

## A CARISTI FIXED POINT THEOREM FOR COMPLETE QUASI-METRIC SPACES BY USING $mw$ -DISTANCES

CARMEN ALEGRE AND JOSEFA MARÍN

Instituto Universitario de Matemática Pura y Aplicada  
Universitat Politècnica de València  
Camí de Vera s/n, 46022 Valencia, Spain  
E-mail: calegre@mat.upv.es, jomarinn@mat.upv.es

**Abstract.** In this paper we give a quasi-metric version of Caristi's fixed point theorem by using  $mw$ -distances. Our theorem generalizes a recent result obtained by Karapinar and Romaguera in [7].

**Key Words and Phrases:** fixed point,  $w$ -distance,  $mw$ -distance, quasi-metric, complete quasi-metric space, Caristi's fixed point theorem.

**2010 Mathematics Subject Classification:** 47H10, 54H25, 54E50.

**Acknowledgements:** The authors thank the support of the Ministry of Economy and Competitiveness of Spain, Grants MTM2012-37894-C02-01 and MTM2015-64373-P (MINECO, FEDER,UE).

### 1. INTRODUCTION AND PRELIMINARIES

In 1976, Caristi [3] stated the following result which is one of the most important generalizations of the Banach contraction principle.

**Theorem A.** (Caristi fixed point theorem) *Let  $T$  be a self mapping of a complete metric space  $(X, d)$ . If there exists a lower semicontinuous function  $\varphi : X \rightarrow \mathbb{R}^+$  such that*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \tag{1.1}$$

*for all  $x \in X$ , then  $T$  has a fixed point.*

It is well known that this theorem is equivalent to Ekeland variational principle ([5]) which is nowadays an important tool in nonlinear analysis. Due to its application, Caristi's fixed point theorem has been investigated, extended, generalized and improved in several directions. Very recently, in [7] Karapinar and Romaguera proved, among other interesting results, the following quasi-metric generalization of Theorem A.

**Theorem B.** ((1)→ (2) in Theorem 2 of [7]) *Let  $T$  be a self mapping of a right  $K$ -sequentially complete quasi-metric space  $(X, d)$ . If there exists a proper bounded below and nearly lower semicontinuous for  $\tau_d$ ,  $\varphi : X \rightarrow \mathbb{R} \cup \{\infty\}$  such that*

$$d(Tx, x) + \varphi(Tx) \leq \varphi(x), \text{ for all } x \in X,$$

*then there exists  $z \in X$  such that  $\varphi(Tz) = \varphi(z)$  and  $d(Tz, z) = 0$ .*

On the other hand, in [6] Kada et al. introduced the notion of  $w$ -distance on a metric space  $(X, d)$  as follows.

A function  $q : X \times X \rightarrow \mathbb{R}^+$  is a  $w$ -distance on  $(X, d)$  if it satisfies the following conditions:

- (W1)  $q(x, y) \leq q(x, z) + q(z, y)$ , for all  $x, y, z \in X$ ;
- (W2)  $q(x, \cdot) : X \rightarrow \mathbb{R}^+$  is lower semicontinuous for  $\tau_d$  for all  $x \in X$ ;
- (W3) for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $q(x, y) \leq \delta$  and  $q(x, z) \leq \delta$  then  $d(y, z) \leq \varepsilon$ .

Clearly the metric  $d$  is a  $w$ -distance on  $(X, d)$ .

In Theorem 2 of [6], the authors obtained the following generalization of Theorem A by using  $w$ -distances.

**Theorem C.** *Let  $T$  be a self mapping of a complete metric space  $(X, d)$  and let  $q$  a  $w$ -distance on  $(X, d)$ . If there exists a lower semicontinuous function  $\varphi : X \rightarrow \mathbb{R}^+$  such that*

$$q(x, Tx) \leq \varphi(x) - \varphi(Tx),$$

*for all  $x \in X$ , then  $T$  has a fixed point.*

Later on, Park in [10] extended the notion of  $w$ -distance to quasi-metric spaces and this concept has been used in some directions in order to obtain fixed point results on complete quasi-metric spaces ([2], [8], [9]).

Since a quasi-metric  $d$  is not in general a  $w$ -distance on the quasi-metric space  $(X, d)$ , in [1] we introduced the notion of  $mw$ -distance which generalizes the concept of quasi-metric and we obtained fixed point theorems for generalized contractions with respect to  $mw$ -distances on complete quasi-metric spaces.

**Definition 1.1.** (Definition 3 of [1]) An  $mw$ -distance on a quasi-metric space  $(X, d)$  is a function  $q : X \times X \rightarrow \mathbb{R}^+$  satisfying the following conditions:

- (W1)  $q(x, y) \leq q(x, z) + q(z, y)$  for all  $x, y, z \in X$ ;
- (W2)  $q(x, \cdot) : X \rightarrow \mathbb{R}^+$  is lower semicontinuous on  $(X, \tau_{d^{-1}})$  for all  $x \in X$ ;
- (mW3) for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $q(y, x) \leq \delta$  and  $q(x, z) \leq \delta$  then  $d(y, z) \leq \varepsilon$ .

Note that the concepts of  $w$ -distance and  $mw$ -distance are independent (see examples of [1]) both in quasi-metric spaces and metric spaces.

In this paper we prove a quasi-metric version of Caristi's fixed point theorem by using  $mw$ -distances which generalizes Theorem B. We also obtain a generalization of Theorem A similar to Theorem C but using  $mw$ -distances instead of  $w$ -distances.

We start by recalling several notions and properties of the theory of quasi-metric spaces. Our basic reference is [4].

A quasi-metric on a set  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}^+$  such that for all  $x, y, z \in X$ :  
 (i)  $d(x, y) = d(y, x) = 0$  if and only if  $x = y$ ; (ii)  $d(x, y) \leq d(x, z) + d(z, y)$ .

A quasi-metric space is a pair  $(X, d)$  such that  $X$  is a set and  $d$  is a quasi-metric on  $X$ .

Each quasi-metric  $d$  on a set  $X$  induces a  $T_0$  topology  $\tau_d$  on  $X$  which has as a base the family of open balls  $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$ , where  $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$  for all  $x \in X$  and  $\varepsilon > 0$ .

If  $d$  is a quasi-metric on  $X$  then  $\tau_d$  is a  $T_1$  topology if and only if  $d(x, y) = 0$  implies  $x = y$ .

Given a quasi-metric  $d$  on  $X$ , the function  $d^{-1}$  defined by  $d^{-1}(x, y) = d(y, x)$  for all  $x, y \in X$ , is also a quasi-metric on  $X$ , called conjugate quasi-metric, and the function  $d^s$  defined by  $d^s(x, y) = \max\{d(x, y), d(y, x)\}$  for all  $x, y \in X$ , is a metric on  $X$ .

There exist several different notions of Cauchy sequence and quasi-metric completeness in the literature (see e.g. [4]). Here we will consider the following ones.

A sequence  $(x_n)_{n \in \mathbb{N}}$  in a quasi-metric  $(X, d)$  is said to be left (right) K-Cauchy if for each  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $d(x_n, x_m) \leq \varepsilon$  whenever  $n_0 \leq n \leq m$  ( $n_0 \leq m \leq n$ ).

A quasi-metric space  $(X, d)$  is  $d^{-1}$ -complete if every left K-Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$  in  $(X, d)$  converges with respect to the topology  $\tau_{d^{-1}}$ , i.e., there exists  $z \in X$  such that  $d(x_n, z) \rightarrow 0$ .

Note that our notion of  $d^{-1}$ -completeness of  $(X, d)$  coincides with the usual notion of right  $K$ -sequential completeness of  $(X, d^{-1})$ .

## 2. THE RESULTS

The following lemma is necessary to prove our main result (Theorem 2.1 below).

**Lemma 2.1.** *Let  $(X, d)$  be a quasi-metric space,  $q$  an  $mw$ -distance on  $(X, d)$  and  $(x_n)_{n \in \mathbb{N}}$  a sequence in  $X$ . If for each  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $q(x_n, x_m) \leq \varepsilon$  whenever  $n_0 \leq n < m$ , then  $(x_{2n})_{n \in \mathbb{N}}$  and  $(x_{2n-1})_{n \in \mathbb{N}}$  are left K-Cauchy sequences in  $(X, d)$ .*

*Proof.* Let  $\varepsilon > 0$ . By (mW3), there exists  $\delta > 0$  such that if  $q(y, x) \leq \delta$  and  $q(x, z) \leq \delta$  then  $d(y, z) \leq \varepsilon$ .

By hypothesis, there exists  $n_0$  such that  $q(x_n, x_m) \leq \delta$  whenever  $n_0 \leq n < m$ . Then,  $q(x_{2n}, x_{2n+1}) \leq \delta$  and  $q(x_{2n+1}, x_{2m}) \leq \delta$  whenever  $n_0 \leq n < m$ . Consequently,  $d(x_{2n}, x_{2m}) \leq \varepsilon$  whenever  $n_0 \leq n \leq m$ .

In a similar way it is proved that  $(x_{2n-1})_{n \in \mathbb{N}}$  is a left K-Cauchy sequence.  $\square$

Recall that if  $X$  is a nonempty set, a function  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  is said to be proper if there exists  $x \in X$  such that  $f(x) < \infty$ .

In [7], the authors introduced the notion of nearly lower semicontinuity which is a generalization of the concept of lower semicontinuity. A proper function  $f : X \rightarrow \mathbb{R} \cup \{\infty\}$  is nearly semicontinuous on the quasi-metric space  $(X, d)$  if whenever  $(x_n)_{n \in \mathbb{N}}$  is a sequence of distinct points of  $X$  that  $\tau_d$  converges to some  $x \in X$  then  $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$ .

**Theorem 2.1.** *Let  $T$  be a self mapping of a  $d^{-1}$ -complete quasi-metric space  $(X, d)$  and let  $q$  be an mw-distance on  $(X, d)$ . If there exists a proper bounded below and nearly lower semicontinuous function for  $\tau_{d^{-1}}$ ,  $\varphi : X \rightarrow \mathbb{R} \cup \{\infty\}$  such that*

$$q(x, Tx) + \varphi(Tx) \leq \varphi(x), \text{ for all } x \in X,$$

*then there exists  $z \in X$  such that  $\varphi(Tz) = \varphi(z)$  and  $q(z, Tz) = 0$ .*

*Proof.* For each  $x \in X$  let

$$S(x) = \{y \in X : q(x, y) + \varphi(y) \leq \varphi(x)\}.$$

Since  $Tx \in S(x)$ , we have that  $S(x) \neq \emptyset$  for all  $x \in X$ . Let

$$i(x) = \inf\{\varphi(y) : y \in S(x)\}.$$

Let  $x_1 \in X$  such that  $\varphi(x_1) < \infty$ . There exists  $x_2 \in S(x_1)$  such that  $\varphi(x_2) \leq i(x_1) + 1$ . Following this process we obtain a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  such that

$$x_{n+1} \in S(x_n),$$

$$\varphi(x_{n+1}) < \infty,$$

and

$$\varphi(x_{n+1}) \leq i(x_n) + \frac{1}{n}$$

for all  $n \in \mathbb{N}$ . Since  $q(x_n, x_{n+1}) + \varphi(x_{n+1}) \leq \varphi(x_n)$ , the sequence  $(\varphi(x_n))_{n \in \mathbb{N}}$  is non-increasing. So,  $\lim_{n \rightarrow \infty} \varphi(x_n)$  exists. Put  $l = \lim_{n \rightarrow \infty} \varphi(x_n)$ .

Now we prove that  $(x_{2n})_{n \in \mathbb{N}}$  is a left K-Cauchy sequence in  $(X, d)$ .

If  $m > n$ , then

$$q(x_n, x_m) \leq \sum_{i=n}^{m-1} q(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} (\varphi(x_i) - \varphi(x_{i+1})) = \varphi(x_n) - \varphi(x_m)$$

Since  $(\varphi(x_n))_{n \in \mathbb{N}}$  is a Cauchy sequence, given  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that if  $n_0 \leq n \leq m$  then  $\varphi(x_n) - \varphi(x_m) < \varepsilon$ . Therefore  $q(x_n, x_m) \leq \varepsilon$  whenever  $n_0 \leq n < m$ . From Lemma 2.1,  $(x_{2n})_{n \in \mathbb{N}}$  is a left K-Cauchy sequence.

Without loss of generality, we distinguish the following two cases.

Case 1. The sequence  $(x_{2n})_{n \in \mathbb{N}}$  is eventually constant. Then there exists  $n_0 \in \mathbb{N}$  such that  $x_{2n} = x_{2n_0}$  for all  $n \geq n_0$ . Since

$$\varphi(x_{2n+2}) - \frac{1}{2n} \leq \varphi(x_{2n+1}) - \frac{1}{2n} \leq i(x_{2n}) \leq \varphi(x_{2n+1}) \leq \varphi(x_{2n}),$$

then

$$\varphi(x_{2n_0}) - \frac{1}{2n} \leq i(x_{2n_0}) \leq \varphi(x_{2n_0}),$$

for all  $n \geq n_0$ . Taking limits, we obtain that  $i(x_{2n_0}) = \varphi(x_{2n_0})$ . Since  $Tx_{2n_0} \in S(x_{2n_0})$ , then  $i(x_{2n_0}) \leq \varphi(Tx_{2n_0}) \leq \varphi(x_{2n_0})$ , so  $\varphi(Tx_{2n_0}) = \varphi(x_{2n_0})$  and  $q(x_{2n_0}, Tx_{2n_0}) = 0$ .

Case 2.  $x_{2n} \neq x_{2m}$  for all  $n, m \in \mathbb{N}$  with  $n \neq m$ . Since  $(X, d)$  is  $d^{-1}$ -complete there exists  $z \in X$  such that  $(x_{2n})$  converges to  $z$  in  $(X, \tau_{d^{-1}})$ .

Next we show that  $z \in S(x_{2n})$  for all  $n \in \mathbb{N}$ .

Let  $n \in \mathbb{N}$  and let  $\varepsilon > 0$ . Since  $q(x_{2n}, \cdot)$  is a lower semicontinuous function on  $(X, \tau_{d^{-1}})$  and  $\varphi$  is a nearly lower semicontinuous function on  $(X, \tau_{d^{-1}})$ , there exists  $m_0 > n$  such that if  $m \geq m_0$  then

$$q(x_{2n}, z) - q(x_{2n}, x_{2m}) < \varepsilon$$

and

$$\varphi(z) - \varphi(x_{2m}) < \varepsilon.$$

Then

$$q(x_{2n}, z) < q(x_{2n}, x_{2m}) + \varepsilon \leq \varphi(x_{2n}) - \varphi(x_{2m}) + \varepsilon < \varphi(x_{2n}) - \varphi(z) + 2\varepsilon.$$

Therefore

$$q(x_{2n}, z) + \varphi(z) \leq \varphi(x_{2n}),$$

i.e.,  $z \in S(x_{2n})$  for all  $n \in \mathbb{N}$ .

Since  $0 \leq q(x_{2n}, z) \leq \varphi(x_{2n}) - \varphi(z)$ , we have that  $\varphi(z) \leq \varphi(x_{2n})$ , for all  $n \in \mathbb{N}$ . So  $\varphi(z) \leq l$ .

Since  $\varphi(z) \geq i(x_{2n})$ , for all  $n \in \mathbb{N}$ , and  $l = \lim_{n \rightarrow \infty} i(x_n)$  because

$$\varphi(x_{n+1}) \leq i(x_n) + \frac{1}{n} \leq \varphi(x_{n+1}) + \frac{1}{n},$$

we obtain that  $\varphi(z) \geq l$ . Hence  $l = \varphi(z)$ .

On the other hand,

$$q(x_{2n}, Tz) \leq q(x_{2n}, z) + q(z, Tz) \leq \varphi(x_{2n}) - \varphi(z) + \varphi(z) - \varphi(Tz) = \varphi(x_{2n}) - \varphi(Tz).$$

Therefore,  $Tz \in S(x_{2n})$  for all  $n \in \mathbb{N}$ .

By using a similar argument to the one given above we obtain that  $l = \varphi(Tz)$ . Hence  $\varphi(z) = \varphi(Tz)$  and, consequently,  $q(z, Tz) = 0$ .  $\square$

Since every quasi-metric  $d$  on  $X$  is an  $mw$ -distance on  $(X, d)$ , we obtain the following corollary.

**Corollary 2.1.** *Let  $T$  be a self mapping of a  $d^{-1}$ -complete quasi-metric space  $(X, d)$ . If there exists a proper bounded below and nearly lower semicontinuous function for  $\tau_{d^{-1}}$ ,  $\varphi : X \rightarrow \mathbb{R} \cup \{\infty\}$  such that  $d(x, Tx) + \varphi(Tx) \leq \varphi(x)$ , for all  $x \in X$ , then there exists  $z \in X$  such that  $\varphi(Tz) = \varphi(z)$  and  $d(z, Tz) = 0$ .*

This corollary is equivalent to Theorem B because a quasi-metric space  $(X, d)$  is right  $K$ -sequentially complete if and only if  $(X, d^{-1})$  is  $d$ -complete. Theorem B can be obtained directly from Theorem 2.1 taking  $q = d^{-1}$ .

On the other hand, since the class of the nearly lower semicontinuous functions on a metric space  $(X, d)$  coincides with the class of the lower semicontinuous functions on  $(X, d)$ , we obtain a generalization of Caristi's fixed point theorem in the same direction as Theorem B.

**Corollary 2.2.** *Let  $T$  be a self mapping of a complete metric space  $(X, d)$  and let  $q$  be an  $mw$ -distance on  $(X, d)$ . If there exists a proper bounded below and lower semicontinuous function  $\varphi : X \rightarrow \mathbb{R} \cup \{\infty\}$  such that  $q(x, Tx) + \varphi(Tx) \leq \varphi(x)$ , for all  $x \in X$ , then  $T$  has a fixed point.*

*Proof.* By Theorem 2.1, there exists  $z \in X$  such that  $\varphi(Tz) = \varphi(z)$  and  $q(z, Tz) = 0$ . Now we are going to prove that  $z = Tz$ .

Given  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $q(y, x) \leq \delta$  and  $q(x, z) \leq \delta$  then  $d(y, z) \leq \varepsilon$ . Since  $q(x_n, z) \leq \varphi(x_n) - \varphi(z)$ , for all  $n \in \mathbb{N}$  and  $l = \varphi(z)$ , there exists  $n_0 \in \mathbb{N}$  such that  $q(x_n, z) \leq \delta$  for all  $n \geq n_0$ . Since  $q(z, Tz) = 0 < \delta$ , we have that  $d(x_n, Tz) < \varepsilon$  for all  $n \geq n_0$ . Therefore  $(x_n)_{n \in \mathbb{N}}$  converges to  $Tz$ . Consequently  $Tz = z$ .  $\square$

**Remark 2.1.** As mentioned above Caristi's fixed point theorem for metric spaces is a generalization of the Banach contraction principle. This is because if  $T$  is a contractive self mapping of a metric space  $(X, d)$ , then  $\varphi(x) = \frac{1}{1-r}d(x, Tx)$ , where  $r$  is the contractivity constant, is a lower semicontinuous function on  $X$  and  $d(x, Tx) + \varphi(Tx) \leq \varphi(x)$ . This is not the case in the quasi-metric framework. In fact, the Banach contraction principle is not fulfilled if the complete metric space is replaced by a  $d^{-1}$ -complete quasi-metric space. For instance, if  $X = \{1/n : n \in \mathbb{N}\}$  and  $d$  is the quasi-metric on  $X$  given by  $d(x, x) = 0$  and  $d(x, y) = x$  if  $x \neq y$  then  $(X, d)$  is  $d^{-1}$ -complete and the self mapping of  $X$  given by  $Tx = x/2$  is contractive but it has not fixed point. Note that  $T$  is not a Caristi type mapping because if that was the case, by Corollary 2.1, there exists  $z \in X$  such that  $d(z, Tz) = 0$  and then  $T$  has a fixed point since  $(X, \tau_d)$  is a  $T_1$  topological space.

**Remark 2.2.** As was expected, in Theorem 2.1 the condition  $q(x, Tx) + \varphi(Tx) \leq \varphi(x)$ , for all  $x \in X$ , can not be replaced by the condition  $q(Tx, x) + \varphi(Tx) \leq \varphi(x)$ , for all  $x \in X$ . Indeed, if  $X = \{1/n : n \in \mathbb{N}\}$ ,  $d$  is the quasi-metric on  $X$  given by  $d(x, y) = y - x$  if  $x \leq y$  and  $d(x, y) = 1$  if  $x > y$ ,  $q = d$  and  $\varphi$  is a function on  $X$  given by  $\varphi(x) = x$ , the self mapping of  $X$  given by  $Tx = x/2$ , satisfies that

$$q(Tx, x) + \varphi(Tx) = \frac{x}{2} + \frac{x}{2} = \varphi(x)$$

and nevertheless  $Tz \neq z$  for every  $z \in X$ .

Finally, we give a characterization of  $d^{-1}$ -completeness in terms of the quasi-metric version of Caristi's fixed point theorem given in Theorem 2.1. For this purpose, we give the following definition.

**Definition 2.1.** Let  $T$  a self mapping of the quasi-metric space  $(X, d)$ . We say that  $T$  is  $(q, \varphi)$ -Caristi if  $q$  is an  $mw$ -distance on  $(X, d)$  and  $\varphi : X \rightarrow \mathbb{R} \cup \{\infty\}$  is a proper bounded below and nearly lower semicontinuous function for  $\tau_{d^{-1}}$  such that  $q(x, Tx) + \varphi(Tx) \leq \varphi(x)$ , for all  $x \in X$ .

The following example shows that if  $q$  is an  $mw$ -distance on the quasi-metric space  $(X, d)$  and  $\varphi : X \rightarrow \mathbb{R} \cup \{\infty\}$  is a proper bounded below and nearly lower semicontinuous function for  $\tau_{d^{-1}}$ , then there exist  $(q, \varphi)$ -Caristi self mappings of  $X$  which are not  $(d, \varphi)$ -Caristi.

**Example 2.1.** Let  $X = \mathbb{N}$  and let  $d$  be the quasi-metric on  $X$  given by  $d(x, x) = 0$  and  $d(x, y) = x$  for all  $x, y \in X$ . Clearly,  $\tau_d$  is the discrete topology on  $X$  and  $\tau_{d^{-1}} = \tau_d$ . Let  $q$  be the  $mw$ -distance on  $(X, d)$  given by  $q(1, 1) = 0$  and  $q(x, y) = 1/2$  otherwise. Define  $T : X \rightarrow X$  as  $T1 = 1$  and  $Tx = x - 1$  for all  $x > 1$ . If we consider the

function  $\varphi : X \rightarrow \mathbb{R}$  given by  $\varphi(x) = x$ , then  $\varphi$  is nearly lower semicontinuous for  $\tau_{d^{-1}}$ ,  $q(1, T1) = 0 = \varphi(1) - \varphi(T1)$  and if  $x > 1$ , then

$$q(x, Tx) = 1/2 < 1 = \varphi(x) - \varphi(Tx).$$

Therefore  $T$  is  $(q, \varphi)$ -Caristi. Nevertheless  $T$  is not  $(d, \varphi)$ -Caristi because  $d(x, Tx) > \varphi(x) - \varphi(Tx)$ , for all  $x > 1$ .

**Theorem 2.2.** *Let  $(X, d)$  be a quasi-metric space. Then  $(X, d)$  is  $d^{-1}$ -complete if and only for every  $(q, \varphi)$ -Caristi self mapping  $T$  of  $X$  exists  $z \in X$  such that  $\varphi(z) = \varphi(Tz)$  and  $q(z, Tz) = 0$ .*

*Proof.* From Theorem 2.1 we have the direct. For the converse, we suppose that  $X$  is not  $d^{-1}$ -complete. Then  $(X, d^{-1})$  is not right  $K$ -sequentially complete. By (2)→(1) of Theorem 2 of [7], there exist a self mapping  $T$  of  $X$  and a proper bounded below and nearly lower semicontinuous function for  $\tau_{d^{-1}}$ ,  $\varphi : X \rightarrow \mathbb{R} \cup \{\infty\}$  such that  $d^{-1}(Tx, x) + \varphi(Tx) \leq \varphi(x)$ , for all  $x \in X$  and  $\varphi(Tz) \neq \varphi(z)$  for all  $z \in X$ . Therefore  $T$  is a  $(d, \varphi)$ -Caristi self mapping of  $X$  such that for all  $z \in X$ ,  $\varphi(Tz) \neq \varphi(z)$  and this is a contradiction.  $\square$

#### REFERENCES

- [1] C. Alegre, J. Marín, *Modified  $w$ -distances on quasi-metric spaces and a fixed point theorem on complete quasi-metric spaces*, Topology and its Appl., 203(2016), 32-41.
- [2] C. Alegre, J. Marín, S. Romaguera, *A fixed point theorem for generalized contractions involving to  $w$ -distances on complete quasi-metric spaces*, Fixed Point Theory Appl., 40(2014), 1-8.
- [3] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., 215(1976), 241-251.
- [4] S. Cobzas, *Functional Analysis in Asymmetric Normed Spaces*, Birkhäuser, Springer Basel, 2013.
- [5] I. Ekeland, *On the variational principle*, J. Math. Soc., 1(1979), 443-474.
- [6] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica, 44(1996), 381-391.
- [7] E. Karapinar, S. Romaguera, *On the weak form of Ekeland's Variational Principle in quasi-metric spaces*, Topology and its Appl., 184(2015), 54-60.
- [8] J. Marín, S. Romaguera, P. Tirado, *Weakly contractive multivalued maps and  $w$ -distances on complete quasi-metric spaces*, Fixed Point Theory Appl., 2(2011), 1-9.
- [9] J. Marín, S. Romaguera, P. Tirado, *Generalized contractive set-valued maps on complete pre-ordered quasi-metric spaces*, J. Functions Spaces Appl., 2013, Article ID 269246 (2013), 6 pages.
- [10] S. Park, *On generalizations of the Ekeland-type variational principles*, Nonlinear Anal., 39(2000), 881-889.

*Received: July 27, 2015; Accepted: December 15, 2015.*

