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AMENABLE LOCALLY COMPACT SEMIGROUPS AND A FIXED POINT PROPERTY

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Abstract. For a locally compact semigroup S, we study a general fixed point property in terms of Banach left S-modules. We then use this property to give our main result which is a new characterization for left amenability of a large class of locally compact semigroups; finally, we investigate several examples which lead us to the conjecture that the main result remains true for all locally compact semigroups.
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1. Introduction and preliminaries

The notion of left amenability of semigroups was initiated by Day [4] and pursed by Lau and Takahashi [11, 12, 13], Namioka [14] and Saeidi [19] for discrete semigroups, and by Dzinotyweyi [7], Holmes and Lau [8], Paterson [15], Riazi and Wong [18] and Wong [22, 23, 24] for topological semigroups; see Berglund, Junghenn and Milnes [2], Dales, Lau, and Strauss [3] and Pier [16] for more details; see also Desaulniers, Nemati and the author [5] for a more general setting.

Let us recall that a locally compact semigroup S, is a semigroup with a locally compact Hausdorff topology whose binary operation is jointly continuous. As usual, let $M(S)$ denote the Banach algebra of all complex Radon measures on S with the convolution product $*$ and the total variation norm. The space of all measures μ in $M(S)$ for which the maps $s \mapsto \delta_s * |\mu|$ and $s \mapsto |\mu| * \delta_s$ from S into $M(S)$ are weakly continuous is denoted by $M_a(S)$, where δ_s denotes the Dirac measure at $s \in S$. It is known that $M_a(S)$ is a closed two-sided L-ideal of $M(S)$; see Baker and Baker

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[1], Dzinotyweyi [7] and Pourabbas and Riazi [17]. Denote by $M_p(S)$ the set of all probability measures in $M(S)$ and set

$$P_1(S) := M_a(S) \cap M_p(S).$$

The locally compact semigroup S is called foundation if the set $F(S)$ is dense in S, where

$$F(S) := \bigcup \{\text{supp}(\upsilon) : \ \upsilon \in P_1(S)\}.$$

In this section, we give a result on left invariant nets which is need in the sequel. In the next section, we study a fixed point property in terms of Banach left S-modules to characterize left amenability of foundation semigroups; we finally give several examples which lead us to the conjecture that this result is valid for arbitrary locally compact semigroups.

Definition 1.1. Let S be a locally compact semigroup. For $\mathfrak{M} \subseteq M_p(S)$, we say that a net $(\upsilon_\lambda)_{\lambda\in\Lambda}$ in $M_a(S)$ is *$\mathfrak{M}$-left invariant* if

$$\mu * \upsilon_\lambda - \upsilon_\lambda \to 0$$

for all $\mu \in \mathfrak{M}$ in the norm topology of $M_a(S)$.

Similarly, we define a *weakly $\mathfrak{M}$-left invariant net*; i.e., the convergence of the limit in the definition of $\mathfrak{M}$-left invariance is only assumed to hold in the weak topology instead of in the norm topology.

Proposition 1.2. *Let S be a locally compact semigroup and let $\mathfrak{M} \subseteq M_p(S)$. Then the convex hull of any weakly $\mathfrak{M}$-left invariant net in $P_1(S)$ contains an $\mathfrak{M}$-left invariant net.*

Proof. Let $(\upsilon_\gamma)_{\gamma\in\Gamma}$ be a weakly $\mathfrak{M}$-left invariant net of measures in $P_1(S)$, and for each finite subset $\mathfrak{F} = \{\upsilon_1, ..., \upsilon_m\}$ of $M_a(S)$, set

$$\mathcal{C}_{\mathfrak{F}} := \{(\upsilon_1 * \upsilon - \upsilon, ..., \upsilon_m * \upsilon - \upsilon) : \ \upsilon \in \text{co}(\{\upsilon_\gamma : \gamma \in \Gamma\})\ \},$$

where $\text{co}(\{\upsilon_\gamma : \gamma \in \Gamma\})$ denotes the convex hull of the set $\{\upsilon_\gamma : \gamma \in \Gamma\}$. Now, consider the m-times ℓ^1-direct sum $\ell^1\text{-}\oplus_{n=1}^m M_a(S)$ of the Banach space $M_a(S)$, and note that 0 is in the weak closure of $\mathcal{C}_{\mathfrak{F}}$ in $\ell^1\text{-}\oplus_{n=1}^m M_a(S)$; in fact, the net $(\upsilon * \upsilon_\gamma - \upsilon_\gamma)_{\gamma\in\Gamma}$ in $M_a(S)$ converges to zero weakly for all $\upsilon \in P_1(S)$, and hence the net

$$((\upsilon_1 * \upsilon_\gamma - \upsilon_\gamma, ..., \upsilon_m * \upsilon_\gamma - \upsilon_\gamma))_{\gamma\in\Gamma} \subseteq \mathcal{C}_{\mathfrak{F}}$$

in $\ell^1\text{-}\oplus_{n=1}^m M_a(S)$ converges to zero weakly. Since $\mathcal{C}_{\mathfrak{F}}$ is convex, it follows that the norm closure of $\mathcal{C}_{\mathfrak{F}}$ in $\ell^1\text{-}\oplus_{n=1}^m M_a(S)$ contains zero. Thus, for each $\varepsilon > 0$, there exists a measure $\upsilon_{(\varepsilon,\mathfrak{F})} \in P_1(S)$ such that for each $\upsilon \in \mathfrak{F}$,

$$\|\upsilon * \upsilon_{(\varepsilon,\mathfrak{F})} - \upsilon_{(\varepsilon,\mathfrak{F})}\| < \varepsilon.$$

Now, let Λ be the set of all $\lambda := (\varepsilon, \mathfrak{F})$ for which $\varepsilon > 0$ and $\mathfrak{F} \subseteq M_a(S)$ is a finite set. Then Λ is a directed set by setting $(\varepsilon', \mathfrak{F}') \succcurlyeq (\varepsilon, \mathfrak{F})$ if and only if $\varepsilon' \leq \varepsilon$ and $\mathfrak{F}' \supseteq \mathfrak{F}$. So, the net $(\upsilon_\lambda)_{\lambda\in\Lambda}$ is an $\mathfrak{M}$-left invariant net. □

2. Left fixed points for S-modules

Let E be a Banach space, and let $\mathcal{B}(E^{**})$ denote the Banach space of bounded linear operators on E^{**}. By the weak* operator topology on $\mathcal{B}(E^{**})$, we shall mean the locally convex topology of $\mathcal{B}(E^{**})$ determined by the family

$$T \mapsto |\langle T(\Psi), \phi\rangle|$$

of seminorms on $\mathcal{B}(E^{**})$, where $\Psi \in E^{**}, \phi \in E^*$; this means that a net $(T_\gamma) \subseteq \mathcal{B}(E^{**})$ converges to zero in the weak* operator topology if and only if for each $\Psi \in E^{**}$ and $\phi \in E^*$,

$$\langle T_\gamma(\Psi), \phi\rangle \longrightarrow 0.$$

Now, let E be a Banach left S-module; that is, a Banach space E equipped with a map from $S \times E$ into E, denoted by $(s, \xi) \mapsto s \cdot \xi$ ($s \in S, \xi \in E$) such that

$$s \cdot (t \cdot \xi) = (st) \cdot \xi$$

for all $s, t \in S$ and $\xi \in E$, the map $s \mapsto s \cdot \xi$ is continuous of S into E for all $\xi \in E$, and the map $\xi \mapsto s \cdot \xi$ is a bounded linear operator on E for all $s \in S$; i.e., there is a constant $C > 0$ with

$$\|s \cdot \xi\| \leq C\ \|\xi\|$$

for all $s \in S$ and $\xi \in E$. In this case, we define

$$\langle \phi \cdot \mu, \xi\rangle = \int_S \langle \phi, s \cdot \xi\rangle\ d\mu(s)$$

and

$$\langle \mu \cdot \Psi, \phi\rangle = \langle \Psi, \phi \cdot \mu\rangle$$

for all $\xi \in E$, $\phi \in E^*$, $\Psi \in E^{**}$ and $\mu \in M(S)$. Any Banach left S-module E equipped with the map

$$(\mu, \xi) \mapsto \mu \cdot \xi$$

for all $\mu \in M(S)$ and $\xi \in E$ can be considered as a Banach left $M(S)$-module.

We denote by $\mathcal{P}(S, E^{**})$ the closure of the set

$$\{T_\upsilon :\ \upsilon \in P_1(S)\}$$

in the weak* operator topology of $\mathcal{B}(E^{**})$, where the operator $T_\upsilon \in \mathcal{B}(E^{**})$ for each $\Psi \in E^{**}$ is defined by

$$T_\upsilon(\Psi) = \upsilon \cdot \Psi.$$

Let us remark that $P_1(S)$ with the convolution multiplication is a semigroup. In particular, the set

$$\{T_\upsilon :\ \upsilon \in P_1(S)\}$$

is a subsemigroup of the semigroup $\mathcal{B}(E^{**})$ with the ordinary multiplication of linear operators, and as easily verified, so is its closure $\mathcal{P}(S, E^{**})$ in the weak* operator topology of $\mathcal{B}(E^{**})$.

Definition 2.1. Let S be a locally compact semigroup, let E be a Banach left S-module, and let $\mathfrak{M} \subseteq M_p(S)$. We say that E has an $\mathfrak{M}$-*left fixed point* if there exists $T \in \mathcal{P}(S, E^{**})$ such that

$$T_\mu T = T \qquad (\mu \in \mathfrak{M});$$

we call such an operator T an $\mathfrak{M}$-*left fixed point* for E .

It should be noted that the notion of left fixed point is already in Holmes and Lau [8] which establishes a fixed point theorem for semigroups of certain mappings on a compact convex subset of a locally convex space.

Lemma 2.2. *Let S be a locally compact semigroup and let $\mathfrak{M} \subseteq M_p(S)$. If there exists an $\mathfrak{M}$-left invariant net in $P_1(S)$, then every left Banach S-module E has an $\mathfrak{M}$-left fixed point.*

Proof. Choose a constant $C > 0$ satisfying

$$\| s \cdot \xi \| \leq C \| \xi \|$$

for all $s \in S$ and $\xi \in E$, and note that for each $\upsilon \in P_1(S)$ and $\Psi \in E^{**}$,

$$\|T_\upsilon(\Psi)\| = \|\upsilon \cdot \Psi\| \leq C \, \|\Psi\|.$$

Hence, the set

$$\{T_\upsilon : \ \upsilon \in P_1(S)\} \subseteq \mathcal{B}(E^{**})$$

is bounded by C, and therefore its closure $\mathcal{P}(S, E^{**})$ is also bounded by C.

Now, let $(\upsilon_\lambda)_{\lambda \in \Lambda}$ be an $\mathfrak{M}$-left invariant net in $P_1(S)$. Then, without loss of generality, we may assume that there exists an operator $T \in \mathcal{P}(S, E^{**})$ such that

$$T_{\upsilon_\lambda} - T \to 0$$

in the weak* operator topology of $\mathcal{B}(E^{**})$. Indeed, considering the projective tensor product $E^{**}\widehat{\otimes}E^*$, we make the canonical identification of the dual space $(E^{**}\widehat{\otimes}E^*)^*$ of $E^{**}\widehat{\otimes}E^*$ with the Banach space $\mathcal{B}(E^{**})$; moreover, the weak* operator topology of $\mathcal{B}(E^{**})$ coincides with the weak* topology of $(E^{**}\widehat{\otimes}E^*)^*$ on bounded subsets of $\mathcal{B}(E^{**})$; since $\mathcal{P}(S, E^{**})$ is a bounded subset of $\mathcal{B}(E^{**})$, it is compact in the weak* operator topology of $\mathcal{B}(E^{**})$ by the Banach-Alaoglu theorem; see for example [6], Corollary VIII.2.2; in particular, the net

$$(T_{\upsilon_\lambda}) \subseteq \mathcal{P}(S, E^{**})$$

has a cluster point $T \in \mathcal{P}(S, E^{**})$ in the weak* operator topology of $\mathcal{B}(E^{**})$.

Next, let $\mu \in \mathfrak{M}$ and let $T^*_\mu : E^{***} \longrightarrow E^{***}$ denote the adjoint operator of $T_\mu \in \mathcal{B}(E^{**})$. Then for each $\phi \in E^*$ and $\Psi \in E^{**}$,

$$\begin{aligned} \langle T^*_\mu(\phi), \Psi\rangle &= \langle \phi, T_\mu(\Psi)\rangle \\ &= \langle \phi, \mu \cdot \Psi\rangle \\ &= \langle \phi \cdot \mu, \Psi\rangle, \end{aligned}$$

whence

$$T^*_\mu(\phi) = \phi \cdot \mu.$$

In particular, $T_\mu^*(\phi) \in E^*$. This shows that

$$\langle (T_\mu T_{\upsilon_\lambda} - T_\mu T)(\Psi), \phi\rangle = \langle (T_{\upsilon_\lambda} - T)(\Psi), T_\mu^*(\phi)\rangle \longrightarrow 0.$$

It follows that

$$T_\mu T_{\upsilon_\lambda} - T_\mu T \to 0$$

in the weak* operator topology of $\mathcal{B}(E^{**})$. Moreover, for each $\lambda \in \Lambda$,

$$\|T_\mu T_{\upsilon_\lambda} - T_{\upsilon_\lambda}\| \leq C\,\|\mu * \upsilon_\lambda - \upsilon_\lambda\|$$

which shows that

$$T_\mu T_{\upsilon_\lambda} - T_{\upsilon_\lambda} \to 0$$

in the norm topology and of course in the weak* operator topology of $\mathcal{B}(E^{**})$. But, for each $\lambda \in \Lambda$,

$$T_\mu T - T = (T_\mu T - T_\mu T_{\upsilon_\lambda}) + (T_\mu T_{\upsilon_\lambda} - T_{\upsilon_\lambda}) + (T_{\upsilon_\lambda} - T)$$

whence $T_\mu T = T$ as required. □

Let us recall that $M_a(S)^{**}$ with the first Arens product $\diamond$ defined by

$$\langle \mathsf{F} \diamond \mathsf{G}, f\rangle = \langle \mathsf{F}, \mathsf{G}\, f\rangle$$

for $f \in M_a(S)^*$ and $\mathsf{F}, \mathsf{G} \in M_a(S)^{**}$ is a Banach algebra, where $\mathsf{G}f \in M_a(S)^*$ is defined by

$$\langle \mathsf{G}f, \upsilon\rangle = \langle \mathsf{G}, f\,\upsilon\rangle$$

for all $\upsilon \in M_a(S)$; here, $f\mu \in M_a(S)^*$ is defined by

$$\langle f\mu, \upsilon\rangle = \langle f, \mu * \upsilon\rangle$$

for all $\mu \in M(S)$ and $\upsilon \in M_a(S)$. For each $\upsilon \in M_a(S)$, let υ also denote the functional in $M_a(S)^{**}$ defined by the formula

$$f \mapsto \langle f, \upsilon\rangle \qquad (f \in M_a(S)^*);$$

this induces a linear isometric embedding of $M_a(\mathcal{S})$ into $M_a(S)^{**}$. In particular, $\mathsf{F} \diamond \upsilon$, $\sigma \diamond \mathsf{F}$ and $\sigma \diamond \upsilon$ are make sense as elements of $M_a(S)^{**}$ for all $\sigma, \upsilon \in M_a(S)$ and $\mathsf{F} \in M_a(S)^{**}$; moreover, $\sigma \diamond \upsilon = \sigma * \upsilon$.

An element M in the second dual $M_a(S)^{**}$ of $M_a(S)$ is said to be a mean on $M_a(S)^*$ if

$$\|\mathsf{M}\| = \langle \mathsf{M}, \varphi_1\rangle = 1,$$

where $\varphi_1 \in M_a(S)^*$ is defined for each $\upsilon \in M_a(S)$ by

$$\langle \varphi_1, \upsilon\rangle = \upsilon(S).$$

Definition 2.3. Let S be a locally compact semigroup. For $\mathfrak{M} \subseteq M_p(S)$, we say that a mean M on $M_a(S)^*$ is $\mathfrak{M}$-*left invariant* if

$$\langle \mathsf{M}, f\mu\rangle = \langle \mathsf{M}, f\rangle$$

for all $\mu \in \mathfrak{M}$ and $f \in M_a(S)^*$; we also say that S is $\mathfrak{M}$-*left amenable* if there exists an $\mathfrak{M}$-left invariant mean on $M_a(S)^*$.

Before we give the main result of this paper, let us remark that $M_a(S)$ equipped with the map

$$(s, \upsilon) \mapsto s \cdot \upsilon$$

defined by

$$s \cdot \upsilon = \delta_s * \upsilon \qquad (\upsilon \in M_a(S), s \in S),$$

is a Banach left S-module; note that in this case we have

$$f \cdot \mu = f\mu \qquad \text{and} \qquad \upsilon \cdot \mathsf{F} = \upsilon \diamond \mathsf{F}$$

for all $\mu \in M(S), \upsilon \in M_a(S),\ f \in M_a(S)^*$ and $\mathsf{F} \in M_a(S)^{**}$.

Theorem 2.4. *Let S be a foundation semigroup with identity and let $\mathfrak{M} \subseteq M_p(S)$. Then the following assertions are equivalent.*

(a) *S is $\mathfrak{M}$-left amenable.*

(b) *Every Banach left S-module E has an $\mathfrak{M}$-left fixed point.*

(c) *The Banach left S-module $M_a(S)$ has an $\mathfrak{M}$-left fixed point.*

Proof. Suppose that (a) holds, and let M be an $\mathfrak{M}$-left invariant mean on $M_a(S)^*$. Since S is a foundation semigroup with identity, it follows from [20] that $M_a(S)$ is the predual of a von Neumann algebra; see also [21]. Thus $P_1(S)$ is weak* dense in the set of all means on $M_a(S)^*$; see [9], Lemma 2.1. So, there is a net $(\upsilon_\gamma)_{\gamma\in\Gamma}$ in $P_1(S)$ such that $\upsilon_\gamma \to \mathsf{M}$ in the weak* topology of $M_a(S)^{**}$. For each $\mu \in \mathfrak{M}$ and $f \in M_a(S)^*$ we have

$$\begin{aligned} \langle \mu * \upsilon_\gamma - \upsilon_\gamma, f\rangle &= \langle \upsilon_\gamma, f\mu - f\rangle \\ &\to \langle \mathsf{M}, f\mu - f\rangle \end{aligned}$$

Therefore, for each $\mu \in \mathfrak{M}$,

$$\mu * \upsilon_\gamma - \upsilon_\gamma \to 0$$

in the weak topology of $M_a(S)$. That is, $(\upsilon_\gamma)_{\gamma\in\Gamma}$ is a weakly $\mathfrak{M}$-left invariant net in $P_1(S)$, and hence there is an $\mathfrak{M}$-left invariant net in the set

$$\mathrm{co}(\{\upsilon_\gamma :\ \gamma \in \Gamma\}) \subseteq P_1(S)$$

by Proposition 1.2. Now, appeal to Lemma 2.2 to conclude that (b) holds.

That (b) implies (c) is trivial.

Now, suppose that (c) holds. To prove (a) choose an element T of $\mathcal{P}(S, M_a(S)^{**})$ with

$$T_\mu T = T$$

for all $\mu \in \mathfrak{M}$, and find a net $(\sigma_\beta)_{\beta\in\daleth}$ in $P_1(S)$ such that

$$\langle T_{\sigma_\beta}(\mathsf{F}), f\rangle \to \langle T(\mathsf{F}), f\rangle$$

for all $\mathsf{F} \in M_a(S)^{**}$ and $f \in M_a(S)^*$. Without loss of generality, we may assume that $(\sigma_\beta)_{\beta\in\daleth}$ converges to a mean M in $M_a(S)^{**}$.

We show that M is an $\mathfrak{M}$-left invariant mean on $M_a(S)^*$. For this end, recall from the assumption that S is a foundation semigroup with identity, and hence $M_a(S)$ has a bounded approximate identity $(u_\iota)_{\iota\in I}$ of measures in $P_1(S)$; see [20], Theorem 5.16. Let U be a weak* cluster point of $(u_\iota)_{\iota\in I}$ in $M_a(S)^{**}$. Then

$$\sigma \diamond \mathsf{U} = \sigma$$

for all $\sigma \in M_a(S)$ by the weak* continuity properties of the first Arens product $\diamond$, and hence for each $\mu \in \mathfrak{M}$ and $f \in M_a(S)^*$ we have

$$\begin{aligned}\langle \mathsf{M}, f\mu\rangle &= \lim_\beta \langle \mu * \sigma_\beta \diamond \mathsf{U}, f\rangle \\ &= \lim_\beta \langle (T_\mu T_{\sigma_\beta})(\mathsf{U}), f\rangle \\ &= \langle (T_\mu T)(\mathsf{U}), f\rangle \\ &= \langle T(\mathsf{U}), f\rangle \\ &= \lim_\beta \langle T_{\sigma_\beta}(\mathsf{U}), f\rangle \\ &= \lim_\beta \langle \sigma_\beta \diamond \mathsf{U}, f\rangle \\ &= \lim_\beta \langle \sigma_\beta, f\rangle \\ &= \langle \mathsf{M}, f\rangle.\end{aligned}$$

That is, M is an $\mathfrak{M}$-left invariant mean on $M_a(S)^*$ as required. □

We say that a Banach left S-module E has a *left fixed point* if it has a δ_S-left fixed point, where

$$\delta_S := \{\delta_s : s \in S\};$$

recall that S is called *left amenable* if there exists a left invariant mean on $M_a(S)^*$; that is, a δ_S-left invariant mean on $M_a(S)^*$.

Our last result is the following consequence of Theorem 2.4.

Corollary 2.5. *Let S be a foundation semigroup with identity. Then the following assertions are equivalent.*

(a) *S is left amenable.*

(b) *Every Banach left S-module E has a left fixed point.*

(c) *The Banach left S-module $M_a(S)$ has a left fixed point.*

This result should be compared with Theorem 5.1 of Lau and Paterson [10] in the case where S is a locally compact group.

We now give some motivating examples of foundation or non-foundation semigroups with or without identity for which our main result is valid.

Example 2.6. Let S be the set $[0, 1]$ with the locally compact Hausdorff topology induced from the real line and let $\mathfrak{M} \subseteq M_p(S)$.

(a) *A foundation semigroup with identity.* Endow S with the operation $xy = \min\{x + y, 1\}$ for all $x, y \in S$ and note that S is a locally compact semigroup with identity. Since

$$M_a(S) = L^1([0, 1]) \oplus \mathbb{C}\,\delta_1,$$

it follows that S is foundation. Moreover,

$$P_1(S) = \{f \in L^1([0, 1]) : \ f \geq 0, \|f\|_1 = 1\} \cup \{\delta_1\}$$

and

$$T_{\delta_1} \in \mathcal{P}(S, E^{**}),$$

for all Banach left S-modules E. Therefore, E has an $\mathfrak{M}$-left fixed point and S is $\mathfrak{M}$-left amenable. In fact, T_{δ_1} is an $\mathfrak{M}$-left fixed point for E and δ_1 is an $\mathfrak{M}$-left invariant mean on $M_a(S)^*$.

(b) *A foundation semigroup without identity.* Endow S with the operation $xy = 0$ for all $x, y \in S$ and note that S is a locally compact semigroup without identity. Since

$$M_a(S) = M(S),$$

it follows that S is foundation. Moreover,

$$P_1(S) = M_p(S) \text{ and } \mathcal{P}(S, E^{**}) = \{0\}$$

for all Banach left S-modules E. Therefore, E has no $\mathfrak{M}$-left fixed point and S is not $\mathfrak{M}$-left amenable.

(c) *A non-foundation semigroup with identity.* Endow S with the usual multiplication and note that S is a locally compact semigroup with identity. Since

$$M_a(S) = \mathbb{C}\,\delta_0,$$

it follows that S is non-foundation. Moreover,

$$P_1(S) = \{\delta_0\} \text{ and } \mathcal{P}(S, E^{**}) = \{T_{\delta_0}\}$$

for all Banach left S-modules E. Therefore, E has an $\mathfrak{M}$-left fixed point and S is $\mathfrak{M}$-left amenable. In fact, T_{δ_0} is an $\mathfrak{M}$-left fixed point for E and δ_0 is an $\mathfrak{M}$-left invariant mean on $M_a(S)^*$.

(d) *A non-foundation semigroup without identity.* Endow S with the operation $xy = y$ for all $x, y \in S$ and note that S is a locally compact semigroup without identity. Since

$$M_a(S) = \{0\},$$

it follows that S is non-foundation. Moreover,

$$P_1(S) = \emptyset \text{ and } \mathcal{P}(S, E^{**}) = \emptyset$$

for all Banach left S-modules E. That is, E has no $\mathfrak{M}$-left fixed point and S is not $\mathfrak{M}$-left amenable.

Motivated by these examples, we give the following conjecture.

Conjecture 2.7. *Let S be an arbitrary locally compact semigroup and $\mathfrak{M} \subseteq M_p(S)$. Does $\mathfrak{M}$-left amenability of S is equivalent to that every Banach left S-module E has an $\mathfrak{M}$-left fixed point?*

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