \ t \in [0,m], \\ x(t) = \psi_1(t), \dot{x}(t) = \psi_2(t), \ for \ t \in [-\tau,0] \end{gathered}$$

where

(F1) $A : [0,m] \to R^{N\times N}$ is a integrable matrix function such that

$$\langle A(t)u, u\rangle \geq a(t)\|u\|$$

for a.a. $t \in [0,m]$ and suitable function $a \in L^1([0,m];R_+)$, and $\psi_1(t), \psi_2(t) \in C([-\tau,0];R^N)$;
(F2) $F:[0,m]\times R^N \times C([-\tau,0];R^N) \to 2^{R^N}$ is an upper carathéodory function with compact convex value such that for all $v_1 \in F(t,y_1,z_1), v_2 \in F(t,y_2,z_2)$

$$\|v_1 - v_2\| \leq \alpha(t)(\|y_1 - y_2\| + \|z_1 - z_2\|),\ for\ a.a.\ t \in [0,m]$$

with $\int_0^m \alpha(t)dt < \frac{1}{2}$ and for all $y_1, y_2 \in R^N, z_1, z_2 \in C([-\tau,0];R^N)$;
(F3) there exists an integrable function $\beta : [0,m] \to [0,+\infty)$ with $\int_0^m \beta(s)ds$ sufficiently small such that

$$|H(t,x,y)| \leq \beta(t)(1 + \exp^{\|x\|} + \exp^{\|y\|_0})$$

for all $(t,x,y) \in I \times R^N \times C([-\tau,0];R^N)$.

Let $v = \dot{x}$, then we define the map $B : C([-\tau,m];R^N) \to C^1([-\tau,m];R^N)$ by $B(v) = \psi_1(0)+\int_0^t v(s)ds$, for $t\in[0,m]$, and $B(v)(t) = \psi_1(t)$ for $t \in [-\tau,0)$. Obviously, the map B is a linear continuous map. Hence, the second order differential inclusion is equivalent to the following first order differential inclusion:

$$\dot{v}(t) + A(t)v \in F(t,v,v_t)\ for\ t \in [0,m], \qquad (3.8)$$
$$v(t) = \psi_2(t),\ for\ t \in [-\tau,0].$$

In order to guarantee *a prior* boundedness of solution set, we only need to show that the equation $u(t) = \|\psi_2\|_0 + \int_0^t \beta(s)(1+2e^{u(s)})ds$ for a.a. $t \in [0,m]$ has only one solution. Thus, we have

$$\frac{2e^u}{1+2e^u} = \frac{2e^{u(0)}}{1+2e^{u(0)}} e^{\int_0^t \beta(s)ds},$$

which implies it has a uniformly solution if $\int_0^t \beta(s)ds < 1 + \frac{1}{2e^{\|\psi_2\|_0}}$. It is easy to check that F satisfies all conditions of Theorem 3.1. Therefore, since B is a linear continuous map from $C([-\tau,m];R^N)$ to $C^1([-\tau,m];R^N)$, by Theorem 3.1, the solution set of problem (3.3) is a R_δ compact set in $C^1([-\tau,m];R^N)$.

Example 3.2. Inspired by [12], we consider a class of neural networks described by the system of differential equations

$$\dot{x} = -Ax(t) + Bg(x) + B^\tau g(x(t-\tau)) + I\ for\ a.a.\ t \in [0,m] \qquad (3.9)$$
$$x(t) = \psi(t)\ for\ t \in [-\tau,0]$$

where $x = (x_1, x_2, \dots, x_N)^T \in R^N$ is the vector of neuron state;
$A = \text{diag}(a_1, a_2, \dots, a_N)$ is an $N \times N$ constant diagonal matrix where $a_i > 0$, $i = 1,2,\dots,N$, are the neuron self-inhibitions;
$B = (b_{ij})$ and $B^\tau = (b^\tau_{ij})$ are $N\times N$ constant positive definite matrices which represent the neuron interconnection matrix and the delayed neuron interconnection matrix, respectively; and $\tau > 0$ is the constant delay in the neuron response. Moreover, $g(x) = (g_1(x_1), g_2(x_2), \dots, g_N(x_N))^T : R^N \to R^N$ is a map where $g_i (i = 1,2,\dots,N)$, represents the neuron input-output activation and $I(t) = (I_1, I_2, \dots, I_N)^T \in R^N$ is the vector of constant neuron inputs. Let $\lambda_{\max}$ denote the largest eigenvalue of $B = (b_{ij})$ and $B^\tau = (b^\tau_{ij})$.
We suppose that the activations belong to the following set of discontinuous functions.

Assumption 1. $g_i \in \mathcal{G}$, for any $i = 1, 2, \ldots, N$, where $\mathcal{G}$ denotes the class of functions from $\mathbb{R}$ to $\mathbb{R}$ which are monotone nondecreasing and have at most a finite number of jump discontinuities in every compact interval.

Then in order to obtain solutions, we need replace the original single-valued problem by a multivalued one in which we have filled the gaps at the discontinuity points. We note that if g satisfies Assumption 1, then any $g_i(i = 1, 2, \ldots, N)$, possesses only isolated jump discontinuities where g_i is not necessary defined. Hence, for all $x \in R^N$, we set

$$\Phi[g(x)] = \big([\bar{g}_1(x_1), \hat{g}_1(x_1)], [\bar{g}_2(x_2), \hat{g}_2(x_2)], \ldots, [\bar{g}_N(x_N), \hat{g}_N(x_N)]\big)$$

where $\bar{g}_i(x_i) \leq \liminf_{\varepsilon \to x_i} g_i(\varepsilon)$, $\hat{g}_i(x_i) \geq \limsup_{\varepsilon \to x_i} g_i(\varepsilon)$. Thus the differential equations (3.12) become the following differential inclusions:

$$\dot{x} \in -Ag(x) + B\Phi[g(x)] + B^{\tau}\Phi[g(x_\tau)] + I \ \ for\ a.a\ t \in [0, m]. \tag{3.10}$$

By [12], the inclusion has at least a solution in the sense of Filippov, which is particularly useful in the engineering applications. Since it can be proved that solutions in the sense of Filippov are good approximation of solutions obtained with the neuron activations being Lipschitz functions, it is necessary to know the property of solutions set. In order to get the topological structure of problem (3.12), we will strengthen the condition on g.

(F6)1) For any $s \in R$, $s \to \bar{g}_i(s)$ and $s \to \hat{g}_i(s)(i = 1, 2, \ldots, n)$ are Lipschitz continuous where the Lipschitz constant satisfies $0 < L < \frac{1}{2\lambda_{\max}}$;

2) there exist $\beta \in L^\infty([0, m])$ such that

$$|g_i(x)| \leq \beta |x|.$$

On the basis of the hypothesis (H6), we can define the mapping $F_i(t, \cdot) : C[0, m] \to 2^{L^1[0,m]}$ as

$$F_i(t, x) = \{v \in L^1[0, m] : \bar{g}_i(x) \leq v(t) \leq \hat{g}_i(x)\}.$$

If the hypothesis (F6) holds, we will claim that $F_i(t, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is continuous for every fixed $t \in [0, m]$.

For every $u_1, u_2 \in \mathbb{R}$, every fixed $t \in [0, m]$, let $f_1 \in F_i(t, u_1)$, then

$$\begin{aligned} d_R(f_1, F_i(t, u_2)) &= \inf_{f_2 \in F(t,u_2)} |f_1(t) - f_2(t)| \\ &= d_R(f_1(t), [\bar{g}_i(u_2(t)), \hat{g}_i(u_2(t))]). \end{aligned}$$

So

$$\begin{aligned} \sup_{f_1 \in F_i(t,u_1)} d_R(f_1, F_i(t, u_2)) &\leq \max\{|\bar{g}_i(u_2(t)) - \bar{g}_i(u_1(t))|, |\hat{g}_i(u_2(t)) - \hat{g}_i(u_1(t))|\} \\ &\leq |\bar{g}_i(u_2(t)) - \bar{g}_i(u_1(t))| + |\hat{g}_i(u_2(t)) - \hat{g}_i(u_1(t))|. \end{aligned} \tag{3.11}$$

Similarly, the above inequalities also hold for $\sup_{f_2 \in F(t,u_2)} d_R(f_2, F_i(t, u_1))$. So

$$\begin{aligned} d_H(F_i(t, u_1), F_i(t, u_2)) &\leq |\bar{g}_i(u_2(t)) - \bar{g}_i(u_1(t))| + |\hat{g}_i(u_2(t)) - \hat{g}_i(u_1(t))| \\ &\leq 2L|u_2(t) - u_1(t)|. \end{aligned} \tag{3.12}$$

From (F6)(1), the last sign of equality holds. So, for every $x_1, x_2 \in R^N$,

$$d_H(\Phi[g(x_1)], \Phi[g(x_2)]) \leq 2L\|x_1(t) - x_2(t)\|. \tag{3.13}$$

Letting $H(x,x_\tau) = B\Phi[g(x)] + B_\tau\Phi[g(x_\tau)] + I$, we have $H(x,x_\tau) : R^N \times R^N \to R^N$ for a.a. $t \in [0,m]$. From (3.16), it is easy to check that

$$d_H(H(s_1,w_1), H(s_2,w_2)) \le 2L\lambda_{\max}(\|s_1(t) - s_2(t)\| + \|w_1 - w_2\|)$$

for all $(s_1,w_1),(s_2,w_2) \in R^N \times R^N$ which implies the assumption (H2) holds on H.

Rewrite problem (3.12) in the following equivalent evolution inclusion form:

$$\begin{aligned} \dot{x} + Ax \in H(t,x,x_\tau) \ \ &for\ a.a.\ [0,m] \\ x(t) = \psi(t)\ for\ t \in [-\tau,0]. \end{aligned} \tag{3.14}$$

Thus, by Theorem 3.1, the solution set of problem (3.12) is a compact R_δ set in $C([-\tau,m];R^N)$.

4. Inverse limit method and the topological structure on noncompact intervals

To study the Cauchy problem (3.1) defined on the right half-line, we shall use the inverse limit method. Let us recall some notions of the inverse system, for details see e.g. [14], [13]. We mean a inverse system $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$, where Σ is a set directed by the relation $\le$, X_α is, for all $\alpha \in \Sigma$, a metric space and $\pi_\alpha^\beta : X_\beta \to X_\alpha$ is a continuous function, for all $\alpha,\beta \in \Sigma$ with $\alpha \le \beta$. Moreover, for each $\alpha \le \beta \le \gamma$,

$$\pi_\alpha^\alpha = id_{X_\alpha}\ and\ \pi_\alpha^\beta \circ \pi_\beta^\gamma = \pi_\alpha^\gamma.$$

Let $\lim_{\leftarrow} S$ denote the limit of inverse system S, is defined by

$$\lim_{\leftarrow} S = \{((x_\alpha) \in \prod_{\alpha\in\Sigma} X_\alpha | \pi_\alpha^\beta(x_\beta) = x_\alpha,\ for\ all\ \alpha \le \beta\}.$$

If we denote by $\pi_\alpha : \lim_{\leftarrow} S \to X_\alpha$ the restriction of the projection $p_\alpha : \Pi_{\alpha\in\Sigma} X_\alpha \to X_\alpha$ onto $\alpha-$th axis, then it is obtained that $\pi_\alpha = \pi_\alpha^\beta \pi_\beta$, for all $\alpha \le \beta$.

Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $S' = \{X_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ be two multivalued inverse systems. A family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ is a multivalued mapping of S to S' consisting of a monotone function $\sigma : \Sigma' \to \Sigma$ and multivalued mappings $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')} \to 2^{X'_\alpha}$ with the property

$$\pi_{\alpha'}^{\beta'} \circ \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \circ \pi_{\sigma(\alpha')}^{\sigma(\beta')}$$

for all $\alpha' \le \beta'$. Mapping $\{\sigma, \varphi_{\sigma(\alpha')}\}$ induces a limit mapping $\varphi : \lim_{\leftarrow} S \to 2^{\lim_{\leftarrow} S'}$ satisfying, for all $\alpha' \in \Sigma'$,

$$\pi_{\alpha'}\varphi = \varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}.$$

For more details about the inverse limit method, see, e.g., [4], [14]. Now we summarize some useful properties of limits of inverse system.

Proposition 4.1. (see [3], [14], [24]) *Let $S = \{X_m, \pi_m^p, \mathbb{N}\}$ and $S' = \{Y_m, \pi_m^p, \mathbb{N}\}$ be two (multivalued) inverse systems satisfying $X_m \subset Y_m$. If $\varphi : \lim_{\leftarrow} S \to \lim_{\leftarrow} S'$ is a limit map induced by a mapping $\{id, \varphi_m\}$, where $\varphi_m : X_m \to Y_m$, and if* **Fix**(φ_m) *are, for all $m \in N$, R_δ -sets, then the fixed-point set* **Fix**(φ) *of φ is an R_δ-set, too.*

Proposition 4.2. (see [21, Proposition 2.3]) *Let $S = \{X_m, \pi^p_m, \mathbb{N}\}$ be an inverse system. If for each $m \in \mathbb{N}$, X_m is nonempty and compact(resp. relatively compact), then the limit* $\lim_{\leftarrow} S$ *is also nonempty and compact(resp. relatively compact).*

Next, we take some examples for the inverse system needed later. For each $p, m \in \mathbb{N}_+$ with $p \geq m$, consider a projection $\pi^p_m : C([0,p];R^N) \to C([0,m];R^N)$, defined by

$$\pi^p_m(u) = u|^m_0, \ u \in C([0,p];R^N).$$

It is readily checked that $\{C([0,m];R^N), \pi^p_m, \mathbb{N}_+\}$ is an inverse system and its limit is isometrically homeomorphic to $\widehat{C}([0,\infty);R^N)$, so for convenience we set

$$\widehat{C}([0,\infty);R^N) = \lim_{\leftarrow}\{C([0,m];R^N), \pi^p_m, \mathbb{N}_+\}.$$

Let $L_{loc}([0,\infty);R^N)$ be the separated locally convex space consisting of all locally Bocher integrable functions from $\mathbb{R}_+$ to $\mathbb{R}^N$ endowed with a family of seminorms $\{\|\cdot\|^m_1, \ m \in \mathbb{N}_+\}$, defined by

$$\|u\|^m_1 = \int_0^m \|u(s)\|ds, \ m \in \mathbb{N}_+.$$

Similarly, we also obtain that $\{L^1([0,m];R^N), {\pi'}^p_m, \mathbb{N}_+\}$ is an inverse system where $p \geq m$ and

$${\pi'}^p_m(f) = f|^m_0, \ f \in L^1([0,p];R^N).$$

Moreover, it is clear that

$$L_{loc}([0,\infty), R^N) = \lim_{\leftarrow}\{L^1([0,m], R^N), {\pi'}^p_m, \mathbb{N}_+\}.$$

Assume that $\{C([0,m];R^N), \pi^p_m, \mathbb{N}_+\}$ and $\{L^1([0,m];R^N), {\pi'}^p_m, \mathbb{N}_+\}$ are the inverse systems. It follows that the family $\{id, P_m\}$ is a map from $\{L([0,m];R^N), {\pi'}^p_m, \mathbb{N}_+\}$ into $\{C([0,m];R^N), \pi^p_m, \mathbb{N}_+\}$. Indeed, it is easy to infer that

$$\pi^p_m(P_m(f)) = P_m({\pi'}^p_m(f)) \ for \ all \ f \in L([0,p];R^N) \ and \ m \leq p.$$

So the family $\{id, P_m\}$ induces a limit mapping $P_\infty : L_{loc}([0,\infty);R^N) \to \widehat{C}([0,\infty);R^N)$ such that $P_\infty |^m_0 = P_m(f |^m_0)$ for each $f \in L_{loc}([0,\infty);R^N)$ and $m \in \mathbb{N}_+$.

Let $\Gamma(\psi)$ denote the set of all continuous-solutions to the problem (3.1) on $[-\tau,\infty)$ for each $\psi(t) \in C([-\tau,0];R^N)$. We are in the position to present our main result for problem (3.1) on the noncompact intervals in this section.

Theorem 4.1. *If hypotheses (H1)-(H4) are satisfied, then $\Gamma(\psi)$ is an $R_\delta-$set for each $\psi \in C([-\tau,0];R^N)$.*

Proof. For each $x \in Q^m_\psi$, let $F^m_\psi : Q^m_\psi \to 2^{Q^m_\psi}$ be a multivalued map defined by

$$F^m_\psi(x) = P_m \circ (N(x))$$

(the operator N is defined as in Theorem 3.1), where

$$Q^m_\psi = \{x \in C([-\tau,m];R^N) : x(t) = \psi(t), \ \text{for} t \in [-\tau,0] \text{ and } \|x\| \leq u_\psi(t) \text{ for } t \in [0,m]\}.$$

Obviously, $\mathbf{Fix}(F^m_\psi)$ is the solution set of problem (3.1). From Theorem 3.1, $\mathbf{Fix}(F^m_\psi)$ is an R_δ-set. Moreover, note that $\{Q^m_\psi, \pi^p_m, \mathbb{N}_+\}$ is an inverse system, and then

$$Q^\infty_\psi = \{u \in \widehat{C}([-\tau,\infty), R^N), u(t) = \psi(t), \ \text{for } t \in [-\tau,0], \ \text{and } \|u\| \leq x_\psi(t) \text{ for } t \in \mathbb{R}_+\}$$

$$= \lim_{\leftarrow}\{Q_\psi^m, \pi_m^p, \mathbb{N}_+\}.$$

In order to apply Proposition 4.1, what follows to show that the family $\{id, F_\psi^m\}$ is a map of the inverse system $\{Q_\psi^m, \pi_m^p, \mathbb{N}_+\}$ into itself. Let $p, m \in \mathbb{N}_+$ with $p \geq m$ and $u \in Q_\psi^p$, we claim that

$$N(u \mid_{-\tau}^m) \mid_0^m = \{f \mid_0^m : f \in N(u) \mid_0^p\}.$$

The case $p = m$ is obvious, for the case $p > m$, it is easy to check that

$$\{f \mid_0^m, f \in N(u) \mid_0^p\} \subset N(u \mid_{-\tau}^m) \mid_0^m .$$

It remains to show the reverse inclusion. For $f \in N(u \mid_{-\tau}^m) \mid_0^m$ and $g \in N(u) \mid_0^p$, we set

$$\widehat{f} = f(t)\chi_{[0,m]}(t) + g(t)\chi_{(m,p]}(t) \ for\ a.a.\ t \in [0,p].$$

where $\chi(t)$ is the characteristic function. Obviously, it is easy to see that $\widehat{f} \in N(u) \mid_0^p$, which implies $N(u \mid_{-\tau}^m) \mid_0^m \subset \{f \mid_0^m : f \in N(u) \mid_0^p\}$ as desired. Therefore, the map $\{F_\psi^m\}_{m=1}^\infty$ induces the limit mapping $F_\psi^\infty : Q_\psi^\infty \to Q_\psi^\infty$. The fixed-point set of the mapping F_ψ^∞ is the solution set of problem (3.1) defined on the right half. Since $\mathbf{Fix}F_\psi^m$ is an R_δ set for every $m \in \mathbb{N}_+$, according to Proposition 4.1, the set $\mathbf{Fix}F_\psi^\infty$ is an R_δ set, as claimed, which completes the proof.

5. The nonlocal problem on noncompact intervals

In this section, we are concerned with the existence and topological structure of continuous solutions to the following problem:

$$\begin{aligned} &\dot{x} + D(t)x(t) \in H(t, x, x_t),\ for\ a.a.\ I, \\ &x(t) \in \varphi(x)\ for\ t \in [-\tau, 0], \end{aligned} \tag{5.1}$$

where $I := [0, \infty)$. To present our main result,we also need the following conditions.
(H5) $\varphi : \widehat{C}([-\tau,\infty); R^N) \to C([-\tau,0]; R^N)$ is u.s.c. with convex and compact value such and satisfies

(i) $|\varphi(x)| = \{\|v\|_0 : v \in \varphi(x)\} \leq \|\psi\|_0$ for each

$$x \in \Delta_u := \{x \in \widehat{C}([-\tau,\infty), R^N),\ \|x\| \leq u_\psi(t)\ for\ all\ t \in I,\ and$$
$$\|x\| \leq u_\psi(0) = \|\psi\|_0\ for\ all\ t \in [-\tau, 0]\}$$

where $\psi(t) \in C([-\tau,0]; R^N)$, $u_\psi(t)$ is the unique solution of (3.4).

(ii) If $\Theta \subseteq \Delta_u$ is relatively compact in $\widehat{C}([-\tau,\infty); R^N)$, $\varphi(\Theta)$ is relatively compact in $C([-\tau,0]; R^N)$.

Theorem 5.1. *If hypotheses (H1)-(H5) hold, the problem (5.1) has at least one solution.*

Proof. For given $\psi(t) \in C([-\tau,0]; R^N)$, let $\lambda_1 = \|\psi\|_0$, what follows to denote

$$\Lambda_{\lambda_1} := \{\nu \in C([-\tau,0]; R^N) :\ \|\nu\|_0 \leq \lambda_1\}.$$

Keeping some notions in Theorem 3.1 and Theorem 4.1, let us define the multi-valued mapping $\Gamma : \Lambda_{\lambda_1} \to 2^{\widehat{C}([-\tau,\infty), R^N)}$ by $\Gamma(\psi_1) = \mathbf{Fix}F_{\psi_1}^\infty$ for each $\psi_1 \in \Lambda_{\lambda_1}$, then we will show that Γ is an R_δ-map.

Claim 1. *$\Gamma(\psi_1)$ is an R_δ-map for each $\psi_1 \in \Lambda_{\lambda_1}$.*

As proved in Theorem 4.1, $\Gamma(\psi_1)$ is an R_δ-set for each $\psi_1 \in \Lambda_{\lambda_1}$. By Definition 2.8, it suffices to show the upper semi-continuity of Γ. Let Q_α be a nonempty and closed subset of $\widehat{C}(J;R^N)$ where $J := [-\tau,\infty)$. By Proposition 2.2, it suffices to prove that

$$\Gamma^{-1}(Q_\alpha) = \{\psi \in C([-\tau,0];R^N) : \Gamma(\psi) \cap Q_\alpha \neq \emptyset\}$$

is closed. Let $\{\psi_n\}_{n\geq 1} \subseteq \Gamma^{-1}(Q_\alpha)$ and assume $\psi_n \to \psi$ in Λ_{λ_1}. Let $x_n \in \Gamma(\psi_n) \cap Q_\alpha$ for $n \geq 1$, then by the *a prior* estimation of solution in Theorem 3.1, we have $\{x_n\}_{n\geq 1}$ is uniformly bounded in $\widehat{C}(J;R^N)$. Due to the well known Arzelá-Ascoli theorem, we obtain that there exist a subsequence, without of generality, we assume that $x_n \to x$ in Q_α. From Theorem 3.1 in [25], we have

$$\dot{x} + D(t)x \in \overline{\text{conv}}\overline{\lim}\{\dot{x}_n + D(t)x_n\}_{n\geq 1} \subseteq \overline{\text{conv}}\overline{\lim} N(x_n) \subseteq N(x) \ for\ a.a.\ t \in I,$$

which means $x \in \Gamma(\psi)$. Hence, $x \in \Gamma(\psi) \cap Q_\alpha$ i.e. $\Gamma^{-1}(Q_\alpha)$ is closed in $\widehat{C}(J;R^N)$. This completes the proof of u.s.c of Γ. Therefore, $\Gamma : \Lambda_{r_1} \to 2^{\Delta_u}$ is an R_δ- mapping.

Claim 2. *The composition operator* $\Gamma \circ \varphi : \Delta_u \to \Delta_u$ *is also an* R_δ*-mapping.*

From (H5), we know that $\varphi : \Delta_u \to 2^{\Lambda_{r_1}}$ is u.s.c. with compact and convex value. So from the hierarchy for nonempty subsets of a metric space

$$\text{compact} \ + \ \text{convex} \subset \mathrm{R}_\delta \ \text{set},$$

we have that for every $u \in \Delta_u$, $\varphi(u)$ is R_δ-set, so φ is also an R_δ-map. Next, we claim that $\Gamma(\Lambda_{r_1}) \subset \Delta_u$. From the fact that $\varphi : \Delta_u \to 2^{\Lambda_{r_1}}$, we have $\varphi(\Delta_u) \subseteq \Lambda_{r_1}$. Note that $\Gamma(\Lambda_{r_1}) \subseteq \Delta_u$, therefore, $\Gamma(\varphi(\Delta_u)) \subseteq \Delta_u$. Then the following composition is well-defined: $\Gamma_\psi \circ \varphi : \Delta_u \to \Delta_u$. By Proposition 2.6, $\Gamma_\psi \circ \varphi$ is a R_δ-map from Δ_u to Δ_u.

Claim 3. **Fix**$\Gamma \circ \varphi \neq \emptyset$.

What follows to find solutions in Δ_u. To this end, let us show that the multivalued mapping $\Gamma_\psi \circ \varphi$ has a fixed point in Δ_u. It is noted that Δ_u and Λ_{r_1} being convex subset of $\widehat{C}(J;R^N)$ and $C([-\tau,0];R^N)$ respectively, are AR-spaces. Next, we shall show that the set Δ_u is relatively compact in $\widehat{C}(J;R^N)$. Let $m \in \mathbb{N}$ and Λ_r be defined by (4.1) with r instead of r_1. It is noted that

$$\Gamma(\Lambda_r)|_{-\tau}^m \subset \{x \in C([-\tau,m];R^N), x(t) = P(f)(t),\ \|x\| \leq u_r(t),$$
$$for\ a.a.\ t \in [-\tau,m],\ f \in N(Q_\psi^\infty)\ |_0^m\}.$$

So, we can find that $\Gamma(\Lambda_r)|_{-\tau}^m$ is relatively compact in $C([-\tau,m];R^N)$. Now, noticing $\Gamma(\Lambda_r)\ |_{-\tau}^m \subset \Delta_r\ |_{-\tau}^m$, it follows that $\Delta_u|_{-\tau}^m$ is relatively compact. For the arbitrariness of m and by Proposition 4.2, we obtain that Δ_u is relatively compact in $C(J;R^N)$. Hence, due to (H5), $\varphi(\Delta_u)$ is relatively compact in $C([-\tau,0];R^N)$ by the arbitrariness of $m > 0$. Let $\Lambda = \overline{\text{conv}}\varphi(\Delta_u)$, and Λ is compact in $C([-\tau,0];R^N)$. Since Γ is u.s.c. with compact and convex values, by Proposition 2.6, we obtain the compactness of $\Gamma(\Lambda)$, so we conclude from the result $\Gamma \circ (\varphi(\Delta_u)) \subset \Gamma(\Lambda)$.

Since $\Gamma \circ \varphi$ is a R_δ-mapping, therefore, thanks to Theorem 2.1, we conclude that there exists a fixed point of $\Gamma \circ \varphi$ in Δ_u. Moreover, it is readily checked that $x(t) \in \Gamma \circ \varphi(x)$ and $\max\{\|x(t)\|, \|x_t\|_0\} \subset \Delta_u$ for each $t \in \mathbb{R}_+$, which implies that x is a continuous solution of the nonlocal problem (5.1). The proof is completed.

To study the topological structure of solution set in problem (5.1), we need more conditions on multifunction φ.

(H6) For all $x_1, x_2 \in C(J; R^N)$, and all $v_1 \in \varphi(x_1), v_2 \in \varphi(x_2)$, then

$$\|v_1 - v_2\|_0 \leq L\|x_1 - x_2\|_C,$$

where $0 < L < \frac{1}{3}$.

Theorem 5.2. *If hypotheses (H1)-(H6) hold, the solution set of problem (5.1) is an R_δ-set.*

Proof. The proof is similar to that of Theorem 5.1, we only show the differences here. From the above proof, we know that the composition operator $\Gamma \circ \varphi$ is a R_δ mapping from Δ_u to Δ_u. We next show that the multivalued mapping

$$\mathbf{Fix}(\Gamma \circ \varphi) \mid_{-\tau}^{m} = \{x \in C([-\tau, m]; R^N) : x \in \Gamma \circ \varphi(x) \mid_{-\tau}^{m}\}$$

is an R_δ set, or equivalently show that the map $\Gamma \circ \varphi \mid_{-\tau}^{m}$ is a contraction. In fact, for any $u, v \in C([-\tau, m]; R^N)$, there exist $x_u \in \Gamma \circ \varphi(u) \mid_{-\tau}^{m}$, $x_v \in \Gamma \circ \varphi(v) \mid_{-\tau}^{m}$ which means that there exists $x_u(t) = \psi_u(t) \in \varphi(u)$, $x_v(t) = \psi_v(t) \in \varphi(v)$ for $t \in [-\tau, 0]$ and integrable selections $f_u(\cdot) \in H(t, u, u_t)$ and $f_v(\cdot) \in H(t, u, u_t)$ such that

$$\begin{aligned} d(\Gamma \circ \varphi(u), \Gamma \circ \varphi(v)) &= \|x_u - x_v\|_C \\ &\leq \|\psi_u(t) - \psi_v(t)\|_0 + \int_0^m \|f_u(s) - f_v(s)\| ds \\ &\leq (L + 2\int_0^m \mu(s) ds)\|x - y\|_C. \end{aligned} \tag{5.2}$$

Since $0 < \omega := (L + 2\int_0^m \mu(s)ds) < 1$, then the operator $\Gamma \circ \varphi \mid_{-\tau}^{m}$ is a desired contraction with a Lipschitz constant $\omega \in [0, 1)$. Finally, since $\Gamma \circ \varphi \mid_{-\tau}^{m}$ is a contraction with compact and convex values, the set $\mathbf{Fix}(\Gamma \circ \varphi) \mid_{-\tau}^{m}$ is, according to Proposition 2.5, a nonempty, compact AR-space which completes the proof that $\mathbf{Fix}(\Gamma \circ \varphi) \mid_{-\tau}^{m}$ is an R_δ set. By Proposition 4.1, the fixed-point set $\mathbf{Fix}(\Gamma \circ \varphi)$ of $\Gamma \circ \varphi$ is an R_δ-set, too.

Next, we will prove that the solution set of problem (5.1) may be also an R_δ set if the multivalued nonlinearity φ is nonconvex-valued. We need the following hypothesis for the problem (5.1).

(H7) $\varphi : \widehat{C}(J; R^N) \to C([-\tau, 0]; R^N)$ is l.s.c. with closed value satisfying (H5)(i)(ii), and H satisfies the following property: for all $t \in I$, $(x, x_t) \in R^N \times C([-\tau, 0], R^N)$, there exists a constant $r > 0$, such that

$$\langle x, f \rangle < 0,$$

where $f \in H(t, x, x_t)$ in case of $\|x\| > r$.

Theorem 5.3. *If hypotheses (H1)-(H4) and (H6),(H7) holds, the solutions set of problem (5.1) is an R_δ-set.*

Proof. We first show the problem (5.1) has at least one solution. The proof is similar to Theorem 5.1. Firstly, the multivalued map $\varphi : \widehat{C}(J; R^N) \to C([-\tau, 0]; R^N)$ has nonempty closed, decomposability values in $C([-\tau, 0]; R^N)$ and is l.s.c.. We apply the Bressan-Colombo continuous selection theorem (see [5]) and obtain a continuous map $g : \widehat{C}(J; R^N) \to C([-\tau, 0]; R^N)$ satisfying $g(x) \in \varphi(x)$. To finish our proof, we need

to solve the fixed point problem: $x \in \Gamma \circ g(x)$.
We next show that the single continuous g also satisfies the assumption (H5)(i)(ii). Obviously, (H5)(i) holds on g. Because of the continuity of function g, we have that, for each subset $\Theta \subseteq \Delta_r$ where Δ_r is defined as in Theorem 5.1 with r instead of $u(t)$, if Θ is relatively compact in $\widehat{C}(J; R^N)$, $g(\Theta)$ is relatively compact in $C([-\tau, 0]; R^N)$. Thus we obtain that $g : \Delta_r \to \Lambda_r$ is an R_δ-map. By Theorem 5.1, we obtain that Γ is an R_δ-map from Λ_r to Δ_r. Next, we claim that $\Gamma(\Lambda_r) \subset \Delta_r$. In fact, if this is not the case, then we can assume that there exist $\psi \in \Lambda_r$, $u \in \Gamma(\psi)$ and $t_0 > 0$ such that $\|u(t_0)\| > r$. Therefore, we can find $k \in (0, t_0]$ such that $\|u(t)\| \geq r$ on $t \in [t_0 - k, t_0]$, and $\|u(t_0 - k)\| = r$. Since u is continuous and $\|u(0)\| \leq r$, we have

$$\int_{t_0-k}^{t_0} \langle \dot{u}, u \rangle ds + \int_{t_0-k}^{t_0} \langle D(t)u, u \rangle ds = \int_{t_0-k}^{t_0} \langle f, u \rangle ds$$

where $f \in H(t, u, u_t)$. By (H1) and (H7), it follows that $r < \|u(t_0)\| \leq \|u(t_0-k)\| = r$, which is a contradiction. Next we shall show that $\Gamma \circ g : \Delta_r \to \Delta_r$ has a fixed point, which means the problem (5.1) has at least one solution. As in Theorem 5.1, letting $\Lambda = \overline{\text{conv}} g(\Delta_r)$, we can see that Λ is compact in $C([-\tau, 0]; R^N)$. Since Γ is u.s.c. with compact values, we obtain the compactness of $\Gamma(\Lambda)$. Therefore, we conclude from the result $\Gamma \circ (g(\Delta_r)) \subset \Gamma(\Lambda)$ and Theorem 2.1 that there exists a fixed point u of $\Gamma \circ g$ in Δ_r, which implies that there exists a solution of the nonlocal problem (5.1).

For the topological structure of problem (5.1), under the assumption (H6), we can obtain that the solution set of problem (5.1) is an R_δ-set. The following process of proof is similar that of Theorem 5.2 omitted here. The proof is completed.

Remark 5.1. Through the proof of the above theorem, we know that under some conditions the differential inclusion problems can reduce to single differential equation problems by the continuous selection theorem. Thus, under the appropriate conditions, Theorem 5.2 still holds if φ is a single continuous function.

6. Applications

As samples of applications, we present an example of a time delay boundary control system defined on right half-line, with a priori feedback. Let $T = [0, \infty)$, $\dot{z} = (\dot{z}_1, \dot{z}_2, \ldots, \dot{z}_N)$. We consider the following control system:

$$\begin{aligned} &\dot{z}(t) + A(t)z(t) = g(t, z(t), z_t)u_1(t), \ for\ a.a.\ t \in T \\ &z(t) = u_2(t)\ for\ t \in [-\tau, 0], \\ &u_1(t) \in U_1(t, z(t)),\ u_2(t) \in U_2(t, z(t))\ for\ a.a.\ t \in T, \end{aligned} \tag{6.1}$$

where $A : T \to R^{N \times N}$ is a integrable positive semi-definite matrix, $g(t, z, s) : T \times R^N \times C([-\tau, 0]; R^N) \to R$ is a Carathéodory function, $z_t = z(t + \theta)$, $\theta \in [-\tau, 0]$, and $U_1(t, z), U_2(t, z) \to 2^{R^N}$ are two multivalued maps.

The hypotheses on the data (6.1) are the following:

(H8) $g : T \times R^N \times C([-\tau, 0]; R^N) \to R$ is a carathéodory function such that for almost all $t \in T$, $x, y \in R^N$ and $s_1, s_2 \in C([-\tau, 0]; R^N)$,

$$\|g(t, x, s_1) - g(t, y, s_2)\| \leq \eta(t)(\|x - y\| + \|s_1 - s_2\|_0),$$

where $\eta(t) \in L^1_+(T)$, with $\|\eta\|_{L^1} < \frac{1}{3}$.

(H9) $U_1 : T \times R^N \to R^N$ and $U_2 : C([-\tau, \infty), R^N) \to C([-\tau, 0]; R^N)$ are two upper Carathéodory functions with compact convex values and satisfies:
(i) for each subset Θ is relatively compact in $C([-\tau, \infty); R^N)$, $U_2(\Theta)$ is relatively com- pact in $C([-\tau, 0]; R^N)$;
(ii) for every $t \in T$ and all $s \in R^N, v \in C([-\tau, 0]; R^N)$, $|U_1(t, s)| \leq \alpha$, $|U_2(v)| \leq \|\psi\|$, where $0 \leq \alpha < 1$, $\psi(t) \in C([-\tau, 0]; R^N)$.

Let $\varphi : \widehat{C}([-\tau, \infty); R^N) \to C([-\tau, 0]; R^N)$ be a multifuction defined by

$$\varphi(z) = \{v \in C([-\tau, 0]; R^N) : v(t) = u(t), u(t) \in U_2(z(t))\}.$$

Using hypotheses (H9), it is straightforward to check that φ satisfies hypotheses (H5). Also, we define $G : T \times R^N \times C([-\tau, 0]; R^N) \to R^N$ as

$$G(t, x, x_t) = \{v \in R^N : v = g(t, z(t), z_t)u_1(t), u_1(t) \in U_1(t, z(t)),\ a.a.\ t \in T\}.$$

Using hypotheses (H8), we check that G satisfies hypotheses (H2) and (H3). Rewrite problem (6.1) in the following equivalent evolution inclusion form:

$$\begin{aligned} &\dot{z} + A(t)z \in G(t, z, z_t),\ for\ a.a.\ t \in T, \\ &z(t) \in \varphi(z)\ for\ t \in [-\tau, 0]. \end{aligned} \tag{6.2}$$

We can apply Theorem 5.1 on problem (6.1) and obtain:

Theorem 6.1. *If the hypothesis (H8) and (H9) holds, then the control problem* (6.1) *has at least one solution.*

If we strengthen our hypothesis on the continuity of $U_2(z)$, we can also obtain that the attainable set is an R_δ-set for the control problem (6.1) .

(H10) For all $s_1, s_2 \in C(T; R^N)$ and all $v_1 \in U_2(s_1), v_2 \in U_2(s_2)$,

$$\|v_1 - v_2\|_0 \leq \gamma \|s_1 - s_2\|_C,$$

where $0 < \gamma < \frac{1}{3}$. By (H10), we can apply Theorem 5.2 and obtain the following theorem:

Theorem 6.2. *If the hypothesis (H8)-(H10) hold, then the attainable set of problem* (6.1) *is an R_δ set in $C(T; R^N)$.*

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