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UNBOUNDED SOLUTIONS FOR BOUNDARY VALUE PROBLEMS OF RIEMANN LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS ON THE HALF-LINE

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Abstract. In this study, a boundary value problem is worked out for fractional differential equations on the half line. Here, some results on the existence of positive solutions are obtained for the fractional boundary value problem and its monotone iterative scheme is established by using the monotone iterative technique.
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1. Introduction

Fractional calculus is a field of mathematical analysis, which deals with the investigation and applications of integrals and derivatives of an arbitrary order. In fact, fractional calculus has numerous applications in various disciplines of science, engineering, economy, and other fields; see for instance, the monographs of Kilbas *et al.* [7], Podlubny [15], and Samko *et al.*[16] are commonly cited for the theory of fractional derivatives and integral and applications to differential equations of fractional order.

Recently, boundary value problems of fractional-order differential equations have been extensively investigated and a variety of results on the topic has been established. A great deal of work on fractional boundary value problems involves local/nonlocal boundary conditions. Much attention has been focused on the study of the existence and multiplicity of solutions or positive solutions for boundary value problems. For more details, we refer the reader to [2], [3], [4], [5], [6], [12], [14] and the references therein. However, to our knowledge, it is rare for work to be done on solutions of fractional differential equations on the half-line (see [1], [8], [9], [10], [11], [13], [17], [18], [19]). For example, Su and Zhang in [17] studied the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_{0^+}^{\alpha-1}u(t)), \quad t \in J := [0, +\infty), \\ u(0) = 0, \quad D_{0^+}^{\alpha-1}u(+\infty) = u_\infty, \quad u_\infty \in \mathbb{R}, \end{cases}$$

where $1 < \alpha \leq 2$, $f \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, D_{0+}^{α} and $D_{0+}^{\alpha-1}$ are the standard Riemann-Liouville fractional derivatives and $D_{0+}^{\alpha-1}u(\infty) := \lim_{t\to+\infty} D_{0+}^{\alpha-1}u(t)$.

In [8], Liang and Zhang considered the existence of three positive solutions for the following m-point fractional boundary value problem on an infinite interval

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(u(t)) = 0, \quad 0 < t < +\infty, \\ u(0) = u'(0) = 0, \quad D^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < +\infty$, $\beta_i \geq 0$ $(i = 1, 2, ..., m-2)$ satisfying

$$0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha).$$

The method involves applications of a fixed point theorem due to Leggett-Williams.

In [11], Liu and Jia investigated the following nonlocal boundary value problem for fractional differential equation of the form

$$\begin{cases} {}^{c}D_{0+}^{\alpha}[p(t)u'(t)] + q(t)f(t, u(t)) = 0, \quad t > 0, \\ p(0)u'(0) = 0, \\ \lim_{t\to\infty} u(t) = \int_0^{\infty} g(s)u(s)ds, \end{cases}$$

where ${}^{c}D_{0+}^{\alpha}$ is the standard Caputo fractional derivative, $0 < \alpha < 1$ is a constant, f, g, p and q are given functions. Applying the fixed point theory and the upper and lower solutions method, a new result on the existence of at least three distinct nonnegative solutions under some conditions was established.

In [1], Agarwal *et al.* discussed the existence of solutions for the boundary value problems (BVP for short) for fractional order differential equations of the form

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t)), \quad \text{for each } t \in J := [0, +\infty),\ 1 < \alpha \leq 2, \\ y(0) = 0,\ \text{y is bounded on } [0, +\infty), \end{cases}$$

where D^{α} is the Riemann-Liouville fractional derivative, $f : J\times\mathbb{R} \to \mathbb{R}$ is a given function. Results are based on the nonlinear alternative of Leray-Schauder type combined with the diagonalization method.

In this paper, we are concerned with the following boundary value problem

$$\begin{cases} D_{0+}^{\alpha}x(t) + f(t, x(t)) = 0, \quad t \in (0, +\infty), \\ \lim_{t\to 0} t^{2-\alpha}x(t) = \lim_{t\to\infty} D_{0+}^{\alpha-1}x(t) = \int_0^{\infty} g(s)x(s)ds, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α.

We assume that the following conditions hold:

(H1) $f \in \mathcal{C}[(0, +\infty) \times [0, +\infty), [0, +\infty)]$, f may be singular at $t = 0$ and $f(t, 0) \not\equiv 0$ on any subinterval of $(0, +\infty)$;

(H2) $g \in L^1([0, +\infty))$ and

$$\int_0^{\infty} g(s)[\frac{s^{\alpha-1}}{\Gamma(\alpha)} + s^{\alpha-2}]ds < 1.$$

Our goal is to establish the existence results of unbounded (positive) solutions for the fractional boundary value problem on the half-line. By applying the monotone iterative technique, the existence of positive solutions under some conditions was established and successive iterative schemes for approximating solutions were obtained. Here, we do not require the existence of lower and upper solutions.

2. Preliminaries

In this section, we give some basic definitions and lemmas which are useful for the presentation of our main result.

Definition 2.1. [7, 15] The Riemann Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. [7, 15] The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha} f(t) = \Big(\frac{d}{dt}\Big)^n I_{0^+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where n is the smallest integer greater than or equal to α, provided that the right-hand side is defined pointwise.

Lemma 2.3. [7] *The equality $D_{0^+}^{\gamma} I_{0^+}^{\gamma} f(t) = f(t)$ with $\gamma > 0$ holds for $f \in L^1(0, 1)$.*

Lemma 2.4. [7] *Let $\alpha > 0$. If we assume $u \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where n is the smallest integer greater than or equal to α.

Lemma 2.5. [7] *Let $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order α ($\alpha > 0$) that belongs to $C(0, 1) \cap L(0, 1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where n is the smallest integer greater than or equal to α.

Lemma 2.6. [7] *Let $\alpha > 0$ then*

(i) *If $\mu > -1$, $\mu \neq \alpha - i$ with $i = 1, 2, ..., [\alpha] + 1$, $t > 0$ then*

$$D_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}.$$

(ii) *For $i = 1, 2, ..., [\alpha] + 1$, we have $D_{0^+}^{\alpha} t^{\alpha-i} = 0$.*

Let $\mathcal{C}(0, +\infty)$ be the set of all continuous functions on $(0, +\infty)$. Choose $\sigma > -1$ and

$$X = \{x \in \mathcal{C}(0, +\infty) : \frac{t^{2-\alpha}}{1 + t^{\sigma+2}} x(t) \text{ is bounded on } (0, +\infty)\}.$$

For $x \in X$, define the norm by

$$||x|| = \sup_{t\in(0,+\infty)} \frac{t^{2-\alpha}}{1+t^{\sigma+2}}|x(t)|.$$

It is easy to show that X is a real Banach space. Also we define the cone $P \subset X$ as follows:

$$P = \{x \in X : x(t) \geq 0, t \in (0,+\infty)\}.$$

For the sake of convenience, let us set

$$L = \frac{1}{\psi\Gamma(\alpha)}\Big(1 + \int_0^\infty t^{\alpha-1}g(t)dt\Big), \quad N = \frac{2L\Gamma(\sigma_1+1)}{k^{\sigma_1+1}},$$

$$\psi = 1 - \int_0^\infty g(s)\Big(\frac{s^{\alpha-1}}{\Gamma(\alpha)} + s^{\alpha-2}\Big)ds.$$

Lemma 2.7. *Suppose that $h : (0,+\infty) \to \mathbb{R}$ is a given function satisfying that there exist numbers $k, M > 0$ and $\sigma > -1$ with $|h(t)| \leq Mt^\sigma e^{-kt}$. Then $x \in X$ is a solution of the problem*

$$\begin{cases} D^\alpha_{0^+}x(t) + h(t) = 0, \quad t \in (0,+\infty), \\ \lim_{t\to 0} t^{2-\alpha}x(t) = \lim_{t\to\infty} D^{\alpha-1}_{0^+}x(t) = \int_0^\infty g(s)x(s)ds \end{cases} \tag{2.1}$$

if and only if $x \in X$ and

$$x(t) = \int_0^\infty G(t,s)h(s)ds.$$

Here,

$$G(t,s) = G_0(t,s) + G_1(t,s), \tag{2.2}$$

where

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq +\infty, \\ t^{\alpha-1}, & 0 \leq t \leq s \leq +\infty, \end{cases}$$

and

$$G_1(t,s) = \Big[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + t^{\alpha-2}\Big]\frac{1}{\psi}\int_0^\infty G_0(t,s)g(t)dt.$$

Obviously, $G(t,s)$ is continuous and $G(t,s) \geq 0$ for $(t,s) \in (0,+\infty) \times (0,+\infty)$.

Proof. We may apply Lemma 2.5 to reduce BVP (2.1) to an equivalent integral equation

$$x(t) = -I^\alpha h(t) - c_1t^{\alpha-1} - c_2t^{\alpha-2}, \tag{2.3}$$

where c_1 and c_2 are arbitrary constants. Since

$$\Big|t^{2-\alpha}\int_0^t (t-s)^{\alpha-1}h(s)ds\Big| \leq t^{\sigma+2}M\int_0^1 (1-\tau)^{\alpha-1}\tau^\sigma d\tau \to 0, \ t \to 0, \tag{2.4}$$

and from (2.3), we have

$$t^{2-\alpha}x(t) = -t^{2-\alpha}I^\alpha h(t) - c_1t - c_2,$$

together with the given boundary condition in (2.1), we find that

$$c_2 = -\int_0^\infty g(s)x(s)ds.$$

Since

$$\Big|\int_0^\infty h(s)ds\Big| \le \frac{M}{k^{\sigma+1}}\int_0^\infty \tau^\sigma e^{-\tau}d\tau = \frac{M}{k^{\sigma+1}}\Gamma(\sigma+1), \tag{2.5}$$

and

$$D_{0^+}^{\alpha-1}x(t) = -I^1 h(t) - c_1\Gamma(\alpha).$$

Then, boundary condition in (2.1) implies $c_1 = -\frac{1}{\Gamma(\alpha)}\big[I^1h(\infty) + \int_0^\infty g(s)x(s)ds\big]$. Substituting the values of c_1 and c_2 into (2.3) gives

$$x(t) = -I^\alpha h(t) + \Big[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + t^{\alpha-2}\Big]\int_0^\infty g(s)x(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}I^1h(\infty), \tag{2.6}$$

where

$$\begin{aligned}\int_0^\infty g(s)x(s)ds = &\frac{1}{1-\int_0^\infty g(s)\Big[\frac{s^{\alpha-1}}{\Gamma(\alpha)} + s^{\alpha-2}\Big]ds}\Big(-\int_0^\infty g(s)I^\alpha h(s)ds\\ &+\int_0^\infty h(\tau)d\tau\int_\tau^\infty \frac{g(s)s^{\alpha-1}}{\Gamma(\alpha)}ds\Big).\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6), we have

$$\begin{aligned}x(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty h(s)ds\\ &- \Big[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + t^{\alpha-2}\Big]\frac{1}{\psi}\frac{1}{\Gamma(\alpha)}\int_0^\infty h(s)\int_s^\infty g(t)(t-s)^{\alpha-1}dtds\\ &+ \Big[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + t^{\alpha-2}\Big]\frac{1}{\psi}\int_0^\infty h(s)ds\int_0^\infty g(s)\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds\\ &= \int_0^\infty G(t,s)h(s)ds,\end{aligned} \tag{2.8}$$

where $G(t,s)$ is defined by (2.2). It is easy to show that $G(t,s) \ge 0$ for $(t,s) \in (0,+\infty)\times(0,+\infty)$. Now, we prove $x \in X$. From (2.8) together with $|h(t)| \le Mt^\sigma e^{-kt}$, we know that $x \in \mathcal{C}(0,+\infty)$. Observe that

$$\frac{t^{2-\alpha}}{1+t^{\sigma+2}}|x(t)| = \Big|\int_0^\infty \frac{t^{2-\alpha}}{1+t^{\sigma+2}}G(t,s)h(s)ds\Big|.$$

One see that

$$\frac{t^{2-\alpha}}{1+t^{\sigma+2}}G(t,s) = \frac{t^{2-\alpha}}{1+t^{\sigma+2}}G_0(t,s) + \frac{t^{2-\alpha}}{1+t^{\sigma+2}}G_1(t,s).$$

Then

$$\frac{t^{2-\alpha}}{1+t^{\sigma+2}}G_0(t,s) \le \frac{t}{\Gamma(\alpha)(1+t^{\sigma+2})} \le \frac{1}{\Gamma(\alpha)},$$

and

$$\begin{aligned}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}G_1(t,s) &\leq \Big[\frac{t}{\Gamma(\alpha)(1+t^{\sigma+2})}+\frac{1}{1+t^{\sigma+2}}\Big]\frac{1}{\psi}\int_0^{\infty}G_0(t,s)g(t)dt\\ &\leq \Big[\frac{1}{\Gamma(\alpha)}+1\Big]\frac{1}{\psi}\Big(\int_0^{s}\frac{t^{\alpha-1}}{\Gamma(\alpha)}g(t)dt+\int_s^{\infty}\frac{[t^{\alpha-1}-(t-s)^{\alpha-1}]}{\Gamma(\alpha)}g(t)dt\Big)\\ &\leq \Big[\frac{1}{\Gamma(\alpha)}+1\Big]\frac{1}{\psi}\int_0^{\infty}\frac{t^{\alpha-1}}{\Gamma(\alpha)}g(t)dt.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}G(t,s) &\leq \frac{1}{\Gamma(\alpha)}+\Big[\frac{1}{\Gamma(\alpha)}+1\Big]\frac{1}{\psi}\int_0^{\infty}\frac{t^{\alpha-1}}{\Gamma(\alpha)}g(t)dt\\ &\leq \frac{1}{\psi\Gamma(\alpha)}\Big(1+\int_0^{\infty}t^{\alpha-1}g(t)dt\Big).\end{aligned} \tag{2.9}$$

So, we obtain

$$\begin{aligned}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}|x(t)| &= \Big|\int_0^{\infty}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}G(t,s)h(s)ds\Big|\\ &\leq L\Big|\int_0^{\infty}h(s)ds\Big|\\ &\leq \frac{LM}{k^{\sigma+1}}\Gamma(\sigma+1).\end{aligned}$$

Hence, $x \in X$. Conversely, if $x \in X$ satisfies (2.8), then we can show easily that $x \in X$ and satisfies (2.1). The proof is completed. □

Let us define an operator A on P by

$$(Ax)(t) = \int_0^{\infty}G(t,s)f(s,x(s))ds.$$

Observe that the BVP (1.1) has a solution if and only if the operator A has a fixed point.

Lemma 2.8. *Assume that (H_1) and (H_2) hold and f satisfies that*

- *for each $r > 0$ there exist constants $k, M_r > 0$ and $\sigma_1 \in (-1,\sigma)$ such that*

$$0 \leq f(t,\frac{1+t^{\sigma+2}}{t^{2-\alpha}}x) \leq M_r t^{\sigma_1}e^{-kt} \text{ for all } t \in (0,+\infty), |x| \leq r.$$

Then $A : P \to P$ is well defined and completely continuous.

Proof. We divide the proof into several steps.

Step **1.** We prove that $A : P \to P$ is well defined and maps bounded sets into bounded sets. For $x \in P$, we find $x(t) \geq 0$ for all $t \in (0,+\infty)$ and there exists $r \geq 0$ such that

$$||x|| = \sup_{t\in(0,+\infty)}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}|x(t)| \leq r.$$

Then there exist constants $\sigma_1 \in (-1,\sigma)$ and $M_r > 0$ such that

$$0 \leq f(t,\frac{1+t^{\sigma+2}}{t^{2-\alpha}}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}x) \leq M_r t^{\sigma_1}e^{-kt} \text{ for all } t \in (0,+\infty), ||x|| \leq r.$$

Since f is nonnegative together with Lemma 2.7, we know that $Ax(t) \geq 0$ for all $t \in (0, +\infty)$. From the definition of A, we have $Ax \in \mathcal{C}(0, +\infty)$.
On the other hand, from (2.9)

$$\begin{aligned} ||(Ax)|| &= \sup_{t\in(0,+\infty)} \frac{t^{2-\alpha}}{1+t^{\sigma+2}}\Big|\int_0^\infty G(t,s)f(s,x(s))ds\Big| \\ &\leq L\int_0^\infty f(s,x(s))ds \\ &\leq LM_r\int_0^\infty s^{\sigma_1}e^{-ks}ds < +\infty. \end{aligned}$$

So $Ax \in P$ and $A : P \to P$ is well defined. Similarly, it is easy to show that A maps bounded sets into bounded sets.
Step **2.** Now we show that the operator A is continuous.
We consider $\{x_n\}_{n=1}^{\infty} \subset P$ such that $x_n \to x$ as $n \to \infty$. Then there exists $r > 0$ such that $\max_{n\in\mathbb{N}\setminus\{0\}}\{||x_n||, ||x||\} \leq r$ together with

$$0 \leq f(t, \frac{1+t^{\sigma+2}}{t^{2-\alpha}}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}x) \leq M_r t^{\sigma_1}e^{-kt} \text{ for all } \ t \in (0, +\infty).$$

Thus,

$$\Big|\int_0^{+\infty} f(s,x_n(s))ds - \int_0^{+\infty} f(s,x(s))ds\Big| \leq 2M_r\int_0^\infty s^{\sigma_1}e^{-ks}ds < +\infty.$$

By the Lebesgue dominated convergence theorem, we have

$$\Big|\int_0^{+\infty} f(s,x_n(s))ds - \int_0^{+\infty} f(s,x(s))ds\Big| \to 0,\ n \to +\infty.$$

Therefore, by (2.9), we have

$$\begin{aligned} &||(Ax_n) - (Ax)|| \\ &= \sup_{t\in(0,+\infty)} \frac{t^{2-\alpha}}{1+t^{\sigma+2}}\Big|\int_0^\infty G(t,s)f(s,x_n(s))ds - \int_0^\infty G(t,s)f(s,x(s))ds\Big| \\ &\leq L\Big|\int_0^\infty f(s,x_n(s))ds - \int_0^\infty f(s,x(s))ds\Big| \\ &\to 0,\ n \to +\infty. \end{aligned}$$

Hence A is a continuous operator. Recall that $\Omega \subset X$ is relatively compact if

(i) it is bounded,
(ii) $\frac{t^{2-\alpha}}{1+t^{\sigma+2}}\Omega$ is equicontinuous on any closed subinterval $[a, b]$ of $(0, +\infty)$,
(iii) $\frac{t^{2-\alpha}}{1+t^{\sigma+2}}\Omega$ is equiconvergent at $t = 0$ and $t = \infty$.

Step **3.** Let $\Omega = \{x \in P : ||x|| \leq r\}$ be a bounded set in P. We prove that A is an equicontinuous operator on any closed subinterval of $(0, +\infty)$. For $x \in \Omega$, we have

$$||x|| = \sup_{t\in(0,+\infty)} \frac{t^{2-\alpha}}{1+t^{\sigma+2}}|x(t)| \leq r.$$

Then there exist constants $\sigma_1 \in (-1, \sigma)$ and $M_r > 0$ such that

$$0 \le f(t, \frac{1+t^{\sigma+2}}{t^{2-\alpha}} \frac{t^{2-\alpha}}{1+t^{\sigma+2}} x) \le M_r t^{\sigma_1} e^{-kt} \text{ for all } \ t \in (0, +\infty), ||x|| \le r.$$

For $[a, b] \subset (0, +\infty)$, $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, we can get

$$\begin{aligned}
&\Big|\frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}} Ax(t_2) - \frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} Ax(t_1)\Big| \\
&\le M_r \int_0^{+\infty} \Big|\frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}} G(t_2, s) - \frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G(t_1, s)\Big| s^{\sigma_1} e^{-ks} ds \\
&\le M_r \int_0^{+\infty} \Big|\frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}} G_0(t_2, s) - \frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_0(t_1, s)\Big| s^{\sigma_1} e^{-ks} ds \\
&\quad + M_r \int_0^{+\infty} \Big|\frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}} G_1(t_2, s) - \frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_1(t_1, s)\Big| s^{\sigma_1} e^{-ks} ds \\
&\le M_r \int_0^{+\infty} \Big|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_0(t_2, s) - \frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_0(t_1, s)\Big| s^{\sigma_1} e^{-ks} ds \\
&\quad + M_r \int_0^{+\infty} \Big|\frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}} G_0(t_2, s) - \frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_0(t_2, s)\Big| s^{\sigma_1} e^{-ks} ds \\
&\quad + M_r \frac{1}{\psi} \int_0^{\infty} G_0(t, s) g(t) dt \\
&\quad \times |(\frac{t_2}{\Gamma(\alpha)(1+t_2^{\sigma+2})} + \frac{1}{1+t_2^{\sigma+2}}) - (\frac{t_1}{\Gamma(\alpha)(1+t_1^{\sigma+2})} + \frac{1}{1+t_1^{\sigma+2}})| \\
&\quad \times \int_0^{+\infty} s^{\sigma_1} e^{-ks} ds.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_0^{+\infty} \Big|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_0(t_2, s) - \frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_0(t_1, s)\Big| s^{\sigma_1} e^{-ks} ds \\
&\le \Big(\int_0^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{+\infty}\Big) \Big|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_0(t_2, s) - \frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} G_0(t_1, s)\Big| s^{\sigma_1} e^{-ks} ds \\
&= \int_0^{t_1} \frac{t_1^{2-\alpha} |(t_2^{\alpha-1} - t_1^{\alpha-1}) + (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}|}{1+t_1^{\sigma+2}} s^{\sigma_1} e^{-ks} ds \\
&\quad + \int_{t_1}^{t_2} \frac{t_1^{2-\alpha} |(t_2^{\alpha-1} - t_1^{\alpha-1}) - (t_2 - s)^{\alpha-1}|}{1+t_1^{\sigma+2}} s^{\sigma_1} e^{-ks} ds \\
&\quad + \int_{t_2}^{+\infty} \frac{t_1^{2-\alpha} (t_2^{\alpha-1} - t_1^{\alpha-1})}{1+t_1^{\sigma+2}} s^{\sigma_1} e^{-ks} ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{t_1^{2-\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{1+t_1^{\sigma+2}}\int_0^{t_1} s^{\sigma_1}e^{-ks}ds \\
&\quad+\int_0^{t_1}\frac{t_1^{2-\alpha}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|}{1+t_1^{\sigma+2}}s^{\sigma_1}e^{-ks}ds \\
&\quad+\frac{t_1^{2-\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{1+t_1^{\sigma+2}}\int_{t_1}^{t_2} s^{\sigma_1}e^{-ks}ds+\int_{t_1}^{t_2}\frac{t_1^{2-\alpha}(1-s)^{\alpha-1}}{1+t_1^{\sigma+2}}s^{\sigma_1}e^{-ks}ds \\
&\quad+\frac{t_1^{2-\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{1+t_1^{\sigma+2}}\int_{t_2}^{+\infty} s^{\sigma_1}e^{-ks}ds \\
&\le \frac{t_1^{2-\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{1+t_1^{\sigma+2}}\int_0^{t_1} s^{\sigma_1}e^{-ks}ds+\frac{t_1^{2-\alpha}(t_2-t_1)^{\alpha-1}}{1+t_1^{\sigma+2}}\int_0^{t_1} s^{\sigma_1}e^{-ks}ds \\
&\quad+\frac{t_1^{2-\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{1+t_1^{\sigma+2}}\int_{t_1}^{t_2} s^{\sigma_1}e^{-ks}ds+\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}\int_{t_1}^{t_2}(1-s)^{\alpha-1}s^{\sigma_1}e^{-ks}ds \\
&\quad+\frac{t_1^{2-\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{1+t_1^{\sigma+2}}\int_{t_2}^{+\infty} s^{\sigma_1}e^{-ks}ds.
\end{aligned}
$$

Hence,

$$\int_0^{+\infty}\Big|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}G_0(t_2,s)-\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}G_0(t_1,s)\Big|s^{\sigma_1}e^{-ks}ds\to 0,\ t_1\to t_2,$$

and

$$\int_0^{+\infty}\Big|\frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}}G_0(t_2,s)-\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}G_0(t_2,s)\Big|s^{\sigma_1}e^{-ks}ds\to 0,\ t_1\to t_2.$$

Thus, we can get

$$\Big|\frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}}Ax(t_2)-\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}Ax(t_1)\Big|\to 0, t_1\to t_2.$$

Consequently, $A\Omega$ is equicontinuous on $(0,+\infty)$.

Next, we show that operator $A:P\to P$ is an equiconvergent operator at infinity. For each $x\in\Omega$, we have

$$\int_0^{+\infty} f(s,x(s))ds\le M_r\int_0^{+\infty} s^{\sigma_1}e^{-ks}ds<+\infty,$$

and

$$
\begin{aligned}
\Big|\frac{t^{2-\alpha}}{1+t^{\sigma+2}}Ax(t)\Big|&\le\frac{1}{\Gamma(\alpha)}\frac{t^{\sigma_1+2}}{1+t^{\sigma+2}}M_r\int_0^1(1-\tau)^{\alpha-1}\tau^{\sigma_1}d\tau+\frac{1}{\Gamma(\alpha)}\frac{t}{1+t^{\sigma+2}}\frac{M_r}{k^{\sigma_1+1}}\Gamma(\sigma_1+1) \\
&\quad+\Big(\frac{1}{\Gamma(\alpha)}\frac{t}{1+t^{\sigma+2}}+\frac{1}{1+t^{\sigma+2}}\Big)\frac{1}{\psi}\int_0^{\infty}g(s)\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds\frac{M_r}{k^{\sigma_1+1}}\Gamma(\sigma_1+1).
\end{aligned}
$$

The right-hand side of the above inequality tends to 0 uniformly as $t\to+\infty$. Thus $A\Omega$ is equiconvergent at infinity.

Finally, we prove that operator $A : P \to P$ is an equiconvergent operator at 0.

$$\begin{aligned}
&\Big|\frac{t^{2-\alpha}}{1+t^{\sigma+2}}Ax(t) - \frac{1}{\psi}\int_0^\infty \big(\int_0^\infty G_0(t,s)g(t)dt\big)f(s,x(s))ds\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\frac{t^{\sigma_1+2}}{1+t^{\sigma+2}}M_r\int_0^1(1-\tau)^{\alpha-1}\tau^{\sigma_1}d\tau + \frac{1}{\Gamma(\alpha)}\frac{t}{1+t^{\sigma+2}}\frac{M_r}{k^{\sigma_1+1}}\Gamma(\sigma_1+1) \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)}\frac{t}{1+t^{\sigma+2}} + \frac{1}{1+t^{\sigma+2}} - 1\Big|\frac{1}{\psi}\int_0^\infty \big(\int_0^\infty G_0(t,s)g(t)dt\big)f(s,x(s))ds \\
&\le \frac{1}{\Gamma(\alpha)}\frac{t^{\sigma_1+2}}{1+t^{\sigma+2}}M_r\int_0^1(1-\tau)^{\alpha-1}\tau^{\sigma_1}d\tau + \frac{1}{\Gamma(\alpha)}\frac{t}{1+t^{\sigma+2}}\frac{M_r}{k^{\sigma_1+1}}\Gamma(\sigma_1+1) \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)}\frac{t}{1+t^{\sigma+2}} + \frac{1}{1+t^{\sigma+2}} - 1\Big|\frac{1}{\psi}\int_0^\infty g(s)\frac{s^{\alpha-1}}{\Gamma(\alpha)}ds\frac{M_r}{k^{\sigma_1+1}}\Gamma(\sigma_1+1).
\end{aligned}$$

The right-hand side of the above inequality tends to 0 uniformly as $t \to 0$. Then $A\Omega$ is equiconvergent at 0.

Thus, A is completely continuous. The proof is completed. □

3. Main result

In this section, we deal with the existence of positive solutions for the problem (1.1).

Theorem 3.1. *Assume that (H_1) and (H_2) hold, and there exist $\sigma_1 \in (-1,\sigma)$ and $d > 0, k > 0$ satisfying the following conditions:*

(H3) $f(t,u_0) \le f(t,\overline{u_0})$, *for any* $t \in (0,+\infty)$, $0 \le u_0 \le \overline{u_0}$;

(H4) $f\Big(t, \dfrac{1+t^{\sigma+2}}{t^{2-\alpha}}u_0\Big) \le \dfrac{d}{N}t^{\sigma_1}e^{-kt}$, $(t,u_0) \in (0,+\infty)\times[0,d]$.

Then the BVP (1.1) has maximal and minimal positive solutions w^* and v^* on $(0,+\infty)$, such that

$$0 < ||w^*|| \le d, \quad 0 < ||v^*|| \le d.$$

Moreover, for initial values $w_0(t) = d[\frac{t^{\alpha-1}+t^{\alpha-2}}{2}]$, $v_0(t) = 0$, $t \in (0,+\infty)$, define the iterative sequences w_n and v_n by

$$w_n = Aw_{n-1} = A^n w_0, \quad v_n = Av_{n-1} = A^n v_0$$

then

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} A^n w_0 = w^*, \quad \lim_{n\to\infty} v_n = \lim_{n\to\infty} A^n v_0 = v^*.$$

Proof. By Lemma 2.8, $A : P \to P$ is completely continuous. For any $x_1, x_2 \in P$ with $x_1 \le x_2$, from the definition of A and $(H3)$, we know that $Ax_1 \le Ax_2$. Let $P_d = \{x \in P : ||x|| \le d\}$. Next, we show that $A : P_d \to P_d$. If $x \in P_d$, then $||x|| \le d$. Hence

$$0 \le \frac{t^{2-\alpha}x(t)}{1+t^{\sigma+2}} \le d \text{ for } t \in (0,+\infty).$$

By $(H4)$, we know that

$$f\Big(t, \frac{1+t^{\sigma+2}}{t^{2-\alpha}}u_0\Big) \le \frac{d}{N}t^{\sigma_1}e^{-kt}, \ (t,u_0) \in (0,+\infty)\times[0,d].$$

By Lemma 2.7, (2.9) and $(H4)$, we have

$$\begin{aligned}||(Ax)|| &= \sup_{t\in(0,+\infty)} \frac{t^{2-\alpha}}{1+t^{\sigma+2}}\Big|\int_0^\infty G(t,s)f(s,x(s))ds\Big| \le L\int_0^\infty f(s,x(s))ds \\ &\le \frac{dL}{N}\int_0^\infty s^{\sigma_1}e^{-ks}ds = \frac{dL}{N}\frac{\Gamma(\sigma_1+1)}{k^{\sigma_1+1}} \le d.\end{aligned}$$

Hence, we prove that $A: P_d \to P_d$. Let

$$w_0(t) = d\left[\frac{t^{\alpha-1}+t^{\alpha-2}}{2}\right],\ t\in(0,+\infty),$$

then $w_0(t) \in P_d$. Now, we denote a sequence $\{w_n\}$ by the iterative scheme

$$w_{n+1} = Aw_n = A^n w_0, \quad n = 0,1,2,\dots. \tag{3.1}$$

Since $A(P_d) \subset P_d$ and $w_0(t) \in P_d$, we have $w_n \in P_d, n = 0,1,2,\dots$. It follows from the complete continuity of A that $\{w_n\}_{n=1}^{\infty}$is a sequentially compact set in X, then $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in P_d$ such that $w_{n_k} \to w^*$.

On the other hand,

$$\begin{aligned}w_1(t) &= Aw_0(t) \\ &= \int_0^\infty G(t,s)f(s,w_0(s))ds \\ &= \int_0^\infty (G_0(t,s)+G_1(t,s))f(s,w_0(s))ds \\ &\le \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty f(s,w_0(s))ds \\ &\quad + \Big[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+t^{\alpha-2}\Big]\frac{1}{\psi}\int_0^\infty \frac{s^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\int_0^\infty f(s,w_0(s))ds \\ &\le \Big(\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\Big[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+t^{\alpha-2}\Big]\frac{1}{\psi}\int_0^\infty \frac{s^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\Big)\int_0^\infty f(s,w_0(s))ds \qquad (3.2) \\ &= \frac{t^{\alpha-1}\Big[1-\int_0^\infty g(s)s^{\alpha-2}ds\Big]+t^{\alpha-2}\int_0^\infty g(s)s^{\alpha-1}ds}{\Gamma(\alpha)\psi}\int_0^\infty f(s,w_0(s))ds \\ &\le \frac{t^{\alpha-1}+t^{\alpha-2}\Big[\int_0^\infty g(s)s^{\alpha-1}ds+1\Big]}{\Gamma(\alpha)\psi}\int_0^\infty f(s,w_0(s))ds \\ &\le L[t^{\alpha-1}+t^{\alpha-2}]\int_0^\infty f(s,w_0(s))ds \\ &\le \frac{dL}{N}[t^{\alpha-1}+t^{\alpha-2}]\frac{\Gamma(\sigma_1+1)}{k^{\sigma_1+1}} = d[\frac{t^{\alpha-1}+t^{\alpha-2}}{2}] = w_0(t).\end{aligned}$$

So, by (3.2) and $(H3)$, we have

$$w_2(t) = Aw_1(t) \le Aw_0(t) = w_1(t), \quad 0 < t < +\infty.$$

Moreover, we get

$$w_{n+1}(t) \le w_n(t), \quad 0 < t < +\infty, \quad n = 0, 1, 2, \dots.$$

Therefore, $w_n \to w^*$. Applying the continuity of A and $w_{n+1} = Aw_n$, we obtain $Aw^* = w^*$.

Let $v_0(t) = 0$, $0 < t < +\infty$, then $v_0(t) \in P_d$. Let $v_1 = Av_0$, then $v_1 \in P_d$. We denote

$$v_{n+1} = Av_n = A^n v_0, \quad n = 0, 1, 2, \dots.$$

Similar to $\{w_n\}_{n=1}^{\infty}$, we assert that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in P_d$ such that $v_{n_k} \to v^*$. Since $v_1 = Av_0 = A0 \in P_d$, we have

$$v_2(t) = (Av_1)(t) \ge (A0)(t) = v_1(t), \quad 0 < t < +\infty.$$

By induction, it is easy to see that for $n = 1, 2, ...$

$$v_{n+1}(t) \ge v_n(t), \quad 0 < t < +\infty.$$

Thus $v_n \to v^*$. Applying the continuity of A and $v_{n+1} = Av_n$, we get $Av^* = v^*$.

If $f(t, 0) \not\equiv 0$ on any subinterval of $(0, +\infty)$, then the zero function is not the solution of the BVP (1.1). Thus, v^* is a positive solution of BVP (1.1) on $(0, +\infty)$.

We are in a position to show that w^*, v^* are the maximal and minimal positive solutions of the BVP (1.1) in $(0, d[\frac{t^{\alpha-1}+t^{\alpha-2}}{2}]]$. Let $x \in (0, d[\frac{t^{\alpha-1}+t^{\alpha-2}}{2}]]$ be any solution of the BVP (1.1); that is, $Ax = x$. Note that A is nondecreasing and

$$v_0 = 0 \le x(t) \le d\left[\frac{t^{\alpha-1} + t^{\alpha-2}}{2}\right] = w_0(t),$$

then we have $v_1(t) = (Av_0)(t) \le x(t) \le (Aw_0)(t) = w_1(t)$, for all $t \in (0, +\infty)$.
By induction, we have

$$v_n \le x \le w_n, \ n = 1, 2, 3, \dots.$$

Since $w^* = \lim_{n\to+\infty} w_n$, $v^* = \lim_{n\to+\infty} v_n$,

$$v_0 \le v_1 \le ...v_n \le ... \le v^* \le x \le w^* \le ... \le w_n \le ... \le w_1 \le w_0.$$

Thus, w^*, v^* are the maximal and minimal positive solutions of the BVP (1.1) in $(0, d[\frac{t^{\alpha-1}+t^{\alpha-2}}{2}]]$. The proof is completed. □

Remark 3.2. By Theorem 3.1, we note that w^*, v^* are the maximal and minimal positive solutions of the BVP (1.1) in P_d, they may coincide, and then BVP (1.1) has only one solution in P_d.

4. An example

Consider the following boundary-value problem with fractional integral boundary conditions:

$$\begin{cases} D_{0^+}^{\frac{3}{2}} x(t) + f(t, x(t)) = 0, \quad t \in (0, +\infty), \\ \lim_{t\to 0} t^{2-\alpha} x(t) = \lim_{t\to\infty} D_{0^+}^{\frac{1}{2}} x(t) = \frac{1}{4}\int_0^{\infty} e^{-s} x(s) ds, \end{cases} \tag{4.1}$$

where $\alpha = \frac{3}{2}$, $g(s) = \frac{1}{4}e^{-s}$. By calculation, we have

$$\int_0^{\infty} g(s)\left[\frac{s^{\alpha-1}}{\Gamma(\alpha)} + s^{\alpha-2}\right] ds = 0.69 < 1.$$

Choose

$$f(t,x) = t^{-\frac{3}{4}}e^{-t} + \frac{t^{-\frac{1}{4}}e^{-t}}{1+t^{\frac{3}{2}}}x,$$

and $\sigma = -\frac{1}{2}, \sigma_1 = -\frac{3}{4}, k = 1$. By direct computation, we get $L \approx 5.89$ and $N \approx 42.7$. Take $d = 120$, then $(t,x) \in (0,+\infty) \times [0,d]$,

$$f\Big(t, \frac{1+t^{\frac{3}{2}}}{t^{\frac{1}{2}}}x\Big) = t^{-\frac{3}{4}}e^{-t} + t^{-\frac{3}{4}}e^{-t} = 2t^{-\frac{3}{4}}e^{-t} \leq \frac{d}{N}t^{\sigma_1}e^{-kt} = 2.81t^{-\frac{3}{4}}e^{-t},$$

$(t,x) \in (0,+\infty) \times [0,120]$.

Since all the conditions of Theorem 3.1 are satisfied, the conclusion of Theorem 3.1 holds.

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