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GENERALIZED MEIR-KEELER TYPE CONTRACTIONS AND DISCONTINUITY AT FIXED POINT

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Abstract. In this paper, we show that generalized Meir-Keeler type contractive definitions are strong enough to generate a fixed point but do not force the mapping to be continuous at the fixed point. Thus we provide more answers to the open question posed by B.E. Rhoades in the paper *Contractive definitions and continuity*, Contemporary Mathematics 72(1988), 233-245.
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1. Introduction

A fixed point theorem is one which guarantees the existence of a fixed point of mapping under suitable assumptions both on the space and the mapping. Apart from ensuring the existence of a fixed point, it often becomes essential to prove the uniqueness of the fixed point. Besides, from a computational point of view, an algorithm for calculating the value of the fixed point is desirable. Often such algorithms involve iterates of the mapping.

The questions about the existence, uniqueness and approximation of a fixed point provide three significant features of a general fixed point theorem. Most of the fixed point theorems for contractive mappings answer to all the three questions mentioned above [30]. Meanwhile, a more complete study (data dependence, well-posedness, Ulam-Hyers stability, Ostrowski property) was recently proposed in [27].

In 1988, Rhoades [25] compared 250 contractive definitions and showed that majority of the contractive definitions does not require the mapping to be continuous in the entire domain. However, in all the cases the mapping is continuous at the fixed point. He further demonstrated that the contractive definitions force the mapping to be continuous at the fixed point though continuity was neither assumed nor implied by the contractive definitions. In [3] Bryant proved that power contraction (finite compositions of the mapping) of a complete metric space need not imply continuity of the mapping. But the example presented in his paper is continuous at the fixed point.

The question whether there exists a contractive definition which is strong enough to generate a fixed point but does not force the mapping to be continuous at the fixed point was reiterated by Rhoades in [26] as an existing open problem. The question of the existence of contractive mappings which are discontinuous at their fixed points was settled in the affirmative by Pant [18]. In order to achieve his goal he employed a combination of an $(\varepsilon - \delta)$ condition and a ϕ-contractive condition to prove a fixed point in which the fixed point may be a point of discontinuity. In this paper we show that some contractive definitions which include power contractions need not be continuous at the fixed point. Thus we provide more answers to the open question posed in [26].

In [7] Jachymski listed some Meir-Keeler type conditions and established relations between them. Further he gave some new Meir-Keeler type conditions ensuring a convergence of the successive approximations. In all that follows T is a self-mapping on metric space (X, d). For $i \in \{1, \ldots, 5\}$, we consider:

$[A_i]$ for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that, for any $x, y \in X$,

$$\varepsilon \leq m_i(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon;$$

$[B_i]$ for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that, for any $x, y \in X$,

$$\varepsilon < m_i(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) \leq \varepsilon;$$

$[C_i]$ $d(Tx, Ty) < m_i(x, y)$, for any $x, y \in X$ with $m_i(x, y) > 0$, where
$m_1(x, y) = d(x, y)$,
$m_2(x, y) = \max\{d(x, Tx), d(y, Ty)\}$,
$m_3(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$,
$m_4(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \dfrac{d(x, Ty) + d(y, Tx)}{2}\right\}$,
$m_5(x, y) = \max\left\{d(x, y), \dfrac{k[d(x, Tx) + d(y, Ty)]}{2}, \dfrac{k[d(x, Ty) + d(y, Tx)]}{2}\right\}$, $0 \leq k < 1$.

Condition A_1 is studied by Meir-Keeler [16] and B_1 has been considered by Matkowski [14]. By considering the common features of various contractive definitions several authors introduced some new contractive definitions which yielded new fixed point theorems (for various contractive definitions of Meir-Keeler type one may see [4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 28, 29, 31]).

It is well-known that $A_1 \implies A_3 \implies A_4 \implies A_5$ and $A_i \implies (B_i \wedge C_i)$, for $i = 1, 2, 3, 4, 5$ but not conversely.

In this paper, we study the following condition which subsumes most of the conditions stated above.

$$m_6(x, y) = \max\left\{d(x, y), ad(x, Tx) + (1-a)d(y, Ty), (1-a)d(x, Tx) + ad(y, Ty), \frac{b[d(x, Ty) + d(y, Tx)]}{2}\right\},$$

where $0 < a < 1, 0 \leq b < 1$.

Further, we do not assume any kind of continuity condition on the mapping. Our results improve and generalize many fixed point theorems existing in the literature [1, 2, 6, 7, 10, 13, 14, 15, 16, 18, 19].

It may be observed that an $(\varepsilon - \delta)$ contractive condition does not ensure the existence of a fixed point. The following example [19] illustrates this fact.

Example 1.1. Let $X = [0, 2]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = \frac{1+x}{2} \text{ if } x \in [0,1], \quad T(x) = 0 \text{ if } x \in (1, 2].$$

Then T satisfies condition (i) of Theorem 2.1 (below) with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$ but T is a fixed point free mapping.

Therefore, to ensure the existence of fixed points under condition (i) of Theorem 2.1 (below), an additional condition is necessarily required either on δ or on the mapping. These additional conditions may assume various forms:

(A) δ is assumed lower semicontinuous [8];

(B) δ is assumed nondecreasing [20];

(C) By assuming relatively strong conditions on the continuity of mapping [21, 22];

(D) By assuming corresponding ϕ-contractive condition but without additional hypothesis on ϕ and ε [1, 18].

2. Main results

Our first main result is the following.

Theorem 2.1. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < m_6(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$;*

(ii) *$d(Tx, Ty) < m_6(x, y)$, whenever $m_6(x, y) > 0$.*

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $\lim_{x \to z} m_6(x, z) = 0$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = T x_n$ and $c_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \bigcup \{0\}$. Then by (ii)

$$c_n = d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) < m_6(x_{n-1}, x_n) = c_{n-1}.$$

Thus $\{c_n\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $c \geq 0$. If possible, suppose $c > 0$. Then there exists a positive integer $k \in \mathbb{N}$ such that $n \geq k$ implies

$$c < c_n < c + \delta(c).$$

It follows from (i) and $c_n < c_{n-1}$ that $c_n \leq c$, for $n \geq k$, which contradicts the above inequality. Thus we have $c = 0$.

We shall show that $\{x_n\}$ is a Cauchy sequence. Fix an $\varepsilon > 0$. Without loss of generality, we may assume that $\delta(\varepsilon) < \varepsilon$. Since $c_n \to 0$, there exists $k \in \mathbb{N}$ such that $c_n < \frac{1}{2}\delta$, for $n \geq k$.

Following Jachymski [7] we shall use induction to show that, for any $n \in \mathbb{N}$,

$$d(x_k, x_{k+n}) < \varepsilon + \frac{1}{2}\delta. \tag{2.1}$$

Inequality (2.1) holds for $n = 1$. Assuming (2.1) is true for some n we shall prove it for $n + 1$. By the triangle inequality, we have

$$d(x_k, x_{k+n+1}) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+n+1}).$$

Observe that it suffices to show that

$$d(x_{k+1}, x_{k+n+1}) \leq \varepsilon.$$

To show it we shall prove that $M(x_k, x_{k+n}) \leq \varepsilon + \delta$, where

$$m_6(x_k, x_{k+n}) = \max\Big\{ d(x_k, x_{k+n}), ad(x_k, Tx_k) + (1-a)d(x_{k+n}, Tx_{k+n}),$$
$$(1-a)d(x_k, Tx_k) + ad(x_{k+n}, Tx_{k+n}), \frac{b[d(x_k, Tx_{k+n}) + d(x_{k+n}, Tx_k)]}{2} \Big\}.$$

By the induction hypothesis, we get

$$d(x_k, x_{k+n}) < \varepsilon + \frac{1}{2}\delta,$$
$$ad(x_k, x_{k+1}) + (1-a)d(x_{k+n}, x_{k+n+1}) < \frac{a}{2}\delta + \frac{(1-a)}{2}\delta = \frac{1}{2}\delta,$$
$$(1-a)d(x_k, x_{k+1}) + ad(x_{k+n}, x_{k+n+1}) < \frac{(1-a)}{2}\delta + \frac{a}{2}\delta = \frac{1}{2}\delta.$$

Also,

$$\frac{b[d(x_k, Tx_{k+n+1}) + d(x_{k+1}, Tx_{k+n})]}{2}$$
$$\leq \frac{b[d(x_k, x_{k+n}) + d(x_{k+n+1}, x_{k+n}) + d(x_k, x_{k+1}) + d(x_k, x_{k+n})]}{2}$$
$$< \frac{[d(x_k, x_{k+n}) + d(x_{k+n+1}, x_{k+n}) + d(x_k, x_{k+1}) + d(x_k, x_{k+n})]}{2} < \varepsilon + \delta.$$

Thus $M(x_k, x_{k+n}) < \varepsilon + \delta$ so by (ii) $d(x_{k+1}, x_{k+n+1}) \leq \varepsilon$, completing the induction. Hence (2.1) implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. We claim that $Tz = z$. For if $Tz \neq z$, using (ii) we get

$$d(Tz, Tx_n) < \max\Big\{ d(z, x_n), ad(z, Tz) + (1-a)d(x_n, Tx_n), (1-a)d(z, Tz)$$
$$+a(d(x_n, Tx_n), \frac{b[d(z, Tx_n) + d(x_n, Tz)]}{2} \Big\}.$$

On letting $n \to \infty$ this yields,

$$d(Tz, z) \leq \max\Big\{ ad(z, Tz), (1-a)d(z, Tz), \frac{b[d(z, Tz)]}{2} \Big\},$$

that is, $Tz = z$ and z is a fixed point of T. Uniqueness of the fixed point follows easily.

Now, let T be continuous at the fixed point z and $x_n \to z$. Then $Tx_n \to Tz = z$. Hence

$$\lim_n m_6(x_n, z) = \lim_n \max\Big\{ d(x_n, z), ad(x_n, Tx_n) + (1-a)d(z, Tz),$$

$$(1-a)d(x_n, Tx_n) + ad(z, Tz), \frac{b[d(x_n, Tz) + d(z, Tx_n)]}{2}\bigg\} = 0.$$

On the other hand, if $\lim_{x_n \to z} m_6(x_n, z) = 0$, then $d(x_n, Tx_n) \to 0$ as $x_n \to z$. This implies that $Tx_n \to z = Tz$, i.e., T is continuous at z. This concludes the theorem. □

Remark 2.2. The last part of Theorems 2.1 can alternatively be stated as: T is discontinuous at z iff $\lim_{x\to z} m_6(x, z) \neq 0$.

Theorem 2.1 is also true if we take $a = 0, b = 1$ in $m_6(x, y)$.

Corollary 2.3. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < m'_6(x, y) < \varepsilon+\delta$ implies $d(Tx, Ty) \leq \varepsilon$;*
(ii) *$d(Tx, Ty) < m'_6(x, y)$, whenever*

$$m'_6(x, y) = \max\bigg\{d(x, y), d(x, Tx), d(y, Ty), \frac{[d(x, Ty) + d(y, Tx)]}{2}\bigg\} > 0.$$

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $\lim_{x\to z} m'_6(x, z) = 0$.

The following example [18] illustrates the above theorem:

Example 2.4. Let $X = [0, 2]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = 1 \text{ if } x \in [0, 1], \quad T(x) = 0 \text{ if } x \in (1, 2].$$

Then T satisfies the conditions of Theorem 2.1 and has a unique fixed point $x = 1$ at which T is discontinuous. The mapping T satisfies condition (i) with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$. It can also be easily seen that $\lim_{x\to 1} m_6(x, 1) \neq 0$ and T is discontinuous at the fixed point $x = 1$.

The following theorem shows that power contraction allows the possibility of discontinuity at the fixed point. In the next theorem we denote:

$$m''_6(x, y) = \max\bigg\{d(x, y), ad(x, T^q x) + (1-a)d(y, T^q y),$$

$$(1-a)d(x, T^q x) + ad(y, T^q y), \frac{b[d(x, T^q y) + d(y, T^q x)]}{2}\bigg\},$$

where $0 < a < 1$ and $0 \leq b < 1$ and $q \in \mathbb{N}$.

Theorem 2.5. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < m''_6(x, y) < \varepsilon + \delta$ implies $d(T^q x, T^q y) \leq \varepsilon$;*
(ii) *$d(T^q x, T^q y) < m''_6(x, y)$, whenever $m''_6(x, y) > 0$.*

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$.

Proof. By Theorem 2.1, T^q has a unique fixed point $z \in X$; i.e., $T^q(z) = z$. Then $T(z) = T(T^q(z)) = T^q(T(z))$ and so $T(z)$ is a fixed point of T^q. Since the fixed point of T^q is unique, $Tz = z$. To prove the uniqueness, we assume that y is another fixed point of T. Then $Ty = y$ and so $T^q(y) = y$. Again by the uniqueness of the fixed point of T^q, we have $z = y$. Hence z is a fixed point of T.

Recall that the set $O(x;T) = \{T^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of the self-mapping T at the point $x \in X$.

Definition 2.6. A self-mapping T of a metric space (X, d) is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(x;T)$ (for some $x \in X$) $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.

Every continuous self-mapping of a metric space is orbitally continuous, but converse need not be true (see Example 2.4 above).

Theorem 2.7. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < m_6(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \le \varepsilon$;*
(ii) *$d(Tx, Ty) < m_6(x, y)$, whenever $m_6(x, y) > 0$.*

Suppose T is orbitally continuous. Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $\lim_{x \to z} m_6(x, z) = 0$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Then following the proof of Theorem 2.1 we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. Orbital continuity of T implies that $\lim_{n\to\infty} Tx_n = Tz$. This yields $Tz = z$, that is, z is a fixed point of T. Uniqueness of the fixed point follows from (ii). This concludes the theorem.

In the next theorem, we replace the orbital continuity of the mapping T by continuity condition on T^p, where $p \ge 2$ is an integer.

Theorem 2.8. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that such that T^p is continuous for any $x, y \in X$;*

(i) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < m_6(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \le \varepsilon$;*
(ii) *$d(Tx, Ty) < m_6(x, y)$, whenever $m_6(x, y) > 0$.*

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $\lim_{x \to z} m_6(x, z) = 0$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Then following the proof of above theorem we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$ and $T^p x_n \to z$. By continuity of T^p, we have $T^p x_n \to T^p z$. This implies $T^p z = z$. We claim that $Tz = z$. For if $Tz \ne z$, we get

$$
\begin{aligned}
& d(Tz, z) = d(Tz, T^p z) < m_6(z, T^{p-1} z) = d(T^p z, T^{p-1} z); \\
& d(T^p z, T^{p-1} z) < m_6(T^{p-1} z, T^{p-2} z) = d(T^{p-1} z, T^{p-2} z); \\
& \vdots \\
& d(T^2 z, Tz) < m_6(Tz, z) = d(Tz, z),
\end{aligned}
$$

a contradiction. Thus z is a fixed point of T. Uniqueness of the fixed point follows from (ii).

Corollary 2.9. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^q is continuous and satisfy the condition $d(Tx, Ty) < d(x, y)$ for $x \ne y$,* *and* $d(Tx,Ty) \leq \phi[d(x,y)]$ *for any* $x, y \in X$*, where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$*, satisfies the condition*

(ϕ) *for a given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that, for any* t*,*

$$\varepsilon < t < \varepsilon + \delta \text{ implies } \phi(t) \leq \varepsilon.$$

Then T *has a unique fixed point, say* z*, and* $T^n x \to z$ *for each* $x \in X$*.*

Proof. Obviously, such a mapping T satisfies B_1 and C_1, and hence B_6 and C_6. Moreover, T^q is continuous so for $q = 1$ the corollary is same as proved in [7]. For $q \geq 2$ apply Theorem 2.8 (above) to get the result.

Remark 2.10. The above proved theorems unify and improve the results due to Bisht and Pant [1], Boyd and Wong [2], Ćirić [4, 5], Jachymski [7], Kannan [9], Kuczma et al. [10], Maiti and Pal [13], Matkowski [14], Meir and Keeler [16] and Pant [18, 19].

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