Fixed Point Theory, 19(2018), No. 1, 45-56
DOI 10.24193/fpt-ro.2018.1.05
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

Δ-CONVERGENCE THEOREMS FOR INVERSE-STRONGLY MONOTONE MAPPINGS IN CAT(0) SPACES

SATTAR ALIZADEH*, HOSSEIN DEHGHAN** AND FRIDOUN MORADLOU***

*Department of Mathematics, Marand Branch
Islamic Azad University, Marand, Iran
E-mail: salizadeh@marandiau.ac.ir, s_alizadeh_s@yahoo.com

**Department of Mathematics
Institute for Advanced Studies in Basic Sciences (IASBS)
Gava Zang, Zanjan 45137-66731, Iran
E-mail: hossein.dehgan@gmail.com

***Department of Mathematics, Sahand University of Technology
Tabriz, Iran
E-mail: moradlou@sut.ac.ir, fridoun.moradlou@gmail.com

Abstract. In this paper, we first define and study inverse-strongly monotone mappings in general metric spaces. Then, we prove the existence theorem of solutions for variational inequalities involving such mappings. Finally, we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for inverse-strongly monotone mappings in CAT(0) metric spaces.
Key Words and Phrases: inverse-strongly monotone mapping; nonexpansive mapping; variational inequality; fixed point; CAT(0) metric space.
2010 Mathematics Subject Classification: 47H09, 47H10, 47J20, 47J25.

1. Introduction

Let (X, d) be a metric space. Berg and Nikolaev [3] introduced the concept of *quasilinearization* in metric spaces. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is the map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} \left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d) \right), \quad (a, b, c, d \in X). \tag{1.1}$$

It is easily seen that

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle, \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle \text{ and } \langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle,$$

for all $a, b, c, d, x \in X$.

*Corresponding author.

Using the concept of quasilinearization, we may formulate a variational inequality (VI) in metric spaces as finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad \langle \overrightarrow{x^*Tx^*}, \overrightarrow{xx^*} \rangle \geq 0, \quad \forall x \in C, \tag{1.2}$$

where C is a nonempty subset of a metric space X and $T : C \to X$ is a mapping. The set of solutions of the variational inequality problem (1.2) is denoted by $\mathrm{VI}(C, T)$. It is easy to verify that if $X = H$ is a Hilbert space, then VI (1.2) reduces to the following VI: find x^* such that

$$x^* \in C \quad \text{and} \quad \langle x^* - Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

Various forms of VI (1.3) have been extensively studied by many authors (see, e.g., [1, 7, 12, 19, 6] and references therein).

The purpose of this paper is to investigate existence and approximation of solutions for VI (1.2) when X is a complete CAT(0) space and $T : C \to X$ is a non-self inverse-strongly monotone mappings. To the best of our knowledge, this would probably be the first time in the literature that finding solutions of variational inequalities of the kind (1.2) is investigated in the framework of metric spaces without linear structure.

2. Preliminaries

A metric space (X, d) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. For other equivalent definitions and basic properties, we refer the reader to standard texts such as [2, 4]. Complete CAT(0) spaces are often called Hadamard spaces. Let $x, y \in X$ and $\lambda \in [0, 1]$. We write $\lambda x \oplus (1 - \lambda)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = (1 - \lambda)d(x, y) \quad \text{and} \quad d(z, y) = \lambda d(x, y). \tag{2.1}$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is,

$$[x, y] = \{\lambda x \oplus (1 - \lambda)y : \lambda \in [0, 1]\}.$$

A subset C of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$. The metric space X is said to satisfy the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leqslant d(a, b)d(c, d),$$

for all $a, b, c, d \in X$. It is known [3, Corollary 3] that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

The concept of Δ-convergence introduced by Lim [17] in 1976 was shown by Kirk and Panyanak [15] in CAT(0) spaces to be very similar to the weak convergence in Hilbert space setting. Next, we give the concept of Δ-convergence and collect some basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [9] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.
A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Uniqueness of asymptotic center implies that CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y). \tag{2.2}$$

We need following lemmas in the sequel.

Lemma 2.1. [15] *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.2. [10] *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.3. [13] *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in X$.*

Lemma 2.4. [11, Lemma 2.5] *A geodesic space X is a CAT(0) space if and only if the following inequality*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x, z) + (1-\lambda)d^2(y, z) - \lambda(1-\lambda)d^2(x, y), \tag{2.3}$$

is satisfied for all $x, y, z \in X$ and $\lambda \in [0, 1]$.

Let C be a nonempty closed convex subset of a complete CAT(0) space X. It is known that for any $x \in X$ there exists a unique point $u \in C$ such that

$$d(x, u) = \inf_{y\in C} d(x, y).$$

The mapping $P_C : X \to C$ defined by $P_C x = u$ is called the metric projection from X onto C. It follows from Proposition 2.4 of [4] that P_C is nonexpansive and for each $x \in X$,

$$P_C[\lambda P_C x \oplus (1-\lambda)x] = P_C x. \tag{2.4}$$

Recently, Dehghan and Rooin [8] obtained the following characterization of metric projection in CAT(0) metric spaces.

Theorem 2.5. [8, Theorem 2.2] *Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then*

$$u = P_C x \quad \textit{if and only if} \quad \langle \overrightarrow{ux}, \overrightarrow{yu}\rangle \geq 0, \quad \textit{for all } y \in C.$$

3. Inverse-strongly monotone mappings in CAT(0) metric spaces

In this section, we present an appropriate definition of inverse-strongly monotone mappings in metric space. We prove that the fixed points set of such mappings is nonempty closed and convex, which will be useful to study an existence of solutions of variational inequalities in the next section.

Let C be a nonempty subset of a metric space X. Then a mapping $T : C \to X$ is called *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in C$. A point $x \in C$ is called a fixed point of T if $x = Tx$. We denote by $F(T)$ the set of all fixed points of T. Kirk [16, Theorem 5.1] showed that every nonexpansive self-mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Also, $F(T)$ is closed and convex.

To obtain an appropriate definition for inverse-strongly monotone mappings in metric spaces, we first recall their definition in Hilbert spaces. Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$ and C be a nonempty subset of H. A mapping $A : C \to H$ is called an α-inverse-strongly monotone [5, 18] if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{3.1}$$

Putting $T = I - A$, where I is identity map on C, we see that $I - T$ is α-inverse-strongly monotone.

Definition 3.1. Let C be a nonempty subset of a metric space X and $T : C \to X$ be a mapping. Let us formally say that "$I - T$ is *inverse-strongly monotone*" if there exists a positive real number $\alpha > 0$ such that

$$d^2(x, y) - \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \geq \alpha \ \Phi_T(x, y), \quad \forall x, y \in C, \tag{3.2}$$

where $\Phi_T(x, y) = d^2(x, y) + d^2(Tx, Ty) - 2\langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle$. If (3.2) holds, we also say that $I - T$ is a α-inverse-strongly monotone.

Note that $I - T$ is just a symbol. Furthermore, the definition of inverse-strongly monotone mapping finds its origin in Hilbert spaces. If X is a CAT(0) space the for every mapping $T : C \to X$ and every $x, y \in C$, the quantity $\Phi_T(x, y)$ is nonnegative by the Cauchy-Schwarz and AGM inequality.

Example 3.2. Consider $\mathbb{R}^2$ with the usual Euclidean metric d. Let $X = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r, where $d_r(x, y) = d(x, y)$ if x and y are situated on a Euclidean straight line passing through the origin and $d_r(x, y) = d(x, \mathbf{0}) + d(y, \mathbf{0})$ otherwise (see [14] and [20, page 65]). We put $p = (0, 1)$, $q = (1, 0)$ and $C = A \cup B \cup D$, where

$$A = \{(0, t) : t \in [2/3, 1]\}, \quad B = \{(t, 0) : t \in [2/3, 1]\},$$
$$D = \{(t, s) : t + s = 1, t \in (0, 1)\}$$

and define $T : C \to C$ by

$$Tx = \begin{cases} q & x \in A, \\ p & x \in B, \\ x & x \in C. \end{cases}$$

We show that $I - T$ is $\frac{1}{4}$-inverse-strongly monotone. For $\alpha = \frac{1}{4}$ the inequality (3.2) is equivalent to

$$3d_r^2(x,y) + d_r^2(Tx,x) + d_r^2(Ty,y) \geq d_r^2(Tx,Ty) + d_r^2(Tx,y) + d_r^2(x,Ty). \tag{3.3}$$

It is easy to verify that (3.3) holds when both of x and y are in A, B or D.
In the case that $x = (0,t) \in A$ and $y = (s,0) \in B$, the inequality (3.3) turns to the following valid inequality:

$$3(t+s)^2 + (1+t)^2 + (1+s)^2 \geq 4 + (1-s)^2 + (1-t)^2.$$

If $x = (0,t) \in A$ and $y = (u,v) \in D$, then the inequality (3.3) is equivalent to

$$2\left(t + \sqrt{u^2+v^2}\right)^2 + (1+t)^2 \geq 2\left(1 + \sqrt{u^2+v^2}\right)^2,$$

which can be obtained by simple calculation. By the symmetry we conclude that T is $\frac{1}{4}$-inverse-strongly monotone. Note that T is not a nonexpansive mapping. In fact, if $x = (0, 2/3)$ and $y = (2/3, 0)$, then we have

$$d_r(Tx,Ty) = d_r(p,q) = 2 > \frac{4}{3} = d_r(x,y).$$

Lemma 3.3. *Let X be a CAT(0) space, $a,b,c,d \in X$ and $\lambda \in [0,1]$. Then*

$$\begin{aligned} d^2(\lambda a \oplus (1-\lambda)b, \lambda c \oplus (1-\lambda)d) \quad \leq \quad & \lambda^2 d^2(a,c) + (1-\lambda)^2 d^2(b,d) \\ & +2\lambda(1-\lambda)\langle \overrightarrow{ac}, \overrightarrow{bd} \rangle. \end{aligned}$$

Proof. The assertion can be easily obtained by twice using (2.3). □

Proposition 3.4. *Let C be a nonempty convex subset of a CAT(0) space X and $T : C \to X$ be a mapping such that $I - T$ is an α-inverse-strongly monotone. Assume $\lambda \in [0,1]$ and define $T_\lambda : C \to X$ by $T_\lambda x = (1-\lambda)x \oplus \lambda Tx$. If $0 < \lambda < 2\alpha$, then T_λ is nonexpansive and $F(T_\lambda) = F(T)$.*

Proof. By Lemma 3.3 and the fact that $\Phi_T(x,y)$ is nonnegative, we have

$$\begin{aligned} d^2(T_\lambda x, T_\lambda y) &= d^2((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty) \\ &\leq (1-\lambda)^2 d^2(x,y) + \lambda^2 d^2(Tx,Ty) + 2\lambda(1-\lambda)\langle \overrightarrow{xy}, \overrightarrow{TxTy} \rangle \\ &= d^2(x,y) + \lambda^2 \left[d^2(x,y) + d^2(Tx,Ty) - 2\langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle\right] \\ &\quad -2\lambda \left[d^2(x,y) - \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle\right] \\ &\leq d^2(x,y) + \lambda^2 \left[d^2(x,y) + d^2(Tx,Ty) - 2\langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle\right] \\ &\quad -2\alpha\lambda\Phi_T(x,y) \\ &= d^2(x,y) - \lambda\left[2\alpha - \lambda\right]\Phi_T(x,y) \\ &\leq d^2(x,y). \end{aligned}$$

Thus, T_λ is nonexpansive. Since

$$d(x, T_\lambda x) = \lambda d(x, Tx), \tag{3.4}$$

then $T_\lambda x = x$ if and only if $Tx = x$. □

If $I-T$ is α-inverse-strongly monotone and $p \in F(T)$, then by a simple computation we have

$$d^2(Tx, p) \leq d^2(x, p) + (1 - 2\alpha)d^2(x, Tx), \tag{3.5}$$

for all $x \in C$.

Lemma 3.5. *Let $T : C \to X$ be a mapping such that $I - T$ is an α-inverse-strongly monotone. If $F(T) \neq \emptyset$, then $F(P_C T) = F(T)$.*

Proof. Clearly, $F(T) \subset F(P_C T)$. Thus, we only need to show the converse inclusion. Assume that $x = P_C Tx$, then for $p \in F(T)$ it follows from Theorem 2.5 that

$$\begin{aligned} d^2(Tx, p) &= d^2(Tx, x) + d^2(p, x) + 2\langle \overrightarrow{Txx}, \overrightarrow{xp} \rangle \\ &= d^2(Tx, x) + d^2(p, x) + 2\langle \overrightarrow{Tx(P_C Tx)}, \overrightarrow{(P_C Tx)p} \rangle \\ &\geq d^2(Tx, x) + d^2(p, x). \end{aligned} \tag{3.6}$$

On the other hand, by (3.5), we have

$$d^2(Tx, p) \leq d^2(x, p) + (1 - 2\alpha)d^2(Tx, x).$$

This together with (3.6) implies that

$$2\alpha d^2(Tx, x) \leq 0.$$

Therefore, $x = Tx$. □

Remark 3.6. The Lemma 3.5 is valid for nonexpansive mapping.

Theorem 3.7. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X and $T : C \to X$ be a mapping such that $I - T$ is an α-inverse-strongly monotone. Then, $F(T)$ is nonempty, closed and convex.*

Proof. Let $\lambda \in [1 - 2\alpha, 1) \cap [0, 1)$. Using Proposition 3.4 and Remark 3.6, we have

$$F(T) = F(T_\lambda) = F(P_C T_\lambda).$$

Since P_C and T_λ are nonexpansive, then $P_C T_\lambda$ is a nonexpansive self-mapping of C. Therefore, Theorem 5.1 of [16] guarantees that $F(P_C T_\lambda) \neq \emptyset$. It is an easy task to prove that $F(S)$ is closed and convex for nonexpansive mapping $S : C \to C$. □

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that "$I - T$ is demiclosed at zero" if the conditions, $\{x_n\} \subseteq C$, Δ- converges to x^* and $d(x_n, Tx_n) \to 0$ imply $x^* \in F(T)$.

Theorem 3.8. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T : C \to X$ be a mapping. If $I - T$ is an α-inverse-strongly monotone, then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\} \subseteq C$ is Δ-convergent to x^* and $d(x_n, Tx_n) \to 0$. It follows respectively from Lemma 2.2 and (3.4) that $x^* \in C$ and

$$d(x_n, T_\lambda x_n) = (1-\lambda) d(x_n, Tx_n) \to 0, \tag{3.7}$$

where T_λ is as in Proposition 3.4. If $x^* \neq T_\lambda x^*$, then by Opial's property (2.2), we have

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, x^*) &< \limsup_{n\to\infty} d(x_n, T_\lambda x^*) \\ &\leq \limsup_{n\to\infty} [d(x_n, T_\lambda x_n) + d(T_\lambda x_n, T_\lambda x^*)] \\ &\leq \limsup_{n\to\infty} d(x_n, x^*), \end{aligned}$$

which is a contradiction. Hence, $x^* = T_\lambda x^*$ and then by Proposition 3.4 $x^* = Tx^*$. □

4. Existence and convergence theorems

Consider the variational inequality of finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad \langle \overrightarrow{x^*Tx^*}, \overrightarrow{xx^*} \rangle \geq 0, \quad \forall x \in C, \tag{4.1}$$

where C is a closed convex subset of a complete CAT(0) metric space X and $T : C \to X$ is a mapping such that $I - T$ is an α-inverse-strongly monotone.

The purpose for this section is to prove an existence theorem for VI (4.1) and to introduce an iterative algorithm to approximate common element of the set of solutions of the VI (4.1) and the set of fixed points of a nonexpansive mapping. To proceed in this direction, we need the following interesting lemma which can be used for an arbitrary mapping $T : C \to X$.

Lemma 4.1. *Let C be a nonempty convex subset of a complete CAT(0) space X and $T : C \to X$ be a mapping. Then,*

$$VI(C, T) = VI(C, T_\lambda),$$

where $\lambda \in (0, 1]$ and $T_\lambda : C \to X$ is a mapping defined by $T_\lambda x = (1-\lambda)x \oplus \lambda Tx$ for all $x \in C$.

Proof. Let $u \in VI(C, T_\lambda)$. For each $x \in C$, it follows from (2.1) and (2.3) that

$$\begin{aligned} 0 &\leq 2\langle \overrightarrow{uT_\lambda u}, \overrightarrow{xu} \rangle = d^2(T_\lambda u, x) - d^2(u, x) - d^2(T_\lambda u, u) \\ &\leq (1-\lambda) d^2(u, x) + \lambda d^2(Tu, x) - \lambda(1-\lambda) d^2(u, Tu) - d^2(u, x) - \lambda^2 d^2(Tu, u) \\ &= \lambda \left[d^2(Tu, x) - d^2(u, x) - d^2(u, Tu) \right] \\ &= 2\lambda \langle \overrightarrow{uTu}, \overrightarrow{xu} \rangle, \end{aligned}$$

which implies that $u \in VI(C, T)$.

For the converse let $u \in VI(C, T)$. Using Theorem 2.5, we conclude that $u = P_C Tu$.

It follows from (2.4) that

$$\begin{aligned} P_C T_\lambda u &= P_C[(1-\lambda)u \oplus \lambda Tu] \\ &= P_C[(1-\lambda)P_C Tu \oplus \lambda Tu] \\ &= P_C Tu \\ &= u. \end{aligned}$$

Again by Theorem 2.5, we have $u \in VI(C, T_\lambda)$. □

The following theorem guarantees the existence of solution for VI (4.1).

Theorem 4.2. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X and $T : C \to X$ be an α-inverse-strongly monotone. Then $VI(C, T)$ is nonempty, closed and convex.*

Proof. Let $0 < \lambda \leq 2\alpha$. It follows from Lemma 4.1 and Theorem 2.5 that

$$VI(C, T) = VI(C, T_\lambda) = F(P_C T_\lambda),$$

where the last term is nonempty, closed and convex as mentioned in the second paragraph of Section 3. □

Note that we may conclude Theorem 4.2 from Lemma 3.5, Theorems 2.5 and 3.7 by a similar way.

Next, we introduce an iterative scheme to approximate solutions of VI (4.1). Our algorithm generates a sequence $\{x_n\}$ through the recursive formula

$$\begin{cases} y_n = P_C[\beta_n x_n \oplus (1-\beta_n)Tx_n] \\ x_{n+1} = P_C[\alpha_n x_n \oplus (1-\alpha_n)Sy_n], \quad n \geq 0, \end{cases} \tag{4.2}$$

where the initial guess x_0 is arbitrary and $\{\alpha_n\}$ and $\{\beta_n\}$ are real control sequences in the interval $(0, 1)$.

To prove convergence theorem of (4.2),we need the following lemma.

Lemma 4.3. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $\{z_n\}$ be a sequence in X such that*

$$d(z_{n+1}, z) \leq d(z_n, z),$$

for all $z \in C$ and $n \geq 0$. Then, $P_C z_n$ converges to some $u \in C$.

Proof. Put $u_n = P_C z_n$. For any $m > n \geq 1$, we have

$$d^2(u_m, u_n) = d^2(z_m, u_n) - d^2(z_m, u_m) - 2\langle \overrightarrow{z_m u_m}, \overrightarrow{u_m u_n} \rangle.$$

By Theorem 2.5, we know that $\langle \overrightarrow{z_m u_m}, \overrightarrow{u_m u_n} \rangle \geq 0$. Thus,

$$\begin{aligned} d^2(u_m, u_n) &\leq d^2(z_m, u_n) - d^2(z_m, u_m) \\ &\leq d^2(z_n, u_n) - d^2(z_m, u_m). \end{aligned} \tag{4.3}$$

On the other hand,

$$d^2(z_m, u_m) \leq d^2(z_m, u_n) \leq d^2(z_n, u_n),$$

which implies that $\lim_{n\to\infty} d^2(z_n, u_n)$ exists. Therefore, it follows from (4.3) that $\{u_n\}$ is Cauchy sequence and so converges to some $u \in C$. □

Theorem 4.4. *Let C be a nonempty closed convex subset of a complete CAT(0) space X, $T : C \to X$ be a mapping such that $I - T$ is an α-inverse-strongly monotone and $S : C \to X$ be a nonexpansive mapping such that $F(S) \cap VI(C,T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by (4.2). If $\alpha_n \subset [\alpha, \gamma]$ for some $\alpha, \gamma \in (0,1)$ and $\beta_n \subset [\beta, \delta]$ for some $\beta, \delta \in (1-2\alpha, 1)$, then $\{x_n\}$ is Δ-convergent to $q \in F(S) \cap VI(C,T)$, where $q = \lim_{n\to\infty} P_{F(S)\cap VI(C,T)} x_n$.*

Proof. It follows from Theorem 2.5 and Lemma 3.5 that $VI(C,T) = F(P_C T) = F(T)$. Let $p \in F(S) \cap VI(C,T)$. It follows from (2.3), (3.5) and nonexpansiveness of P_C that

$$\begin{aligned} d^2(y_n,p) &= d^2(P_C[\beta_n x_n \oplus (1-\beta_n)Tx_n], P_C p) \\ &\leq d^2(\beta_n x_n \oplus (1-\beta_n)Tx_n, p) \\ &\leq \beta_n d^2(x_n,p) + (1-\beta_n)d^2(Tx_n,p) - \beta_n(1-\beta_n)d^2(x_n,Tx_n) \\ &\leq \beta_n d^2(x_n,p) + (1-\beta_n)\left[d^2(x_n,p) + (1-2\alpha)d^2(x_n,Tx_n)\right] \\ &\quad -\beta_n(1-\beta_n)d^2(x_n,Tx_n) \\ &= d^2(x_n,p) - (1-\beta_n)(\beta_n - (1-2\alpha))d^2(x_n,Tx_n). \end{aligned} \tag{4.4}$$

Since S is nonexpansive, using (2.3) and (4.4), we have

$$\begin{aligned} d^2(x_{n+1},p) &\leq d^2(\alpha_n x_n \oplus (1-\alpha_n)Sy_n, p) \\ &\leq \alpha_n d^2(x_n,p) + (1-\alpha_n)d^2(Sy_n,p) - \alpha_n(1-\alpha_n)d^2(x_n,Sy_n) \\ &\leq \alpha_n d^2(x_n,p) + (1-\alpha_n)d^2(y_n,p) - \alpha_n(1-\alpha_n)d^2(x_n,Sy_n) \\ &\leq \alpha_n d^2(x_n,p) + (1-\alpha_n)\left[d^2(x_n,p) - (1-\beta_n)(\beta_n-(1-2\alpha))d^2(x_n,Tx_n)\right] \\ &\quad - \alpha_n(1-\alpha_n)d^2(x_n,Sy_n) \\ &= d^2(x_n,p) - (1-\alpha_n)(1-\beta_n)(\beta_n - (1-2\alpha))d^2(x_n,Tx_n) \\ &\quad - \alpha_n(1-\alpha_n)d^2(x_n,Sy_n). \end{aligned} \tag{4.5}$$

Thus, by the conditions $0 < \alpha \leq \alpha_n \leq \gamma < 1$ and $(1-2\alpha) < \beta \leq \beta_n \leq \delta < 1$ for all $n \geq 0$, we have $d(x_{n+1},p) \leq d(x_n,p)$, that is, the sequence $\{d(x_n,p)\}$ is decreasing and so $\lim_{n\to\infty} d(x_n,p)$ exists. Moreover, from (4.5), we have

$$\begin{aligned} (1-\gamma)(1-\delta)(\beta-(1-2\alpha))d^2(x_n,Tx_n) &\leq (1-\alpha_n)(1-\beta_n)(\beta_n-(1-2\alpha))d^2(x_n,Tx_n) \\ &\leq d^2(x_n,p) - d^2(x_{n+1},p) \end{aligned}$$

and

$$\begin{aligned} \alpha(1-\gamma)d^2(x_n,Sy_n) &\leq \alpha_n(1-\alpha_n)d^2(x_n,Sy_n) \\ &\leq d^2(x_n,p) - d^2(x_{n+1},p), \end{aligned}$$

which imply that

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} d(x_n,Sy_n) = 0. \tag{4.6}$$

Using (2.1), we obtain $d(y_n,x_n) = (1-\beta_n)d(x_n,Tx_n) \to 0$. Therefore,

$$\begin{aligned} d(x_n,Sx_n) &\leq d(x_n,Sy_n) + d(Sy_n,Sx_n) \\ &\leq d(x_n,Sy_n) + d(y_n,x_n) \to 0. \end{aligned} \tag{4.7}$$

Since $\{x_n\}$ is bounded, it follows from Lemma 2.1 that $\omega_\Delta(x_n) \neq \emptyset$, where

$$\omega_\Delta(x_n) = \{x \in X : x_{n_i} \ \Delta\text{-converges to } x \text{ for some subsequence } \{n_i\} \text{ of } \{n\}\}.$$

Next, we show that $\omega_\Delta(x_n) \subset F(S) \cap VI(C,T)$ and it is singleton. Let $p \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which Δ-converges to p. Using (4.6), (4.7) and Theorem 3.8 (demiclosedness of S and T), we get $p \in F(S) \cap VI(C,T)$ and so $\omega_\Delta(x_n) \subset F(S) \cap VI(C,T)$. Let $p, q \in \omega_\Delta(x_n)$ and let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ which Δ-converge to p and q, respectively. If $p \neq q$, then from (2.2) and the fact that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(S) \cap VI(C,T)$, we have

$$\begin{aligned} \lim_{n\to\infty} d(x_n,p) &= \limsup_{i\to\infty} d(x_{n_i},p) < \limsup_{i\to\infty} d(x_{n_i},q) \\ &= \lim_{n\to\infty} d(x_n,q) = \limsup_{j\to\infty} d(x_{n_j},q) \\ &< \limsup_{j\to\infty} d(x_{n_j},p) = \lim_{n\to\infty} d(x_n,p), \end{aligned}$$

which is a contradiction. Hence, $p = q$ and $\omega_\Delta(x_n) = \{p\}$. Thus $\{x_n\}$ Δ-converges to p.

Put $u_n = P_{F(S)\cap VI(C,T)} x_n$. We show that $p = \lim_{n\to\infty} u_n$. Since $p \in F(S)\cap VI(C,T)$, it follows from Theorem 2.5 that

$$\langle \overrightarrow{u_n x_n}, \overrightarrow{p u_n} \rangle \geq 0.$$

By Lemma 4.3, $\{u_n\}$ converges strongly to some $u \in F(S) \cap VI(C,T)$. Also,

$$\begin{aligned} 0 &\leq \langle \overrightarrow{u_n x_n}, \overrightarrow{p u_n} \rangle \\ &= \langle \overrightarrow{u_n p}, \overrightarrow{p u_n} \rangle + \langle \overrightarrow{p x_n}, \overrightarrow{p u} \rangle + \langle \overrightarrow{p x_n}, \overrightarrow{u u_n} \rangle \\ &\leq \langle \overrightarrow{u_n p}, \overrightarrow{p u_n} \rangle + \langle \overrightarrow{p x_n}, \overrightarrow{p u} \rangle + d(p,x_n) d(u,u_n). \end{aligned}$$

Taking $\limsup_{n\to\infty}$, using Lemma 2.3 and the fact that x_n Δ-converges to p and $u_n \to u$, we obtain

$$0 \leq \langle \overrightarrow{pu}, \overrightarrow{up} \rangle = -d^2(p,u),$$

which gives us $p = u$ and the proof is complete. □

Definition 4.5. Let C be a nonempty subset of a CAT(0) space X. A mapping $T : C \to X$ is called *strict pseudo-contraction* if there exists a constant $0 \leq \kappa < 1$ such that

$$d^2(Tx,Ty) \leq d^2(x,y) + 4\kappa d^2\left(\frac{1}{2}x \oplus \frac{1}{2}Ty, \frac{1}{2}y \oplus \frac{1}{2}Tx\right), \tag{4.8}$$

for all $x, y \in C$. If (4.8) holds, we also say that T is a κ-strict pseudo-contraction.

By the definition of strict pseudo-contraction (4.8) and Lemma 3.3, we have

$$\begin{aligned} d^2(Tx,Ty) &\leq d^2(x,y) + 4\kappa d^2\left(\frac{1}{2}x \oplus \frac{1}{2}Ty, \frac{1}{2}y \oplus \frac{1}{2}Tx\right) \\ &\leq d^2(x,y) + \kappa d^2(x,y) + \kappa d^2(Tx,Ty) + 2\kappa \langle \overrightarrow{xy}, \overrightarrow{TyTx} \rangle, \end{aligned}$$

which is equivalent to

$$\frac{1-\kappa}{2}\left(d^2(x,y)+d^2(Tx,Ty)-2\langle\overrightarrow{TxTy},\overrightarrow{xy}\rangle\right)\leq d^2(x,y)-\langle\overrightarrow{TxTy},\overrightarrow{xy}\rangle. \tag{4.9}$$

Hence, $I-T$ is $\frac{1-\kappa}{2}$-inverse-strongly monotone mapping and we have the following corollary.

Corollary 4.6. *Let C be a nonempty closed convex subset of a complete CAT(0) space X, $T : C \to C$ be a κ-strict pseudo-contraction for some $0 \leq \kappa < 1$ and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)S[\beta_n x_n \oplus (1-\beta_n)Tx_n], \qquad n \geq 0.$$

If $\alpha_n \subset [\alpha,\gamma]$ for some $\alpha,\gamma \in (0,1)$ and $\beta_n \subset [\beta,\delta]$ for some $\beta,\delta \in (\kappa,1)$, then $\{x_n\}$ converges weakly to $q \in F(S)\cap F(T)$, where $q = \lim_{n\to\infty} P_{F(S)\cap F(T)}x_n$.

Remark 4.7. From Theorem 4.4 and Corollary 4.6, we can respectively deduce Theorem 3.1 and Theorem 4.1 of Takahashi and Toyoda [21] as a corollary.

Acknowledgements. The authors would like to thank Marand Branch, Islamic Azad University for the Financial support of this research, which is based on a research project contract. Also, the authors would like to thank the referee(s) for precise consideration and valuable comments.

References

[1] C. Baiocchi, A. Capelo, *Variational and Quasi-Variational Inequalities*, Wiley, New York, 1984.
[2] W. Ballmann, *Lectures on Spaces of Nonpositive Curvature*, in: DMV Seminar Band, vol. 25, Birkhäuser, Basel, 1995.
[3] I.D. Berg, I.G. Nikolaev, *Quasilinearization and curvature of Alexandrov spaces*, Geom. Dedicata, **133**(2008), 195-218.
[4] M. Bridson, A. Haefliger, *Metric Spaces of Nonpositive Curvature*, Springer-Verlag, Berlin, Heidelberg, New York, 1999.
[5] F.E. Browder, W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert Spaces*, J. Math. Anal. Appl., **20**(1967), 197-228.
[6] L.C. Ceng, Q.H. Ansari, J.C. Yao, *On relaxed viscosity iterative methods for variational inequalities in Banach spaces*, J. Comput. Appl. Math., **230**(2009), 813-822.
[7] R.W. Cottle, F. Giannessi, J.L. Lions, *Variational Inequalities and Complementarity Problems: Theory and Applications*, Wiley, New York, 1980.
[8] H. Dehghan, J. Rooin, *A characterization of metric projection in CAT(0) spaces*, arXiv:1311.4174v1, 2013.
[9] S. Dhompongsa, W.A. Kirk, B. Sims, *Fixed points of uniformly lipschitzian mappings*, Nonlinear Anal., **65**(2006), 762-772.
[10] S. Dhompongsa, W.A. Kirk, B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear and Convex Anal., **8**(2007), 35-45.
[11] S. Dhompongsa, B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl., **56**(2008), 2572-2579.
[12] F. Giannessi, G. Mastroeni, X.Q. Yang, *A survey on vector variational inequalities*, Boll. Unione Mat. Ital., **9**(1)(2009), 225-237.
[13] B.A. Kakavandi, *Weak topologies in complete CAT(0) metric spaces*, Proc. Amer. Math. Soc., **141**(2013), 1029–1039.

[14] W.A. Kirk, *Some recent results in metric fixed point theory*, J. Fixed Point Theory Appl., **2**(2007), 195–207.
[15] W.A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68**(2008), 3689-3696.
[16] W. A. Kirk, *Geodesic geometry and fixed point theory*, in Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), vol. 64 of Colecc. Abierta, University of Seville, Secretary Publication, Seville, Spain, (2003), 195-225.
[17] T.C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., **60**(1976), 179-182.
[18] F. Liu, M.Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Analysis, **6**(1998), 313-344.
[19] G. Marino, H.K. Xu, *Explicit hierarchical fixed point approach to variational inequalities*, J. Optim. Theory. Appl., **149**(2011), 61-78.
[20] A. Papadopoulos, *Metric spaces, convexity and nonpositive curvature*, IRMA Lectures in Mathematics and Theoretical Physics, 6. European Mathematical Society (EMS), Zürich, 2005.
[21] W. Takahashi, T. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory. Appl., **118**(2003), 417-428.
[22] W. Takahashi, T. Tamura, *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Anal., **5**(1998), 45-56.
[23] H. Zhou, *Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **69**(2008), 456-462.

Received: September 2, 2015; Accepted: February 12, 2016.