Fixed Point Theory, 19(2018), No. 1, 107-122
DOI 10.24193/fpt-ro.2018.1.09
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A STRONGLY CONVERGENT MODIFICATION OF THE PROXIMAL POINT ALGORITHM IN NONSMOOTH BANACH SPACES

SOUHAIL CHEBBI*, NAJLA ALTWAIJRY* AND HONG-KUN XU**,[1]

*Department of Mathematics, College of Science, King Saud University
P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mails: schebbi@ksu.edu.sa, najla@ksu.edu.sa

**Department of Mathematics, School of Science
Hangzhou Dianzi University, Hangzhou 310018, China
E-mail: xuhk@hdu.edu.cn

Abstract. Rockafellar's proximal point algorithm is known to be not strongly convergent in general in an infinite-dimensional Hilbert space. Effort has thus been made to modify this algorithm so that strong convergence is guaranteed. In this paper we provide a strongly convergent modification of Rockafellar's proximal point algorithm in a uniformly convex Banach space which is not necessarily smooth.

Key Words and Phrases: Maximal monotone operator, proximal point algorithm, strong convergence, generalized projection, uniformly convex Banach space, zero point.

2010 Mathematics Subject Classification: 90C25, 47H05, 90C30, 46B20, 47H09.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let T be a maximal monotone operator with domain $D(T)$ and range $R(T)$ in H. An important problem in the theory of maximal monotone operators is the inclusion problem

$$0 \in Tx \tag{1.1}$$

which has applications in various disciplines (see [4, 5, 7, 17, 30, 32]). A typical example of (1.1) is the minimization problem

$$\min_{x \in H} f(x), \tag{1.2}$$

where $f : H \to \overline{\mathbb{R}} := (-\infty, \infty]$ is a proper, lower semicontinuous, and convex function. It is known that $\hat{x} \in \mathrm{dom}(f)$ solves (1.2) if and only if it is a solution to the inclusion

$$0 \in \partial f(\hat{x}), \tag{1.3}$$

[1] Corresponding author.

where ∂f is the subdifferential of f. It is known that ∂f is a maximal monotone operator in H.

In his seminal paper [29], Rockafellar proposed his proximal point algorithm (PPA) which extends Martinet's algorithm [24] to general Hilbert spaces. Rockafellar's PPA, an iterative method for solving the inclusion problem (1.1), generates a sequence $\{x_n\}$ according to the recursion formula:

$$x_n + e_n \in x_{n+1} + \lambda_n T x_{n+1}, \tag{1.4}$$

where the initial guess $x_0 \in H$ is arbitrarily selected, $\{e_n\}$ is a sequence of errors, and $\{\lambda_n\}$ is a sequence of parameters.

Rockafellar proved the following result.

Theorem 1.1. (Rockafellar [29].) *Assume the solution set S of the inclusion (1.1) is nonempty. Assume the error sequence $\{e_n\}$ satisfies the criterion:*

$$\|e_n\| \le \varepsilon_n \text{ for all } n \text{ and } \sum_{n=0}^{\infty} \varepsilon_n < \infty. \tag{1.5}$$

Assume in addition the parameter sequence $\{\lambda_n\}$ is such that $\inf_{n\ge 0} \lambda_n > 0$. *Then the sequence $\{x_n\}$ generated by PPA (1.4) converges weakly to a point in S.*

Since its publication in Rockafellar [29] in 1976, proximal point methods have become active and popular in optimization and variational inequalities; see [8, 9, 10, 11, 12, 15, 16, 22, 27, 28, 31] and the references therein.

Rockafellar [29] posed a question whether the sequence $\{x_n\}$ generated by his PPA (1.4) can be strongly convergent in the setting of infinite-dimensional Hilbert spaces?

A negative answer to this question was given by Güler [19] in 1991. (A similar and relevant counterexample to the strong convergence of trajectories of contraction semigroups was however constructed [18] as early as in 1979; see also [36]).

Recently, basing upon Hundal's counterexample [20] to the strong convergence of alternating projections [6] onto two intersecting closed convex subsets of an infinite-dimensional Hilbert space, Bauschke, et al [3] constructed another counterexample to Rockafellar's question (see also [2] for more examples based on Hundal's example).

Two questions are thus of interest to investigate:

(1) What conditions are sufficient to guarantee strong convergence of the sequence $\{x_n\}$ generated by PPA (1.4)?
(2) How to modify Rockafellar's PPA (1.4) so that strong convergence is guaranteed?

This latter question was first attacked in [33] and then in [35, 23] in an infinite-dimensional Hilbert space. A strongly convergent modification in the setting of uniformly convex and uniformly smooth Banach spaces was given in [21].

It is the aim of this paper to give another strongly convergent modification of Rockafellar's PPA in the setting of uniformly convex Banach spaces which are not necessarily smooth. The lack of smoothness brings us difficulties in our argument in constructing a generalized projection in contrast to the case of uniform smoothness in [21]. Thus another contribution of this paper is how to find appropriate generalized projections in nonsmooth Banach spaces.

2. Preliminaries

2.1. **Uniformly convex and uniformly smooth Banach spaces.** Let X be a real Banach space with dual X^*. The modulus of convexity of X is defined as

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon\right\}, \quad \varepsilon \in [0,2].$$

Recall that X is said to be uniformly convex if

$$\delta_X(\varepsilon) > 0 \quad \text{for all } \varepsilon \in (0,2].$$

Hilbert spaces H and l^p spaces (and $L^p[a,b]$) for $1 < p < \infty$ are all uniformly convex. As a matter of fact, the moduli of these spaces are

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2}$$

and

$$\delta_{l^p}(\varepsilon) = \delta_{L^p}(\varepsilon) \begin{cases} = 1 - \sqrt[p]{1 - \left(\frac{\varepsilon}{2}\right)^p} & \text{if } 2 \le p < \infty, \\ \ge 1 - \sqrt{1 - (p-1)\left(\frac{\varepsilon}{2}\right)^2} & \text{if } 1 < p < 2. \end{cases}$$

We need inequality characterizations for uniform convexity.

Proposition 2.1. [34] *Let X be a real Banach space. Then either of the following two statements characterizes uniform convexity of X:*

(i) *for each fixed real number $r > 0$, there exists a strictly increasing continuous function $h : [0,\infty) \to [0,\infty)$, $h(0) = 0$, satisfying the property:*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)h(\|x-y\|)$$

for all $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$, and $0 \le t \le 1$;

(ii) *for each fixed real number $r > 0$, there exists a strictly increasing continuous function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, satisfying the property:*

$$\|x + y\|^2 \ge \|x\|^2 + 2\langle y, j\rangle + g(\|y\|)$$

for all $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$, and for $j \in Jx$.

Denote by S_X the unit sphere of a Banach space X (i.e., $S_X = \{x \in X : \|x\| = 1\}$). Then we say that

- X is smooth if the
$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$
exists for every $x, y \in S_X$;
- X is uniformly smooth if the limit (2.1) exists and is attained uniformly in $x, y \in S_X$.

Alternatively we can use duality map to characterize smoothness and uniform smoothness. Recall that the (normalized) duality map $J : X \to X^*$ is defined by

$$Jx = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

It is known that
$$Jx = \partial \frac{1}{2}\|x\|^2, \quad x \in X.$$
(Here ∂ denotes the subdifferential in the sense of convex analysis.) The following facts are well-known.

- X is smooth if and only if J is single-valued.
- X is uniformly smooth if and only of J is norm-to-norm uniformly continuous over bounded sets of X.

More details regarding convexity, smoothness and duality maps can be found in [13].

2.2. **Generalized projections.** Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. It is then well-known that we can define the metric (or nearest point) projection $P_C : X \to C$ by assigning to each point $x \in X$ the unique point $P_C x$ in C which satisfies the property
$$\|x - P_C x\| = \min_{y \in C} \|x - y\|. \tag{2.2}$$
It is also well-known that if X is a Hilbert space H, then P_C is nonexpansive (i.e., $\|P_C x - P_C y\| \le \|x - y\|$ for all $x, y \in H$), which plays a crucial role in Hilbert space techniques. However, metric projections are no longer nonexpansive in Banach spaces with dimensions exceeding one. Instead, the so-called generalized projections seem more prevail than metric projections. To define generalized projections, assume in addition that X is smooth so that the duality map J is single-valued. Consequently, we can define [1, 21] a function φ by
$$\varphi(x, y) = \frac{1}{2}\|x\|^2 - \langle x, Jy \rangle + \frac{1}{2}\|y\|^2. \tag{2.3}$$
Since X is smooth and uniformly convex, $\varphi(\cdot, y)$ is, for each fixed $y \in X$, smooth and (uniformly) convex. Hence the minimization problem
$$\min_{x \in C} \varphi(x, y) = \min_{x \in C} \left(\frac{1}{2}\|x\|^2 - \langle x, Jy \rangle + \frac{1}{2}\|y\|^2 \right) \tag{2.4}$$
is uniquely solvable. Let $Q_C y \in C$ be its unique solution in C. Namely, $Q_C y$ is the only point in C with the property
$$\varphi(Q_C y, y) = \min_{x \in C} \varphi(x, y). \tag{2.5}$$
Thus we have defined an operator $Q_C : X \to C$ which assigns to each $y \in X$ the unique point $Q_C y$ in C via (2.5). This operator Q_C is referred to as the generalized projection from X onto C. It is easily seen that if X is a Hilbert space, then $\varphi(x, y) = \frac{1}{2}\|x - y\|^2$ and the generalized projection Q_C is reduced to the metric projection P_C as defined in (2.2).

Observing that the minimization (2.4) is convex and differentiable with gradient (with respect to the argument x)
$$\nabla \varphi(x, y) = Jx - Jy,$$
we immediately have the following characterization of Q_C.

Proposition 2.2. *Let $z \in C$ and $y \in X$. Then $z = Q_C y$ if and only if there holds the relation:*

$$\langle x - z, Jz - Jy \rangle \geq 0, \quad x \in C. \tag{2.6}$$

Proof. The variational inequality (2.6) is actually the optimality condition for the differentiable convex minimization (2.4). □

2.3. **Maximal monotone operators.** Let X be a real Banach space X with dual X^* and let $J : X \to X^*$ be the (normalized) duality map.

Recall that a (possibly multivalued) operator T with domain $D(T)$ in X and range $R(T)$ in X^*, respectively, is said to be monotone if

$$\langle x - x', \xi - \xi' \rangle \geq 0$$

for all $x, x' \in D(T)$ and $\xi \in Tx$ and $\xi' \in Tx'$. In other words, the graph

$$G(T) := \{(x, \xi) \in X \times X^* : x \in D(T), \xi \in Tx\}$$

is a monotone set in the product space $X \times X^*$.

A monotone operator is said to be maximal monotone if its graph is not contained properly in the graph of any other monotone operator. In other words, a monotone operator T is maximal monotone if and only if the following relation holds:

$$(x', \xi') \in X \times X^*, \ \langle x - x', \xi - \xi' \rangle \geq 0 \ \forall\, (x, \xi) \in G(T) \quad \Longrightarrow \quad (x', \xi') \in G(T).$$

A typical example of maximal monotone operators is the subdifferential ∂f of a proper lower-semicontinuous convex function $f : X \to \overline{\mathbb{R}}$.

Let $\lambda > 0$ and let J_λ^T (or simply J_λ if no confusions arise) be the resolvent of T; that is,

$$J_\lambda^T = (J + \lambda T)^{-1}.$$

It is known that a monotone operator $T : D(T) \subset X \to X^*$ is maximal monotone if and only if the range of the operator $J + rT$ is the whole space X^*; that is, $R(J + rT) = X^*$ for any $r > 0$. In other words, if T is maximal monotone, then the resolvent J_λ^T is single-valued and defined over the entire space X^*.

Let S denote the set of zeros of a maximal monotone operator T; that is,

$$S = T^{-1}0 = \{x \in D(T) : 0 \in Tx\}.$$

Note that if X is smooth, then

$$S = \mathrm{Fix}(J_\lambda^T J) := \{x \in X : J_\lambda^T Jx = x\}.$$

If X is a Hilbert space, the resolvent J_λ^T is nonexpansive:

$$\|J_\lambda^T x - J_\lambda^T y\| \leq \|x - y\|, \quad x, y \in X.$$

Moreover, we can equivalently rewrite Rockafellar's PPA (1.4) as

$$x_{n+1} = J_{\lambda_n}^T (x_n + e_n) = (I + \lambda_n T)^{-1}(x_n + e_n), \quad n = 0, 1, \cdots. \tag{2.7}$$

Throughout the rest of this paper, we will use the notation:

- '$x_n \rightharpoonup x$' means that the sequence x_n is weakly convergent to x;
- '$x_n \to x$' means that the sequence x_n is strongly convergent to x;

- $\omega_w(x_n) = \{x : x \text{ is the weak limit of a subsequence of } \{x_n\}\}$ is the weak ω-limit set of the sequence $\{x_n\}$.

3. Existing strong convergent modifications

Since Rockafellar's PPA (1.4) (or (2.7)) does not have strong convergence in general in an infinite-dimensional Hilbert space, effort has been made to modify PPA (2.7) so as to have strong convergence. Three strong convergent modifications of PPA (2.7) are available in literature.

3.1. **Solodov and Svaiter's modification via metric projections.** Additional projections applied to PPA (2.7) can force strong convergence. This is the idea of Solodov and Svaiter [33]. (Their idea can also be adapted to build up other strongly convergent iterative algorithms for nonlinear operator equations (see [25, 26].) Define a sequence $\{x_n\}$ by the following modified proximal point algorithm (mPPA).

$$\begin{cases} x_0 \in H, \\ y_n = (I + \lambda_n T)^{-1} x_n, \\ v_n = \frac{1}{\lambda_n}(x_n - y_n), \\ V_n = \{z \in H : \langle z - y_n, v_n \rangle \le 0\}, \\ W_n = \{z \in H : \langle z - x_n, x_0 - x_n \rangle \le 0\}, \\ x_{n+1} = P_{V_n \cap W_n} x_0. \end{cases} \tag{3.1}$$

Theorem 3.1. [33] *Let H be a real Hilbert space. Let $\{x_n\}$ be defined by mPPA (3.1). Assume that the parameter sequence $\{\lambda_n\}$ is bounded below from zero. Then $\{x_n\}$ converges strongly to $P_S x_0$, the metric projection of x_0 onto the solution set S.*

3.2. **Modification via additional contractions: the contraction-proximal point algorithm.** Since the resolvents which define Rockafellar's PPA (2.7) are nonexpansive, appropriate convex combinations of these resolvents with contractions turn out to be contractive, which can lead to strong convergence. This is the idea in the papers [35, 23]. More precisely, we define a sequence $\{x_n\}$ by the following so-called contraction-proximal point algorithm:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J^T_{\lambda_n}(x_n) + e_n, \quad n \ge 0, \tag{3.2}$$

where for each n, $\alpha_n \in (0, 1)$, $\lambda_n > 0$ and e_n is an error.

Theorem 3.2. [35, 23] *Let $\{x_n\}$ be generated by the contraction-proximal point algorithm (3.2). Assume that*

(a) $\lim_n \alpha_n = 0$;
(b) $\sum_n \alpha_n = \infty$;
(c) *either* $\sum_n |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_n \alpha_n/\alpha_{n+1} = 1$;
(d) $0 < \bar{\lambda} \le \lambda_n \le \tilde{\lambda}$ $(\forall n)$;
(e) $\sum_n |\lambda_{n+1} - \lambda_n| < \infty$;
(f) $\sum_n \|e_n\| < \infty$.

Then $\{x_n\}$ converges strongly to $P_S u$, the metric projection of u onto the solution set S.

3.3. **Kamimura and Takahashi's modification in uniformly convex and uniformly smooth Banach spaces.** Let X be a uniformly convex and uniformly smooth Banach space and let $T : X \to X^*$ be a maximal monotone operator such that the solution set $S = T^{-1}(0) = \{x \in D(T) : 0 \in Tx\} \neq \emptyset$. Kamimura and Takahashi [21] introduced a modified PPA which generates a sequence $\{x_n\}$ by the algorithm:

$$\begin{cases} x_0 \in X, \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n), \quad v_n \in Ty_n, \\ V_n = \{z \in X : \langle z - y_n, v_n \rangle \leq 0\}, \\ W_n = \{z \in X : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = Q_{V_n \cap W_n} x_0. \end{cases} \tag{3.3}$$

Note: Since T is maximal monotone, $R(J + rT) = X^*$ for all $r > 0$. In particular, $Jx_n \in R(J+r_nT)$ for $r_n > 0$. Thus, there is $y_n \in D(T)$ such that $Jx_n \in Jy_n+r_nTy_n$. Set $v_n := \frac{1}{r_n}(Jx_n - Jy_n)$. Then $v_n \in Ty_n$ and $0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n)$.

Theorem 3.3. [21] *Let X be a uniformly convex and uniformly smooth Banach space. Assume* $\liminf_{n\to\infty} r_n > 0$. *Then the sequence $\{x_n\}$ generated by mPPA (3.3) converges strongly to the generalized projection $Q_S x_0$ of x_0 onto the solution set S.*

4. A new strongly convergent modification of PPA

In this section we present a new strongly convergent modification of Rockafellar's PPA (1.4) in the setting of uniformly convex Banach spaces which are not necessarily smooth. As opposed to Kamimura and Takahashi's modification (3.3), the difficulty lies in how to define a generalized projection Q_C without smoothness of the underlying Banach space X.

4.1. **Generalized projections in nonsmooth Banach spaces.** Let X be a real Banach space and C be a nonempty closed convex subset of X. In this subsection we define a generalized projection Q_C. Towards this we let $J : X \to X^*$ be the normalized duality map. Since we do not assume smoothness of X, J is possibly multivalued. Consequently, we cannot use (2.3) to define the function φ, minimizing which defines the generalized projection Q_C in the smooth case. To overcome possible difficulties arising from nonsmoothness of X (i.e., multivaluedness of J), we introduce a function $\tau(x, y)$ by

$$\tau(x, y) = \max\{\langle x, -j\rangle : j \in Jy\} = \max\{\langle x, j\rangle : j \in J(-y)\}, \quad x, y \in X. \tag{4.1}$$

(i.e., $\tau(x, y)$ is the support function at x to the weakly compact convex set $J(-y)$.) It is clear that $\tau(x, y)$ is reduced to the pairing $-\langle x, Jy\rangle$ if Jy is a singleton.

We then define a function $\psi(x, y)$ on $X \times X$ by

$$\psi(x, y) = \frac{1}{2}\|x\|^2 + \tau(x, y) + \frac{1}{2}\|y\|^2, \quad x, y \in X \tag{4.2}$$

(We also write $\psi_y(x) = \psi(x, y)$ whenever needed.)

Note that if X is smooth, then Jy is a singleton and ψ becomes

$$\psi(x, y) = \frac{1}{2}\|x\|^2 - \langle x, Jy\rangle + \frac{1}{2}\|y\|^2$$

which is the previous function $\varphi(x, y)$ defined in (2.3).

Some properties of τ are available immediately.

Lemma 4.1. *Let X be a Banach space and let $y \in X$. Then $\tau(\cdot, y) : X \to \mathbb{R}$ satisfies the properties:*

(i) • $\tau(x + x', y) \le \tau(x, y) + \tau(x', y)$ *for all* $x, x' \in X$;
  • $\tau(\lambda x, y) = \lambda\tau(x, y)$ *for all* $x \in X$ *and* $\lambda \ge 0$.
  In particular, $\tau(\cdot, y)$ *is convex.*

(ii) $\tau(\cdot, y)$ *is Lipschitz continuous:*

$$|\tau(x, y) - \tau(x', y)| \le \|y\|\|x - x'\|, \quad x, x' \in X.$$

(iii) $\partial_x\tau(x, y) = -J_x^* y$, *where* $\partial_x\tau(x, y)$ *is the subdifferential of* τ *with respect to* x, *and*

$$J_x^* y = \{j^* \in Jy : \tau(x, y) = \langle x, -j^*\rangle\} \tag{4.3}$$

which is closed convex.

Proof. Property (i) follows trivially. To see property (ii), select $j \in Jy$ such that $\tau(x, y) = \langle x, -j\rangle$. We then have

$$\begin{aligned} \tau(x, y) &= \langle x - x', -j\rangle + \langle x', -j\rangle \\ &\le \|y\|\|x - x'\| + \tau(x', y). \end{aligned}$$

Similarly by interchanging x and x' we get $\tau(x', y) \le \|y\|\|x - x'\| + \tau(x, y)$, and (ii) is thus proved.

Finally, property (iii) follows from a more general result (cf. [5, 14]) which states that if K^* is a weak*-compact convex subset of X^* and h is the support function to K^* defined by

$$h(x) = \max\{\langle x, x^*\rangle : x^* \in K^*\}, \quad x \in X,$$

then the subdifferential of h is given by

$$\partial h(x) = \{x^* \in K^* : h(x) = \langle x, x^*\rangle\}.$$

Setting $K^* = -Jy$ yields (4.3). □

Some properties of ψ are included in the following proposition.

Proposition 4.2. *Let X be a real Banach space (not necessarily smooth).*

(i) *There holds the relation*

$$\frac{1}{2}(\|x\| - \|y\|)^2 \le \psi(x, y) \le \frac{1}{2}(\|x\| + \|y\|)^2, \quad x, y \in X.$$

(ii) *For any fixed* $y \in X$, $\psi(x, y)$ *is continuous and convex in* x.

(iii) *Given* $y \in X$, *the subdifferential (with respect to the first argument) of* ψ_y *at* x *is*

$$\partial\psi_y(x) = Jx - J_x^* y,$$

where $J_x^* y$ *is given in (4.3).*

(iv) *If* X *is strictly convex, then, for each* $y \in X$, ψ_y *is strictly convex.*

(v) *If* X *is uniformly convex and* $\{x_n\}$ *and* $\{y_n\}$ *are bounded sequences in* X, *then there holds the relation:*

$$y_n - x_n \to 0 \quad \Longleftrightarrow \quad \psi(x_n, y_n) \to 0.$$

(vi) *If X is uniformly convex, then ψ is locally uniformly convex in x. That is, for any convex bounded subset K of X, there holds the inequality*

$$\psi_y(tx + (1-t)x') \leq t\psi_y(x) + (1-t)\psi_y(x') - t(1-t)h(\|x - x'\|) \tag{4.4}$$

for all $t \in (0,1)$ and $x, y \in K$, where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing continuous function with $h(0) = 0$ (and independent of y).

Proof. (i) This easily follows from the fact that $|\tau(x,y)| \leq \|x\| \cdot \|y\|$.

(ii) This is trivial since $\tau(\cdot, y)$ is continuous and convex.

(iii) By Lemma 4.1(iii), we have

$$\partial\psi_y(x) = \partial\frac{1}{2}\|x\|^2 + \partial\tau(x,y) = Jx - J_x^* y.$$

(iv) If X is strictly convex, $\|\cdot\|^2$ is strictly convex, so is $\psi_y(\cdot)$ for $\tau(\cdot, y)$ is convex.

(v) Assume now X is uniformly convex. Assume $\psi(x_n, y_n) \to 0$. Let $r > 0$ be such that the closed ball $B_r = \{u \in X : \|u\| \leq r\}$ contains all the points of $\{x_n\}$, $\{y_n\}$ and $\{x_n - y_n\}$. Applying Proposition 2.1(ii), we obtain, for each $j_n \in J(y_n)$,

$$\begin{aligned} \|x_n\|^2 &= \|y_n + (x_n - y_n)\|^2 \\ &\geq \|y_n\|^2 + 2\langle x_n - y_n, j_n\rangle + g(\|x_n - y_n\|) \\ &= -\|y_n\|^2 + 2\langle x_n, j_n\rangle + g(\|x_n - y_n\|). \end{aligned}$$

It then follows from the definition of ψ that

$$g(\|x_n - y_n\|) \leq \|x_n\| + 2\tau(x_n, -j_n) + \|y_n\|^2 \leq 2\psi(x_n, y_n) \to 0.$$

Therefore $\|x_n - y_n\| \to 0$.

Conversely assume $\|x_n - y_n\| \to 0$. That $\psi(x_n, y_n) \to 0$ now follows from the following computations:

$$\begin{aligned} 2\psi(x_n, y_n) &= \|x_n\|^2 + 2\tau(x_n, y_n) + \|y_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, j_n\rangle + \|y_n\|^2 \quad \text{for some } j_n \in J(y_n) \\ &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, j_n\rangle \\ &\leq (\|x_n\| - \|y_n\|)(\|x_n\| + \|y_n\|) + 2\|x_n - y_n\|\|y_n\| \\ &\leq 4r\|x_n - y_n\| \to 0. \end{aligned}$$

(vi) Take $r > 0$ so that $B_r \supset K$. Since X is uniformly convex, we can apply Proposition 2.1(i) to get, for $x, x' \in K$,

$$\begin{aligned}
\psi_y(tx+(1-t)x') &= \frac{1}{2}\|tx+(1-t)x'\|^2+\tau(tx+(1-t)x',y)+\frac{1}{2}\|y\|^2 \\
&\le \frac{1}{2}(t\|x\|^2+(1-t)\|x'\|^2-t(1-t)h(\|x-x'\|)) \\
&\quad +t\tau(x,y)+(1-t)\tau(x',y)+\frac{1}{2}\|y\|^2 \\
&= t\left(\frac{1}{2}\|x\|^2+\tau(x,y)+\frac{1}{2}\|y\|^2\right) \\
&\quad +(1-t)\left(\frac{1}{2}\|x'\|^2+\tau(x',y)+\frac{1}{2}\|y\|^2\right) \\
&\quad -\frac{1}{2}t(1-t)h(\|x-x'\|) \\
&= t\psi_y(x)+(1-t)\psi_y(x')-t(1-t)\tilde{h}(\|x-x'\|),
\end{aligned}$$

where $\tilde{h}=(1/2)h$. □

If X is a reflexive strictly convex Banach space, then, for any given $y\in X$, ψ_y is a strictly convex continuous function; moreover, it is also coercive: $\psi_y(x)\to\infty$ whenever $\|x\|\to\infty$. Therefore, if C is a nonempty closed convex subset of X, then there exists a unique point $z\in C$ with the property:

$$\psi(z,y)=\min\{\psi(x,y):x\in C\}. \tag{4.5}$$

This leads to the following definition.

Definition 4.3. Let X be a reflexive strictly convex Banach space and let C be a nonempty closed convex subset of X. Define a map $Q_C:X\to C$ by setting $Q_Cy=z$, where $z\in C$ is the unique point in C that solves the minimization (4.5). Hence, Q_Cy is the only point in C satisfying the property

$$\psi(Q_Cy,y)=\min\{\psi(x,y):x\in C\},\quad y\in X.$$

This operator Q_C is referred to as the generalized projection from X onto C.

Lemma 4.4. *Let X be a Banach space and C be a weakly compact convex subset of X. Let g be a continuous convex function on C. Given $z\in C$. Then z is a minimizer of g over C, i.e.,*

$$g(z)=\min\{g(x):x\in C\} \tag{4.6}$$

if and only if the following optimality condition holds:

$$\max_{\xi\in\partial g(z)}\langle v-z,\xi\rangle\ge 0,\quad \forall v\in C. \tag{4.7}$$

Proof. Denote by $g'_+(z;u)$ the (right) directional derivative at z along direction u. Namely,

$$g'_+(z;u)=\lim_{t\downarrow 0}\frac{g(z+tu)-g(z)}{t}.$$

Note that (cf. [14])

$$g'_+(z;u)=\max_{\xi\in\partial g(z)}\langle u,\xi\rangle.$$

Since g is convex, (4.6) holds if and only if, for any $v \in C$,

$$\begin{aligned} 0 &\leq \lim_{t \downarrow 0} \frac{g(z + t(v - z)) - g(z)}{t} \\ &= g'_+(z; v - z) \\ &= \max_{\xi \in \partial g(z)} \langle v - z, \xi \rangle. \end{aligned}$$

□

A characterization for the generalized projection $Q_C y$ is now ready as follows.

Proposition 4.5. *Let X be a real reflexive strictly convex Banach space and C be a closed convex subset of X. Given $y \in X$ and $z \in C$. Then $z = Q_C y$ if and only if there holds the optimality condition:*

$$\max_{\xi \in Jz - J_z^* y} \langle v - z, \xi \rangle \geq 0, \quad \forall v \in C. \tag{4.8}$$

Proof. Fix $y \in X$. Since the subdifferential of $\psi_y(z) = \frac{1}{2}\|z\|^2 + \tau(z, y) + \frac{1}{2}\|y\|^2$ is given by $\partial \psi_y(z) = \partial \frac{1}{2}\|z\|^2 + \partial \tau(z, y) = Jz - J_z^* y$, we can apply Lemma 4.1 to get that $z = Q_C y$ if and only if there holds, for every $v \in C$,

$$\begin{aligned} 0 &\leq \max_{\xi \in \partial \psi_y(z)} \langle v - z, \xi \rangle \\ &= \max_{\xi \in Jz - J_z^* y} \langle v - z, \xi \rangle. \end{aligned}$$

This is (4.8). □

The lemma below gives us convenience when proving strong convergence of sequences in a uniformly convex Banach space.

Lemma 4.6. *Let X be a real uniformly convex Banach space and K be a nonempty closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in X and $u \in X$. Let $q = Q_K u$ be the generalized projection of u onto K. Assume that $\{x_n\}$ satisfies the conditions*

(i) *$\omega_w(x_n) \subset K$ and*

(ii) *$\psi(x_n, u) \leq \psi(q, u)$ for all n.*

Then $x_n \to q$.

Proof. Since X is reflexive and $\{x_n\}$ is bounded, $\omega_w(x_n)$ is nonempty. Noticing the weak lower semi-continuity of $\psi(\cdot, u)$, we derive from condition (ii) that

$$\psi(v, u) \leq \psi(q, u) \quad \forall v \in \omega_w(x_n).$$

However, since $\omega_w(x_n) \subset K$ and $q = Q_K u$, we must have $v = q$ for all $v \in \omega_w(x_n)$ since q is the unique minimizer of $\psi(\cdot, u)$ over K. Thus $\omega_w(x_n) = \{q\}$ and $x_n \rightharpoonup q$.

To see $x_n \to q$, we observe that the inequality $\psi(x_n, u) \leq \psi(q, u)$ in condition (ii) is actually equivalent to the following one

$$\|x_n\|^2 \leq \|q\|^2 + 2[\tau(q, u) - \tau(x_n, u)]. \tag{4.9}$$

Since $\tau(\cdot, u)$ is weakly lower semicontinuous, the fact that $x_n \rightharpoonup q$ implies that $\tau(q, u) \leq \liminf_{n\to\infty} \tau(x_n, u)$. It then follows from (4.9) that

$$\begin{aligned} \limsup_{n\to\infty} \|x_n\|^2 &\leq \|q\|^2 + 2[\tau(q,u) - \liminf_{n\to\infty} \tau(x_n,u)] \\ &\leq \|q\|^2. \end{aligned}$$

Consequently, $\|x_n\| \to \|q\|$, and the uniform convexity of X implies that $x_n \to q$. □

4.2. **The algorithm and its convergence.** Throughout this subsection, X is a real uniformly convex Banach space and T is a maximal monotone operator with domain $D(T)$ in X and range $R(T)$ in X^*, respectively. Assume $S := T^{-1}(0) \neq \emptyset$.

We now introduce our algorithm which generates a sequence $\{x_n\}$ in the following manner.

(i) The initial guess $x_0 \in X$ is arbitrary.
(ii) Once x_n has been defined, define two half spaces W_n and H_n by

$$W_n = \left\{ z \in X : \max_{\substack{\xi_0 \in J^*_{x_n} x_0 \\ \xi_n \in J x_n}} \langle x_n - z, \xi_0 - \xi_n \rangle \geq 0 \right\} \tag{4.10}$$

and

$$H_n = \{z \in X : \langle z - y_n, v_n \rangle \leq 0\}, \tag{4.11}$$

where (y_n, v_n) satisfies $v_n \in Ty_n$. Then define x_{n+1} as the generalized projection of x_0 onto the intersection of W_n and H_n. That is,

$$x_{n+1} = Q_{W_n \cap H_n} x_0. \tag{4.12}$$

Observe that H_n is well-defined. Indeed, since T is maximal monotone, $R(J+rT) = X^*$ for all $r > 0$. In particular, $Jx_n \subset R(J + r_nT)$ wherever $r_n > 0$. Thus, for any $\xi_n \in Jx_n$, there is $y_n \in D(T)$ such that $\xi_n \in Jy_n + r_nTy_n$. Take $\eta_n \in Jy_n$ such that

$$v_n := \frac{1}{r_n}(\xi_n - \eta_n) \in Ty_n. \tag{4.13}$$

With this pair of (v_n, y_n) we can define H_n.

Lemma 4.7. *For each $n \geq 0$, $T^{-1}0 \subset W_n \cap H_n$.*

Proof. That $H_n \supset T^{-1}0$ is easily seen. Indeed, for $x \in T^{-1}0$, $0 \in Tx$. The monotonicity of T then implies $\langle y_n - x, v_n \rangle \geq 0$. This shows that $x \in H_n$. To prove that $W_n \supset T^{-1}0$, we apply induction on $n \geq 0$.

For $n = 0$, we have (noting $J^*_{x_0} x_0 = Jx_0$)

$$W_0 = \left\{ z \in X : \max_{\substack{\xi_0 \in Jx_0 \\ \xi'_0 \in Jx_0}} \langle x_0 - z, \xi_0 - \xi'_0 \rangle \geq 0 \right\} = X \supset T^{-1}0.$$

Assume now $W_n \supset T^{-1}0$ and we next prove that $W_{n+1} \supset T^{-1}0$. Note that we now have $W_n \cap H_n \supset T^{-1}0$ and thus x_{n+1} is well-defined as the generalized projection of x_0 onto $W_n \cap H_n$; namely, x_{n+1} is given by (4.12).

Therefore, by Proposition 4.1, we have

$$\max_{\substack{\xi_0 \in J^*_{x_{n+1}} x_0 \\ \xi_{n+1} \in J x_{n+1}}} \langle x_{n+1} - v, \xi_0 - \xi_{n+1} \rangle \geq 0, \quad v \in W_n \cap H_n. \tag{4.14}$$

However $T^{-1}0 \subset W_n \cap H_n$, we see that (4.14) holds in particular for all $v \in T^{-1}0$. Now $W_{n+1} \supset T^{-1}0$ follows from the definition (4.10) of W_{n+1}. □

Lemma 4.8. *The sequence* $\{x_n\}$ *is bounded.*

Proof. Take $z \in T^{-1}0 \subset W_n$. It follows that

$$\max_{\substack{\xi_0 \in J^*_{x_n} x_0 \\ \xi_n \in J x_n}} \langle x_n - z, \xi_0 - \xi_n \rangle \geq 0.$$

This implies that

$$\begin{aligned} \|x_n\|^2 &\leq \max_{\substack{\xi_0 \in J^*_{x_n} x_0 \\ \xi_n \in J x_n}} (\langle x_n, \xi_0 \rangle + \langle z, \xi_n - \xi_0 \rangle) \\ &\leq \max_{\substack{\xi_0 \in J^*_{x_n} x_0 \\ \xi_n \in J x_n}} (\|x_n\| \|\xi_0\| + \|z\| \|\xi_n\| + \|z\| \|\xi_0\|) \\ &\leq \|x_n\|(\|x_0\| + \|z\|) + \|x_0\| \|z\|. \end{aligned}$$

This immediately implies the boundedness of $\{x_n\}$. □

Lemma 4.9. *The sequence* $\{\psi(x_n, x_0)\}$ *is increasing and* $\lim_{n\to\infty} \psi(x_n, x_0)$ *exists.*

Proof. By definition of W_n, it is immediately clear that $x_n = Q_{W_n} x_0$. Hence,

$$\psi(x_n, x_0) \leq \psi(x, x_0), \quad x \in W_n.$$

In particular, since $x_{n+1} = Q_{W_n \cap H_n} \in W_n \cap H_n \subset W_n$, we get

$$\psi(x_n, x_0) \leq \psi(x_{n+1}, x_0).$$

Moreover, since $\{\psi(x_n, x_0)\}$ is bounded by Lemma 4.2, $\lim_{n\to\infty} \psi(x_n, x_0)$ exists. □

Lemma 4.10. *The sequence* $\{x_n\}$ *is asymptotically regular; namely,* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof. By Proposition 4.1(vi), we see that the function ψ_{x_0} is locally uniformly convex in the sense that for any bounded set K, there exists a continuous strictly increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(0) = 0$, such that

$$\psi_{x_0}(\lambda u + (1-\lambda)v) \leq \lambda \psi_{x_0}(u) + (1-\lambda)\psi_{x_0}(v) - \lambda(1-\lambda)h(\|u - v\|) \tag{4.15}$$

for all $\lambda \in (0,1)$ and $u, v \in K$. Since $\{x_n\}$ is bounded, we may assume that $\{x_n\} \subset K$. Noting that x_n is the minimizer of ψ_{x_0} on W_n and $x_{n+1} \in W_n$, we derive from (4.15) that

$$\begin{aligned} \psi_{x_0}(x_n) &\leq \psi_{x_0}\left(\frac{x_n + x_{n+1}}{2}\right) \\ &\leq \frac{1}{2}\psi_{x_0}(x_n) + \frac{1}{2}\psi_{x_0}(x_{n+1}) - \frac{1}{4}h(\|x_n - x_{n+1}\|). \end{aligned}$$

It turns out by Lemma 4.2 that

$$h(\|x_n - x_{n+1}\|) \leq 2[\psi_{x_0}(x_{n+1}) - \psi_{x_0}(x_n)] \to 0.$$

This implies that $\|x_n - x_{n+1}\| \to 0$. □

Now we are in a position to state and prove the strong convergence of our modified PPA (4.10)-(4.12).

Theorem 4.11. *Suppose X is a uniformly convex Banach space. Let T be a maximal monotone operator in X such that $S := T^{-1}0$ is nonempty, and let $\{x_n\}$ be the sequence generated by the modified PPA (4.10)-(4.12). Assume that the sequence $\{r_n\}$ of parameters tends to infinity as $n \to \infty$. Then $\{x_n\}$ converges in norm to the generalized projection $Q_S x_0$ of x_0 onto the solution set S.*

Proof. In the definition of v_n, we chose $\xi_n \in Jx_n$ in such a way that

$$\max_{\xi \in Jx_n} \langle y_n, -\xi \rangle = \langle y_n, -\xi_n \rangle.$$

Recall $\eta_n \in Jy_n$ which is the subdifferential of the convex function $\frac{1}{2}\|y\|^2$ at $y = y_n$. We can therefore use the subdifferential inequality to get

$$\|Q_{H_n} x_n\|^2 \geq \|y_n\|^2 + 2\langle Q_{H_n} x_n - y_n, \eta_n \rangle.$$

It follows that

$$\begin{aligned}
&2[\psi(Q_{H_n} x_n, x_n) - \psi(y_n, x_n)] \\
&= \|Q_{H_n} x_n\|^2 + 2\tau(Q_{H_n} x_n, x_n) + \|x_n\|^2 - \|y_n\|^2 - 2\tau(y_n, x_n) - \|x_n\|^2 \\
&= \|Q_{H_n} x_n\|^2 - \|y_n\|^2 + 2[\tau(Q_{H_n} x_n, x_n) - 2\tau(y_n, x_n)] \\
&= \|Q_{H_n} x_n\|^2 - \|y_n\|^2 + 2\left[\max_{\xi \in J(-x_n)} \langle Q_{H_n} x_n, \xi \rangle - \max_{\xi \in J(-x_n)} \langle y_n, \xi \rangle\right] \\
&\geq 2\langle Q_{H_n} x_n - y_n, \eta_n \rangle + 2\left[\max_{\xi \in Jx_n} \langle Q_{H_n} x_n, -\xi \rangle + \langle y_n, \xi_n \rangle\right] \\
&\geq 2\langle Q_{H_n} x_n - y_n, \eta_n \rangle + 2[\langle Q_{H_n} x_n, -\xi_n \rangle - \langle y_n, -\xi_n \rangle] \\
&= 2\langle Q_{H_n} x_n - y_n, \eta_n - \xi_n \rangle \\
&= 2r_n \langle y_n - Q_{H_n} x_n, v_n \rangle \\
&\geq 0 \qquad \text{since } Q_{H_n} x_n \in H_n \text{ and by (4.11).}
\end{aligned}$$

Hence

$$\psi(Q_{H_n} x_n, x_n) \geq \psi(y_n, x_n). \tag{4.16}$$

Noticing that $x_{n+1} \in H_n$, we get from (4.16)

$$\psi(y_n, x_n) \leq \psi(Q_{H_n} x_n, x_n) \leq \psi(x_{n+1}, x_n) \to 0.$$

It turns out from Proposition 4.1(v) that

$$\|y_n - x_n\| \to 0. \tag{4.17}$$

We will use Lemma 4.1 to prove the strong convergence of $\{x_n\}$. To achieve this, we first prove that $\omega_w(x_n) \subset S = T^{-1}0$. So assume $\hat{x} \in \omega_w(x_n)$ and $x_{n_i} \rightharpoonup \hat{x}$, where

$\{x_{n_i}\}$ is a subsequence of $\{x_n\}$. Let $\eta_n \in Jy_n$ satisfy (4.13). Since $\|\xi_n\| = \|x_n\|$ and $\|\eta_n\| = \|y_n\|$ are bounded, we get $v_n \to 0$ in norm as $r_n \to \infty$. Now from the relation

$$v_{n_j} \in Ty_{n_j}$$

and the fact that $y_{n_j} \rightharpoonup \hat{x}$ due to (4.17), the maximal monotonicity of T ensures that $0 \in T\hat{x}$. Namely, $\hat{x} \in T^{-1}0 = S$. Therefore, $\omega_w(x_n) \subset S$.

We observe that condition (ii) of Lemma 4.1 holds, with $K = S$, $u = x_0$ and $q = Q_S x_0$. As a matter of fact, since $x_{n+1} = Q_{W_n \cap H_n} x_0$, we have

$$\psi(x_{n+1}, x_0) = \psi(Q_{W_n \cap H_n} x_0, x_0) = \min_{z \in W_n \cap H_n} \psi(z, x_0) \leq \psi(q, x_0).$$

Therefore, an application of Lemma 4.1 yields $x_n \to Q_S x_0$. □

Acknowledgement. This project was funded by the National Plan for Science, Technology and Innovation (MAARIFAH), King Abdulaziz City for Science and Technology, Kingdom of Saudi Arabia, Award Number (12-MAT2658-02).

References

[1] Y.I. Alber, S. Guerre-Delabriere, *On the projection methods for fixed point problems*, Analysis, **21**(2001), 17-39.

[2] H.H. Bauschke, E. Matouskova, S. Reich, *Projections and proximal point methods: convergence results and counterexamples*, Nonlinear Anal., **56**(2004), 715-738.

[3] H.H. Bauschke, J.V. Burke, F.R. Deutsch, H.S. Hundal, J.D. Vanderwerff, *A new proximal point iteration that converges weakly but not in norm*, Proc. Amer. Math. Soc., **133**(6)(2005), 1829-1835.

[4] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces,* Noordhoff, 1976.

[5] V. Barbu, Th. Precupanu, *Convexity and Optimization in Banach Spaces,* Editura Academiei R.S.R., Bucharest, 1978.

[6] L.M. Bregman, *Finding the common point of convex sets by the method of successive projection* (Russian), Dokl. Akad. Nauk SSSR, **162**(1965), 487-490.

[7] H. Brezis, *Operateurs Maximaux Monotones et Semi-Groups de Contractions dans les Espaces de Hilbert,* North-Holland, Amsterdam, 1973.

[8] R.E. Bruck, S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math., **3**(1977), 459-470.

[9] R.S. Burachik, S. Scheimberg, *A proximal point method for the variational inequality problem in Banach spaces,* SIAM J. Control Optim., **39**(2000), 1633–1649.

[10] J.V. Burke, M. Qian, *A variable metric proximal point algorithm for monotone operators,* SIAM J. Control Optim., **37**(1998), 353–375.

[11] D. Butnariu, G. Kassay, *A proximal-projection method for finding zeros of set-valued operators,* SIAM J. Control Optim., **47**(2008), 2096–2136.

[12] D. Butnariu, A.N. Iusem, *On a proximal point method for convex optimization in Banach spaces,* Numer. Funct. Anal. Optim., **18**(1997), 723–744

[13] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems,* Kluwer Academic Publishers, 1990.

[14] F.H. Clarke, *Optimization and Nonsmooth Analysis,* John Wiley & Sons, 1983.

[15] J. Eckstein, *Nonlinear proximal point algorithms using Bregman functions, with application to convex programming,* Math. Oper. Res., **18**(1993), 202–226.

[16] J. Eckstein, D.P. Bertsekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators,* Mathematical Programming, **55**(1992), 293–318.

[17] E.G. Gol'shtein, N.V. Tret'yakov, *Modified Lagrangians in convex programming and their generalizations,* Mathematical Programming Study, **10**(1979), 86-97.

[18] G. Gripenberg, *On the asymptotic behaviour of nonlinear contraction semigroups,* Math. Scand., **44**(1979), 385-379.
[19] O. Güler, *On the convergence of the proximal point algorithm for convex optimization,* SIAM J. Control Optim., **29**(1991), 403-419.
[20] H. Hundal, *An alternating projection that does not converge in norm,* Nonlinear Anal., **57**(2004), 35-61.
[21] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space,* SIAM J. Optim., **13**(2003), 938-945.
[22] A. Kaplan, R. Tichatschke, *Proximal point approach and approximation of variational inequalities,* SIAM J. Optim., **39**(2000), 1136-1159.
[23] G. Marino, H.K. Xu, *Convergence of generalized proximal point algorithms,* Comm. Applied Anal., **3**(2004), 791-808.
[24] B. Martinet, *Regularisation d'inequations variationelles par approximations successives,* Rev. Francaise Informat. Recherche Operationnelle (Ser. R-3), **4**(1970), 154-158.
[25] C. Matinez-Yanes, H.K. Xu, *Strong convergence of the CQ method for fixed point processes,* Nonlinear Anal., **64**(2006), 2400-2411.
[26] K. Nakajo, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups,* J. Math. Anal. Appl., **279**(2003), 372-379.
[27] O. Nevanlinna, S. Reich, *Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces,* Israel J. Math., **32**(1979), 44-58.
[28] S. Reich, *An iterative procedure for constructing zeros of accretive sets in Banach spaces,* Nonlinear Anal., **2**(1978), 85-92.
[29] R.T. Rockafellar, *Monotone operators and the proximal point algorithm,* SIAM J. Control Optim., **14**(1976), 877-898.
[30] R.T. Rockafellar, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming,* Math. Oper. Res., **1**(1976), 97–116.
[31] S. Burachik, S. Scheimberg, *A proximal point method for the variational inequality problem in Banach spaces,* SIAM J. Control Optim., **39**(2001), 1633-1649.
[32] R.E. Showalter, *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations,* Amer. Math. Soc., 1997.
[33] M.V. Solodov, B.F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space,* Mathematical Programming, Ser. A, **87**(2000), 189-202.
[34] H.K. Xu, *Inequalities in Banach spaces with applications,* Nonlinear Anal., **16**(1991), 1127-1138.
[35] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc., **66**(2002), 240-256.
[36] H.K. Xu, *Asymptotic behavior of a gradient flow,* Comm. Appl. Nonlinear Anal., **7**(2004), 11-17.

Received: April 14, 2016; Accepted: May 25, 2016.