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CONTROLLABILITY RESULTS FOR FRACTIONAL ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

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Abstract. In this paper, we investigate the controllability results for fractional order neutral functional differential inclusions with an infinite delay involving the Caputo derivative in Banach spaces. First, we establishes a set of sufficient conditions for the controllability of fractional order neutral functional differential inclusions with infinite delay in Banach spaces. The main techniques rely on Bohnenblust-Karlin's fixed point theorem, operator semigroups and fractional calculus. Further, we extend this result to study the controllability concept with nonlocal conditions. An example is also given to illustrate our main results.
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1. Introduction

Fractional order semilinear equations are abstract formulations for many problems arising in diffusion process, electrical science, electrochemistry, viscoelasticity, control science, electro magnetic theory and several more. On the other hand, the nonlinear fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, medical and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models. Actually, fractional differential equations are considered as an alternative model to integer differential equations. Some works have done on the qualitative properties of solutions for these equations; see the monographs [7, 30, 29, 35, 39], the papers [2, 3, 49] and the references therein.

Control theory is an area of application-oriented mathematics which deals with the analysis and design of control systems. In particular, the concept of controllability plays an important role in various areas of science and engineering. More precisely, the problem of controllability deals with the existence of a control function, which steers the solution of the system from its initial state to a final state, where the initial and final states may vary over the entire space. Existence and control problems for various types of differential systems and fractional differential systems have been studied by many authors in [1, 4, 5, 6, 8, 9, 10, 16, 17, 18, 19, 21, 27, 25, 34, 37, 40, 41, 42, 43, 44, 45, 46, 48, 47, 51, 52, 53, 57, 58].

Recently, Chang [19] proved the controllability of semilinear mixed Volterra-Fredholm type integro-differential inclusions in Banach spaces by using Bohnenblust-Karlin's fixed point theorem. In [6], Balachandran et al. studied the controllability of fractional integrodifferential systems in Banach spaces by using fractional calculus, semigroup theory and the contraction mapping principle. In [21], Debbouche et al. established the controllability result of a class of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems in a Banach space by using the theory of fractional calculus and the contraction mapping principle. In [57], Yan proved the controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay in Banach spaces by using analytic semigroups and fractional powers of closed operators and nonlinear alternative of Leray-Schauder type for multivalued maps due to D. O'Regan. In [49], Wang et al. established the existence and controllability results for fractional semilinear differential inclusions by using fractional calculation, operator semigroups and Bohnenblust-Karlin's fixed point theorem. Very recently in [41], Ravichandran et al. established the controllability problem for a class of mixed type impulsive fractional integro-differential equations in Banach spaces by using Banach contraction theorem combined with the fractional calculus theory and solution operator under some weak conditions.

Inspired by the above mentioned works, this paper establishes a set of sufficient conditions for the controllability of fractional order functional differential inclusions with infinite delay in Banach spaces of the form

$$ {}^{C}D_t^q[x(t) - g(t, x_t)] \in Ax(t) + F(t, x_t) + Bu(t), \quad t \in J = [0, b] \tag{1.1} $$

$$ x_0 = \phi \in \mathcal{B}_h, \ t \in (-\infty, 0], \tag{1.2} $$

where ${}^{C}D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ in X, $g : J \times \mathcal{B}_h \to X$ is an appropriate function, $F : J \times \mathcal{B}_h \to 2^{\mathcal{B}_h}$ is a nonempty, bounded, closed and convex multivalued map, where $\mathcal{B}_h$ is a phase space defined in section 2. The histories $x_t : (-\infty, 0] \to X$, defined by $x_t(s) = x(t+s)$, $s \leq 0$, belongs to some abstract phase space $\mathcal{B}_h$. The control function $u(\cdot) \in L^2(J, U)$, a Banach space of admissible control functions. Further, B is a bounded linear operator from U to X.

This paper is organized as follows. In Section 3, we establish a set of sufficient conditions for the controllability for fractional order functional differential inclusions in Banach spaces. In Section 4, we establish a set of sufficient conditions for the controllability for fractional order functional differential inclusions with nonlocal conditions. An example is presented in Section 5 to illustrate the theory of the obtained results.

2. Preliminaries

In this section, we mention some notations, definitions, lemmas and preliminary facts needed to establish our main results. Throughout this paper, we denote by X a Banach space with the norm $\|\cdot\|$. Let Y be another Banach space, let $L_b(X, Y)$ denote the space of bounded linear operators from X to Y. We also use $\|f\|_{L^p(J,\mathbb{R}^+)}$ norm of f whenever $f \in L^p(J, \mathbb{R}^+)$ for some p with $1 \leq p \leq \infty$. Let $L^p(J, X)$ denote the Banach space of functions $f : J \to X$ which are Bochner integrable normed by $\|f\|_{L^p(J,X)}$. Let $\mathcal{C}(J, X)$, be the Banach space of continuous functions from J into X with the usual supremum norm $\|x\|_{\mathcal{C}} := \sup_{t \in J} \|x(t)\|$, for $x \in \mathcal{C}$.

In this paper, we assume that $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $T(\cdot)$, then there exist a constant $M \geq 1$ such that $\|T(t)\| \leq M$ for every $t \in J$. Let $0 \in \rho(A)$ is the resolvent set of A. Then it is possible to define the fractional power A^{α} for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^{\alpha})$ with inverse $A^{-\alpha}$ (see [38, Chapter 2]). The following are basic properties of A^{α}.

(i) $D(A^{\alpha})$ is a Banach space with the norm $\|x\|_{\alpha} = \|A^{\alpha}x\|$ for $x \in D(A^{\alpha})$.
(ii) $T(t) : X \to X_{\alpha}$ for $t \geq 0$.
(iii) $A^{\alpha}T(t)x = T(t)A^{\alpha}x$ for each $x \in D(A^{\alpha})$ and $t \geq 0$.
(iv) For every $t > 0$, $A^{\alpha}T(t)$ is bounded on X and there exist $M_{\alpha} > 0$ such that

$$ \|A^{\alpha}T(t)\| \leq \frac{M_{\alpha}}{t^{\alpha}}. $$

(v) $A^{-\alpha}$ is a bounded linear operator for $0 \leq \alpha \leq 1$ in X.

Now we define the abstract phase space $\mathcal{B}_h$, which has been used in [56, 41, 25]. Assume that $h : (-\infty, 0] \to (0, +\infty)$ is a continuous function with

$$l = \int_{-\infty}^{0} h(t)dt < +\infty.$$

For any $a > 0$, we define

$$\mathcal{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\},$$

and equip the space $\mathcal{B}$ with the norm

$$\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} ||\psi(s)||, \quad \forall\ \psi \in \mathcal{B}.$$

Let us define

$$\mathcal{B}_h = \{\psi : (-\infty, 0] \to X \text{ such that for any } c > 0, \psi|_{[-c,0]} \in \mathcal{B}$$

and

$$\int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < +\infty\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds,\ \forall\ \psi \in \mathcal{B}_h,$$

then it is clear that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Now we consider the space

$$\mathcal{B}'_h = \{x : (-\infty, b] \to X \text{ such that } x|_J \in \mathcal{C}(J, X),\ x_0 = \phi \in \mathcal{B}_h\}.$$

Set $\|\cdot\|_b$ be a seminorm in $\mathcal{B}'_h$ defined by

$$\|x\|_b = \|\phi\|_{\mathcal{B}_h} + \sup\{||x(s)|| : s \in [0, b]\},\ x \in \mathcal{B}'_h.$$

Lemma 2.1. *(See*[56]*) Assume $x \in \mathcal{B}'_h$, then for $t \in J$, $x_t \in \mathcal{B}_h$. Moreover,*

$$l|x(t)| \leq \|x_t\|_{\mathcal{B}_h} \leq \|\phi\|_{\mathcal{B}_h} + l \sup_{s\in[0,t]} |x(s)|,$$

where $l = \int_{-\infty}^{0} h(t)dt < +\infty$.

Let us recall the following known definitions. For more details see [39, Chapter 1], [35, Chapter 3] and [29, Chapter 2].

Definition 2.2. The fractional integral of order α with the lower limit zero for a function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t > 0, \quad \alpha > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_{0}^{\infty} t^{\alpha-1}e^{-t}dt$.

Definition 2.3. The Riemann-Liouville fractional derivative of order $\alpha > 0,\ n-1 < \alpha < n,\ n \in N$, is defined as

$$^{(R-L)}D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-\alpha-1}f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.4. The Caputo derivative of order α for a function $f : [0,\infty) \to R$ can be written as

$$D^{\alpha}f(t) = D^{\alpha}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t > 0, \quad n-1 < \alpha < n.$$

Remark 2.5. (i) If $f(t) \in C^n[0,\infty)$, then

$$^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds = I^{n-\alpha}f^{(n)}(t),\ t > 0,\ n-1 < \alpha < n.$$

(ii) The Caputo derivative of a constant is equal to zero.
(iii) If f is an abstract function with values in X, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner's sense.

We also introduce some basic definitions and results of multivalued maps. For more details on multivalued maps, see the books of Deimling [22] and Hu and Papageorgiou [28].

A multivalued map $G : X \to 2^X \setminus \{\emptyset\}$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(C) = \bigcup_{x\in C} G(x)$ is bounded in X for any bounded set C of X, i.e., $\sup_{x\in C}\left\{\sup\{\|y\| : y \in G(x)\}\right\} < \infty$.

Definition 2.6. G is called upper semicontinuous (u.s.c. for short) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set C of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq C$.

Definition 2.7. G is called completely continuous if $G(C)$ is relatively compact for every bounded subset C of X.

If the multivalued map G is completely continuous with nonempty values, then G is u.s.c., if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$. G has a fixed point if there is a $x \in X$ such that $x \in G(x)$. In the following, $BCC(X)$ denotes the set of all nonempty, bounded, closed and convex subset of X.

Definition 2.8. A multi-valued map $G : J \to BCC(X)$ is said to be measurable if, for each $x \in X$, the function $v : J \to \mathbb{R}$, defined by

$$v(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\},$$

belongs to $L^1(J,\mathbb{R})$.

Definition 2.9. The multi-valued map $F : J \times X \to BCC(X)$ is said to be L^1-Caratheodory if

(i) $t \to F(t,u)$ is measurable for each $u \in X$,
(ii) $u \to F(t,u)$ is upper semi continuous for almost all $t \in J$,
(iii) for each $r > 0$, there exists $l_r \in L^1(J,\mathbb{R})$ such that

$$\|F(t,u)\| = \sup\{|f| : f(t) \in F(t,u)\} \le l_r(t)$$

for almost all $t \in J$ and all $\|u\| \in \mathcal{B}_h \le r$.

Lemma 2.10. [31, Lasota and Opial] *Let J be a compact real interval, $BCC(X)$ be the set of all nonempty, bounded, closed and convex subset of X and F be a L^1-Caratheodory multivalued map with $S_{F,x} \neq \emptyset$, where*

$$S_{F,x} = \Big\{ f \in L^1(J,X) : f(t) \in F(t,x_t),\ \textit{for a.e. } t \in J \Big\}$$

is nonempty. Let $\mathscr{F}$ be a linear continuous from $L^1(J,X)$ to $\mathcal{C}$, then the operator

$$\mathscr{F} \circ S_F : \mathcal{C} \to BCC(\mathcal{C}),\ x \to (\mathscr{F} \circ S_F)(x) = \mathscr{F}(S_{F,x}),$$

is a closed graph operator in $\mathcal{C} \times \mathcal{C}$.

Definition 2.11. A continuous function $x : (-\infty, b] \to X$ is said to be a mild solution of (1.1) – (1.2) if $x_0 = \phi \in \mathcal{B}_h$ on $(-\infty, 0]$; the restriction of $x(\cdot)$ to the interval $[0, b]$ is continuous, for $s \in [0,t)$, the function $(t-s)^{q-1}A\mathscr{S}(t-s)g(s,x_s)$ is integrable such that

$$x(t) = \mathscr{T}(t)[\phi(0) - g(0,\phi(0))] + g(t,x_t) + \int_0^t (t-s)^{q-1}A\mathscr{S}(t-s)g(s,x_s)ds$$
$$+ \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f(s)ds + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)Bu(s)ds,\ t \in J,$$

where $\mathscr{T}(\cdot)$ and $\mathscr{S}(\cdot)$ are called characteristic solution operators and given by

$$\mathscr{T} = \int_0^\infty \xi_q(\theta)T(t^q\theta)d\theta, \qquad \mathscr{S} = q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)d\theta,$$

and for $\theta \in (0,\infty)$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\overline{w}_q(\theta^{-\frac{1}{q}}) \ge 0,$$

$$\overline{w}_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q).$$

Here, ξ_q is a probability density function defined on $(0,\infty)$, that is

$$\xi_q(\theta) \ge 0, \quad \theta \in (0,\infty) \quad \text{and} \quad \int_0^\infty \xi_q(\theta)d\theta = 1.$$

Definition 2.12. The system (1.1)-(1.2) is said to be controllable on the interval J if, for every continuous initial function $\phi \in \mathcal{B}_h$, $x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that a mild solution $x(t)$ of (1.1)-(1.2) satisfies $x(b) = x_1$.

The following results of $\mathscr{T}(\cdot)$ and $\mathscr{S}(\cdot)$ will be used throughout this paper.

Remark 2.13. (See[51]) It is not difficult to verify that for $\nu \in [0,1]$.

$$\int_0^\infty \theta^\nu \xi_q(\theta)d\theta = \int_0^\infty \theta^{-q\nu}\overline{w}_q(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+q\nu)}.$$

Lemma 2.14. *(See*[58, 51]*) The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

(*i*) *For any fixed $t \geq 0$, $\mathscr{T}$ and $\mathscr{S}$ are linear and bounded operators, that is for any $x \in X$,*

$$\|\mathscr{T}(t)x\| \leq M\|x\| \text{ and } \|\mathscr{S}(t)x\| \leq \frac{qM}{\Gamma(1+q)}\|x\|.$$

(*ii*) *$\{\mathscr{T}(t), t \geq 0\}$ and $\{\mathscr{S}(t), t \geq 0\}$ are strongly continuous.*

(*iii*) *For $t \in J$ and any bounded subsets $D \subset X$, $t \to \{\mathscr{T}(t)x : x \in D\}$ and $t \to \{\mathscr{S}(t)x : x \in D\}$ are equicontinuous if $\|T(t_2^q(\theta))x - T(t_1^q(\theta))x\| \to 0$ with respect to $x \in D$ as $t_2 \to t_1$ for each fixed $\theta \in [0,\infty]$.*

(*iv*) *For any $x \in X$, $\alpha, \beta \in (0,1)$, we have*

$$AT_q(t)x = A^{1-\beta}T_q(t)A^\beta x, \quad t \in J,$$
$$\|A^\alpha T_q(t)\| \leq \frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}t^{-\alpha q}, \quad 0 < t \leq b.$$

Lemma 2.15. [11, Bohnenblust and Karlin]. *Let $\mathcal{D}$ be a nonempty subset of X, which is bounded, closed, and convex. Suppose $G : \mathcal{D} \to 2^X \setminus \{\emptyset\}$ is u.s.c. with closed, convex values, and such that $G(\mathcal{D}) \subseteq \mathcal{D}$ and $G(\mathcal{D})$ is compact. Then G has a fixed point.*

3. Controllability results

In this section, first we establish a set of sufficient conditions for the controllability of fractional order neutral functional differential inclusions in Banach spaces by using Bohnenblust-Karlin's fixed point theorem. In order to establish the result, we need the following hypotheses:

$\mathbf{H_1}$ (*i*) The strongly continuous semigroup of bounded linear operators $T(t)$ generated by A is compact when $t > 0$ and there exists a constant $M \geq 1$ such that $\sup_{t\in J} \|T(t)\| \leq M$.

(*ii*) For all bounded subsets $D \subset X$ and $x \in D$, $\|T(t_2^q\theta)x - T(t_1^q\theta)x\| \to 0$ as $t_2 \to t_1$ for each fixed $\theta \in (0,\infty)$.

$\mathbf{H_2}$ The function $g : J \times \mathcal{B}_h$ is continuous and there exists a constant $H_1 > 0$, $0 < \alpha < 1$ such that g is X_α valued and

$$\|A^\beta g(t,x) - A^\beta g(t,y)\| \leq H_1\|x-y\|_{\mathcal{B}_h}, \ x,y \in \mathcal{B}_h,$$
$$\|A^\beta g(t,x)\| \leq H_1(1+\|x\|_{\mathcal{B}_h}).$$

$\mathbf{H_3}$ The multivalued map $F : J \times \mathcal{B}_h \to BCC(X)$ is an L^1-Caratheodory function which satisfies the following condition

For each $t \in J$, the function $F(t, \cdot)$ is u.s.c; and for each $x \in \mathcal{B}_h$, the function $F(\cdot, x)$ is measurable. And for each fixed $x \in \mathcal{B}_h$, the set

$$S_{F,x} = \Big\{ f \in L^1(J, X) : f(t) \in F(t, x_t), \text{ for a.e. } t \in J \Big\}$$

is nonempty.

$\mathbf{H_4}$ For each positive number r and $x \in \mathcal{C}$ with $\|x\|_{\mathcal{C}} \le r$, there exists a positive function $l_r : J \to \mathbb{R}^+$ such that

$$\sup \Big\{ \|f\| : f(t) \in F(t, x_t) \Big\} \le l_r(t),$$

for a.e. $t \in J$.

$\mathbf{H_5}$ The function $s \to (t-s)^{q-1} l_r(s) \in L^1(J, \mathbb{R}^+)$ and there exists a $\gamma > 0$ such that

$$\lim_{r \to \infty} \frac{\int_0^t (t-s)^{q-1} l_r(s) ds}{r} = \gamma < +\infty.$$

$\mathbf{H_6}$ The linear operator $B : L^2(J, U) \to L^1(J, X)$ is bounded, $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^b (b-s)^{q-1} \mathscr{S}(b-s) Bu(s) ds$$

has an inverse operator W^{-1} which takes values in $L^2(J, U)/kerW$, where the kernel space of W is defined by $kerW = \{x \in L^2(J, U) : Wx = 0\}$ and there exist two positive constants $M_2, M_3 > 0$ such that $\|B\|_{L_b(U,X)} \le M_2$ and $\|W^{-1}\|_{L_b(X, L^2(J,U)/kerW)} \le M_3$.

Let us now explain and prove the following theorem about the controllability for (1.1)-(1.2).

Theorem 3.1. *Suppose that the hypotheses* $\mathbf{H_1}$-$\mathbf{H_6}$ *are satisfied. Then* (1.1)-(1.2) *is controllable on* J *provided that*

$$\left(1 + \frac{M M_2 M_3 b^q}{\Gamma(1+q)}\right) \left[\Big(M_g \|A^{-\beta}\| + K(q, \beta) M_g \frac{b^{q\beta}}{q\beta} \Big) l + \frac{qM}{\Gamma(1+q)} \gamma \right] < 1. \tag{3.1}$$

Proof. Using the assumption $\mathbf{H_6}$ for an arbitrary function $x(\cdot)$ define the control

$$u(t) = W^{-1} \Bigg[x_b - \mathscr{T}(b)[\phi(0) - g(0, \phi(0))] + g(b, x_b) + \int_0^b (b-s)^{q-1} A \mathscr{S}(b-s) g(s, x_s) ds - \int_0^b (b-s)^{q-1} \mathscr{S}(b-s) f(s) ds \Bigg](t).$$

For any $\varepsilon > 0$, we consider the operator $\Phi^\varepsilon : \mathcal{B}'_h \to 2^{\mathcal{B}'_h}$ defined by $\Phi^\varepsilon x$ the set of $z \in \mathcal{B}'_h$ such that

$$z(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ \mathscr{T}(t)[\phi(0) - g(0, \phi(0))] + g(t, x_t) + \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s) g(s, x_s) ds \\ \quad + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) BW^{-1} \Big[x_b - \mathscr{T}(b) g(0, \phi(0)) - g(b, x_b) \\ \quad - \int_0^b (b-\eta)^{q-1} A\mathscr{S}(b-\eta) g(\eta, x_\eta) ds - \int_0^b (b-\eta)^{q-1} \mathscr{S}(b-\eta) f(\eta) d\eta \Big] (s) ds \\ \quad + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) F(s, x_s) ds, \ t \in J, \end{cases}$$

where $f \in S_{F,x}$. We shall show that the operator Φ^ε has fixed points, which are then a solution of (1.1)-(1.2). Clearly, $x_1 = x(b) \in (\Phi^\varepsilon x)(b)$, which means that (1.1)-(1.2) is controllable.

For $\phi \in \mathcal{B}_h$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ \mathscr{T}(t)\phi(0), \quad t \in J, \end{cases}$$

then $\widehat{\phi} \in \mathcal{B}'_h$. Let $x(t) = y(t) + \widehat{\phi}(t)$, $-\infty < t \le b$. It is easy to see that x satisfies definition 2.11 if and only if y satisfies $y_0 = 0$ and

$$\begin{aligned} y(t) = & -\mathscr{T}(t) g(0, \phi(0)) + g(t, y_t + \widehat{\phi}_t) + \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s) g(s, y_s + \widehat{\phi}_s) ds \\ & + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) BW^{-1} \Big[x_1 - \mathscr{T}(b)[\phi(0) - g(0, \phi(0))] - g(b, y_b + \widehat{\phi}_b) \\ & - \int_0^b (b-\eta)^{q-1} A\mathscr{S}(b-\eta) g(\eta, y_\eta + \widehat{\phi}_\eta) d\eta - \int_0^b (b-\eta)^{q-1} \mathscr{S}(b-\eta) f(\eta) d\eta \Big] (s) ds \\ & + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) f(s) ds, \ t \in J. \end{aligned}$$

Let $\mathcal{B}''_h = \{y \in \mathcal{B}'_h : y_0 = 0 \in \mathcal{B}_h\}$. For any $y \in \mathcal{B}''_h$,

$$\begin{aligned} \|y\|_b &= \|y_0\|_{\mathcal{B}_h} + \sup\{\|y(s)\| : 0 \le s \le b\} \\ &= \sup\{\|y(s)\| : 0 \le s \le b\}, \end{aligned}$$

thus $(\mathcal{B}''_h, \|\cdot\|_b)$ is a Banach space. Set $B_r = \{y \in \mathcal{B}''_h : \|y\|_b \le r\}$ for some $r > 0$, then $B_r \subseteq \mathcal{B}''_h$ is uniformly bounded, and for $y \in B_r$, from Lemma 2.1, we have

$$\begin{aligned} \|y_t + \hat{\phi}_t\|_{\mathcal{B}_h} &\le \|y_t\|_{\mathcal{B}_h} + \|\hat{\phi}_t\|_{\mathcal{B}_h} \\ &\le l(r + M_1|\phi(0)|) + \|\phi\|_{\mathcal{B}_h} = r'. \end{aligned} \tag{3.2}$$

Define the multivalued map $\Psi : \mathcal{B}_h'' \to \mathcal{B}_h''$ defined by Ψy the set of $\overline{z} \in \mathcal{B}_h''$ such that

$$\overline{z}(t) = \begin{cases} 0, \quad t \in (-\infty, 0], \\ -\mathscr{T}(t)g(0,\phi(0)) + g(t, y_t + \widehat{\phi}_t) + \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s) g(s, y_s + \widehat{\phi}_s) ds \\ \quad + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)BW^{-1}\Big[x_1 - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] - g(b, y_b + \widehat{\phi}_b) \\ \quad - \int_0^b (b-\eta)^{q-1} A\mathscr{S}(b-\eta) g(\eta, y_\eta + \widehat{\phi}_\eta) d\eta - \int_0^b (b-\eta)^{q-1}\mathscr{S}(b-\eta) f(\eta) d\eta\Big](s) ds \\ \quad + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s) f(s) ds, \ t \in J. \end{cases}$$

Obviously, the operator Φ^ε has a fixed point if and only if Ψ has a fixed point. So, our aim is to show that Ψ has a fixed point. For the sake of convenience, we subdivide the proof into in several steps.

Step 1. Ψ is convex for each $x \in B_r$. In fact, if φ_1, φ_2 belong to $\Psi(x)$, then there exist f_1, $f_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned} \varphi_i(t) = &-\mathscr{T}(t)g(0,\phi(0)) + g(t, y_t + \widehat{\phi}_t) + \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s) g(s, y_s + \widehat{\phi}_s) ds \\ &+ \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)BW^{-1}\Big[x_1 - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] - g(b, y_b + \widehat{\phi}_b) \\ &- \int_0^b (b-\eta)^{q-1} A\mathscr{S}(b-\eta) g(\eta, y_\eta + \widehat{\phi}_\eta) d\eta - \int_0^b (b-\eta)^{q-1}\mathscr{S}(b-\eta) f_i(\eta) d\eta\Big](s) ds \\ &+ \int_0^t (t-s)^{q-1}\mathscr{S}(t-s) f_i(s) ds, \quad i = 1, 2. \end{aligned}$$

Let $\lambda \in [0,1]$. Then for each $t \in J$, we get

$$\begin{aligned} (\lambda\varphi_1 + (1-\lambda)\varphi_2)(t) = &-\mathscr{T}(t)g(0,\phi(0)) + g(t, y_t + \widehat{\phi}_t) \\ &+ \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s) g(s, y_s + \widehat{\phi}_s) ds \\ &+ \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)BW^{-1}\Bigg[x_b - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] \\ &- g(b, y_b + \widehat{\phi}_b) - \int_0^b (b-\eta)^{q-1} A\mathscr{S}(b-\eta) g(\eta, y_\eta + \widehat{\phi}_\eta) d\eta \\ &- \int_0^b (b-\eta)^{q-1}\mathscr{S}(b-\eta)[\lambda f_1(\eta) + (1-\lambda) f_2(\eta)] d\eta\Bigg](s) ds \\ &+ \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)[\lambda f_1(s) + (1-\lambda) f_2(s)] ds. \end{aligned}$$

It is easy to see that $S_{F,x}$ is convex since F has convex values.
So, $\lambda f_1 + (1-\lambda) f_2 \in S_{F,x}$. Thus,

$$\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \Psi(x).$$

Step 2. We show that there exists some $r > 0$ such that $\Psi(B_r) \subseteq B_r$. If it is not true, then there exists $\varepsilon > 0$ such that for every positive number r and $t \in J$, there exists a function $x_r \in B_r$, but $\Psi(x_r) \notin B_r$, that is,

$$\|\Psi(x_r)(t)\| = \sup\{\|\varphi_r\|_b : \varphi_r \in \Psi(x_r)\} \geq r.$$

For such $\varepsilon > 0$, an elementary inequality can show that

$$\begin{aligned}
r &\leq \|(\Psi y^r)(t)\| \\
&\leq \| - \mathscr{T}(t)g(0,\phi(0))\| + \|g(t, y_t^r + \widehat{\phi}_t)\| + \Big\| \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s) g(s, y_s^r + \widehat{\phi}_s) ds \Big\| \\
&+ \Big\| \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) BW^{-1} \Big[x_1 - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] \\
&- g(b, y_b^r + \widehat{\phi}_b) - \int_0^b (b-\eta)^{q-1} A\mathscr{S}(b-\eta) g(\eta, y_\eta^r + \widehat{\phi}_\eta) d\eta \\
&- \int_0^b (b-\eta)^{q-1} \mathscr{S}(b-\eta) f^r(\eta) d\eta \Big](s) ds \Big\| + \Big\| \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) f^r(s) ds \Big\| \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Let us estimate each term above $I_i, i = 1, \cdots, 5$. By Lemma 2.1 and assumptions $\mathbf{H_1}$-$\mathbf{H_2}$, we have

$$\begin{aligned}
I_1 \leq& M\|A^{-\beta}\|\|A^\beta g(0,\phi)\| \leq MM_g\|A^{-\beta}\|(1 + \|\phi\|_{\mathcal{B}_h}), \\
I_2 \leq& \|A^{-\beta}\|\|A^\beta g(t, y_t^r + \widehat{\phi}_t)\| \leq M_g\|A^{-\beta}\|(1 + \|y_t^r + \widehat{\phi}_t\|_{\mathcal{B}_h}) \leq M_g\|A^{-\beta}\|(1 + r'),
\end{aligned}$$

By a standard calculation involving Lemma 2.14, assumption $\mathbf{H_2}$, Eq. (3.2) and the Holder inequality, we can deduce that

$$\begin{aligned}
I_3 &\leq \int_0^t \Big\| (t-s)^{q-1} A^{1-\beta} \mathscr{S}(t-s) A^\beta g(s, y_s^r + \widehat{\phi}_s) \Big\| ds \\
&\leq \frac{qM_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)} \int_0^t (t-s)^{q-1} (t-s)^{-(1-\beta)q} \|A^\beta g(s, y_s^r + \widehat{\phi}_s)\| ds \\
&\leq K(q,\beta) \int_0^t (t-s)^{q\beta - 1} M_g (1 + \|y_s^r + \widehat{\phi}_s\|_{\mathcal{B}_h}) ds \\
&\leq K(q,\beta) M_g \frac{b^{q\beta}}{q\beta} (1 + r')
\end{aligned}$$

where $K(q,\beta) = \frac{qM_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)}$.

A similar argument involves Lemma 2.14 and assumptions $\mathbf{H_3}$ and $\mathbf{H_4}$; we obtain

$$I_4 \le \frac{qM}{\Gamma(1+q)}\int_0^t (t-s)^{q-1} BW^{-1}\Bigg\|\Bigg[x_1 + \mathscr{T}(t)g(0,\phi(0)) + g(t, y_t^r + \widehat{\phi}_t) + \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s) g(s, y_s^r + \widehat{\phi}_s)ds + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s) f^r(s)ds\Bigg]\Bigg\| ds$$

$$\le \frac{MM_2M_3b^q}{\Gamma(1+q)}\Bigg[\|x_1\| + MM_g\|A^{-\beta}\|(1+\|\phi\|_{\mathcal{B}_h}) + M_g\|A^{-\beta}\|(1+r') + K(q,\beta)M_g\frac{b^{q\beta}}{q\beta}(1+r') + \frac{qM}{\Gamma(1+q)}\int_0^t (t-s)^{\alpha-1} l_r(s)ds\Bigg]$$

On the other hand,

$$I_5 \le \int_0^t (t-s)^{\alpha-1}\|\mathscr{S}(t-s) f^r(s)\| ds \le \frac{qM}{\Gamma(1+q)}\int_0^t (t-s)^{\alpha-1} l_r(s) ds$$

Combining I_1-I_5 yields

$$r \le \frac{MM_2M_3b^q}{\Gamma(1+q)}\|x_1\| + \left(1 + \frac{MM_2M_3b^q}{\Gamma(1+q)}\right)\Bigg[MM_g\|A^{-\beta}\|(1+\|\phi\|_{\mathcal{B}_h}) + M_g\|A^{-\beta}\|(1+r') + K(q,\beta)M_g\frac{b^{q\beta}}{q\beta}(1+r') + \frac{qM}{\Gamma(1+q)}\int_0^t (t-s)^{\alpha-1} l_r(s)ds\Bigg] \tag{3.3}$$

Dividing both sides of (3.3) by r and taking $r \to \infty$, we obtain that

$$\left(1 + \frac{MM_2M_3b^q}{\Gamma(1+q)}\right)\left[\Big(M_g\|A^{-\beta}\| + K(q,\beta)M_g\frac{b^{q\beta}}{q\beta}\Big)l + \frac{qM}{\Gamma(1+q)}\gamma\right] \ge 1$$

which is a contradiction to our assumption. Thus for $q > 0$, for some positive number r and some $f \in S_{F,x}$, $\Psi(B_r) \subseteq B_r$.

Step 3. $\Psi(B_r)$ is equicontinuous. Indeed, let $\varepsilon > 0$ be small, $0 < t_1 < t_2 \le b$. For each $y \in B_r$ and $\overline{z}$ belong to $\Psi_1 y$, there exists $f \in S_{F,x}$ such that for each $t \in J$, we have

$$\|\overline{z}(t_2) - \overline{z}(t_1)\| = \| - (\mathscr{T}(t_2) - \mathscr{T}(t_1))g(0,\phi)\| + \|g(t_2, y_{t_2} + \widehat{\phi}_{t_2}) - g(t_1, y_{t_1} + \widehat{\phi}_{t_1})\| + \Big\|\int_{t_1}^{t_2} (t_2-s)^{q-1} A\mathscr{S}(t_2-s) g(s, y_s + \widehat{\phi}_s)ds\Big\| + \Big\|\int_{t_1-\varepsilon}^{t_1} (t_2-s)^{q-1} A[\mathscr{S}(t_2-s) - \mathscr{S}(t_1-s)] g(s, y_s + \widehat{\phi}_s)ds\Big\| + \Big\|\int_{t_1-\varepsilon}^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] A\mathscr{S}(t_1-s) g(s, y_s + \widehat{\phi}_s)ds\Big\|$$

$$
\begin{aligned}
&+\Big\|\int_0^{t_1-\varepsilon}(t_2-s)^{q-1}A[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)]g(s,y_s+\widehat{\phi}_s)ds\Big\|\\
&+\Big\|\int_0^{t_1-\varepsilon}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]A\mathscr{S}(t_1-s)g(s,y_s+\widehat{\phi}_s)ds\Big\|\\
&+\Big\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathscr{S}(t_2-s)BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&\quad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&\quad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\Big\|\\
&+\Big\|\int_{t_1-\varepsilon}^{t_1}(t_2-s)^{q-1}[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)]BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&\quad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&\quad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\Big\|\\
&+\Big\|\int_{t_1-\varepsilon}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\mathscr{S}(t_1-s)BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&\quad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&\quad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\Big\|\\
&+\Big\|\int_0^{t_1-\varepsilon}(t_2-s)^{q-1}[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)]BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&\quad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&\quad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\Big\|\\
&+\Big\|\int_0^{t_1-\varepsilon}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\mathscr{S}(t_1-s)BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&\quad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&\quad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\Big\|\\
&+\Big\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathscr{S}(t_2-s)f(s)ds\Big\|\\
&+\Big\|\int_{t_1-\varepsilon}^{t_1}(t_2-s)^{q-1}[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)]f(s)ds\Big\|
\end{aligned}
$$

$$+\Big\|\int_{t_1-\varepsilon}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\mathscr{S}(t_1-s)f(s)ds\Big\|$$
$$+\Big\|\int_0^{t_1-\varepsilon}(t_2-s)^{q-1}[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)]f(s)ds\Big\|$$
$$+\Big\|\int_0^{t_1-\varepsilon}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\mathscr{S}(t_1-s)f(s)ds\Big\|.$$

Applying Lemma 2.14 and the Holder's inequality, we obtain

$$\begin{aligned}
\|\overline{z}(t_2)-\overline{z}(t_1)\| &= \|(\mathscr{T}(t_2)-\mathscr{T}(t_1))\|\|g(0,\phi)\| + \|g(t_2,y_{t_2}+\widehat{\phi}_{t_2})-g(t_1,y_{t_1}+\widehat{\phi}_{t_1})\| \\
&\quad +K(q,\beta)\int_{t_1}^{t_2}(t_2-s)^{q\beta-1}\|A^\beta g(s,y_s+\widehat{\phi}_s)\|ds \\
&\quad +\int_{t_1-\varepsilon}^{t_1}(t_2-s)^{q\beta-1}\|A^{1-\beta}\|\|[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)]\|\|A^\beta g(s,y_s+\widehat{\phi}_s)\|ds \\
&\quad +K(q,\beta)\int_{t_1-\varepsilon}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\|A^\beta g(s,y_s+\widehat{\phi}_s)\|ds \\
&\quad +\int_0^{t_1-\varepsilon}(t_2-s)^{q\beta-1}\|A^{1-\beta}\|\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|\|A^\beta g(s,y_s+\widehat{\phi}_s)\|ds \\
&\quad +K(q,\beta)\int_0^{t_1-\varepsilon}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\|A^\beta g(s,y_s+\widehat{\phi}_s)\|ds \\
&\quad +\frac{qMM_2M_3}{\Gamma(1+q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\bigg\|\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))] \\
&\quad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta \\
&\quad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big]\bigg\|ds \\
&\quad +M_2\int_{t_1-\varepsilon}^{t_1}(t_2-s)^{q-1}\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|\bigg\|\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))] \\
&\quad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta \\
&\quad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big]\bigg\|ds \\
&\quad +\frac{qMM_2}{\Gamma(1+q)}\int_{t_1-\varepsilon}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\bigg\|\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))] \\
&\quad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta \\
&\quad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big]\bigg\|ds
\end{aligned}$$

$$+M_2\int_0^{t_1-\varepsilon}(t_2-s)^{q-1}\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|\Big\|\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]$$

$$-g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta$$

$$-\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big]\Big\|ds$$

$$+\frac{qMM_2}{\Gamma(1+q)}\int_0^{t_1-\varepsilon}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\Big\|\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]$$

$$-g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta$$

$$-\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big]\Big\|ds$$

$$+\frac{qM}{\Gamma(1+q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}l_r(s)ds$$

$$+\int_{t_1-\varepsilon}^{t_1}(t_2-s)^{q-1}\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|l_r(s)ds$$

$$+\frac{qM}{\Gamma(1+q)}\int_{t_1-\varepsilon}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]l_r(s)ds$$

$$+\int_0^{t_1-\varepsilon}(t_2-s)^{q-1}\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|l_r(s)ds$$

$$+\frac{qM}{\Gamma(1+q)}\int_0^{t_1-\varepsilon}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]l_r(s)ds.$$

Therefore, for ε sufficiently small, we can verify that the right-hand side of the above inequality tends to zero as $t_2\to t_1$. On the other hand, the compactness of $\mathscr{S}(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus Ψ maps B_r into an equicontinuous family of functions.

Step 4. The set $V(t)=\{(\Psi_1 y)(t): y\in B_r\}$ is pre-compact in X. Obviously, $V(t)$ is pre-compact in $\mathcal{B}_h''$ for $t=0$. Let $0<t\le b$ be fixed and ε be a real number satisfying $0<\varepsilon<t$. For $\delta>0$ and $y\in B_r$, define an operator $\Psi^{\varepsilon,\delta}$ on B_r by $\Psi^{\varepsilon,\delta}y\in\mathcal{B}_h''$ the set of such that

$$\begin{aligned}
\overline{z}^{\varepsilon,\delta} &= -\int_{\delta}^{\infty}\xi_q(\theta)T(t^q\theta)g(0,\phi(0))d\theta + g(t-\varepsilon, y_{t-\varepsilon}+\widehat{\phi}_{t-\varepsilon})\\
&+ q\int_0^{t-\varepsilon}(t-s)^{q-1}A\Big(\int_{\delta}^{\infty}\theta\xi_q(\theta)T((t-s)^q\theta)d\theta\Big)g(s,y_s+\widehat{\phi}_s)ds\\
&+ q\int_0^{t-\varepsilon}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&- g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&- \int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\\
&+ q\int_0^{t-\varepsilon}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)f(s)d\theta ds\\
&= -T(\varepsilon^q\delta)\int_{\delta}^{\infty}\xi_q(\theta)T(t^q\theta-\varepsilon^q\delta)g(0,\phi(0))d\theta + g(t-\varepsilon,y_{t-\varepsilon}+\widehat{\phi}_{t-\varepsilon})\\
&+ T(\varepsilon^q\delta)q\int_0^{t-\varepsilon}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\xi_q(\theta)AT((t-s)^q\theta-\varepsilon^q\delta)g(s,y_s+\widehat{\phi}_s)d\theta ds\\
&+ T(\varepsilon^q\delta)q\int_0^{t-\varepsilon}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta-\varepsilon^q\delta)BW^{-1}\Big[x_1\\
&- \mathscr{T}(b)[\phi(0)-g(0,\phi(0))]-g(b,y_b+\widehat{\phi}_b)\\
&- \int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&- \int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\\
&+ T(\varepsilon^q\delta)q\int_0^{t-\varepsilon}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta-\varepsilon^q\delta)f(s)d\theta ds
\end{aligned}$$

$f\in S_{f,x}$. Since $T(\varepsilon^q\delta)$, $(\varepsilon^q\delta>0)$, is a compact operator, then the set

$$V^{\varepsilon,\delta}(t)=\{(\Psi_1^{\varepsilon,\delta}y)(t): y\in B_r\}$$

is pre-compact in X for every ε, $0<\varepsilon<t$ and for all $\delta>0$. Moreover, for every $y\in B_r$, we have

$$\begin{aligned}
\|\overline{z}(t)-\overline{z}^{\varepsilon,\delta}(t)\| &\le \Big\|\int_{\delta}^{\infty}\xi_q(\theta)T(t^q\theta)g(0,\phi(0))d\theta\Big\| + \Big\|g(t,y_t+\widehat{\phi}_t)-g(t-\varepsilon,y_{t-\varepsilon}+\widehat{\phi}_{t-\varepsilon})\Big\|\\
&+q\Big\|\int_0^t\int_0^{\delta}\theta(t-s)^{q-1}\xi_q(\theta)AT((t-s)^q\theta)g(s,y_s+\widehat{\phi}_s)d\theta ds\Big\|\\
&+q\Big\|\int_{t-\varepsilon}^t\int_{\delta}^{\infty}\theta(t-s)^{q-1}\xi_q(\theta)AT((t-s)^q\theta)g(s,y_s+\widehat{\phi}_s)d\theta ds\Big\|
\end{aligned}$$

$$
\begin{aligned}
&+q\Big\|\int_0^t\int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&\qquad -g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&\qquad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\Big\|\\
&+q\Big\|\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)\\
&\qquad -g(0,\phi(0))]-g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&\qquad -\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\Big\|\\
&+q\Big\|\int_0^t\int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)f(s)d\theta ds\Big\|\\
&+q\Big\|\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)f(s)d\theta ds\Big\|\\
&= J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8.
\end{aligned}
$$

A similar argument as before can show that

$$
\begin{aligned}
J_1 &\le M\|A^{-\beta}\|\|A^\beta g(0,\phi)\|\Big(\int_0^\delta \xi_q(\theta)d\theta\Big) \le M\|A^{-\beta}\|M_g\|(1+\|\phi\|_{\mathcal{B}_h})\Big(\int_0^\delta \xi_q(\theta)d\theta\Big)\\
J_2 &\le \|A^{-\beta}\|\|A^\beta g(t,y_t+\widehat{\phi}_t)-A^\beta g(t-\varepsilon,y_{t-\varepsilon}+\widehat{\phi}_{t-\varepsilon})\|\\
&\le \|A^{-\beta}\|M_g\big(\varepsilon+\|(y_t-y_{t-\varepsilon})+(\widehat{\phi}_t-\widehat{\phi}_{t-\varepsilon})\|_{\mathcal{B}_h}\big)\\
J_3 &\le q\int_0^t\int_0^\delta \|\theta(t-s)^{q-1}\xi_q(\theta)A^{1-\beta}T((t-s)^q\theta)A^\beta g(s,y_s+\widehat{\phi}_s)\|d\theta ds\\
&\le K(q,\beta)\int_0^t(t-s)^{q-1}M_g(1+r')ds\int_0^\delta \theta\varepsilon_q(\theta)d\theta\\
J_4 &\le q\int_{t-\varepsilon}^t\int_\delta^\infty \|\theta(t-s)^{q-1}\xi_q(\theta)A^{1-\beta}T((t-s)^q\theta)A^\beta g(s,y_s+\widehat{\phi}_s)\|d\theta ds\\
&\le K(q,\beta)\int_{t-\varepsilon}^t(t-s)^{q\beta-1}M_g(1+r')ds\int_\delta^\infty \theta\varepsilon_q(\theta)d\theta
\end{aligned}
$$

$$
\begin{aligned}
J_5 &\leq qM\int_0^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&-g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&-\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\\
&\leq qMM_2M_3\int_0^t(t-s)^{q-1}\Bigg[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]-g(b,y_b+\widehat{\phi}_b)\\
&-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&-\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Bigg]ds\int_0^\delta\theta\varepsilon_q(\theta)d\theta\\
J_6 &\leq q\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)BW^{-1}\Big[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]\\
&-g(b,y_b+\widehat{\phi}_b)-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&-\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Big](s)d\theta ds\\
&\leq qMM_2M_3\int_{t-\varepsilon}^t(t-s)^{q-1}\Bigg[x_1-\mathscr{T}(b)[\phi(0)-g(0,\phi(0))]-g(b,y_b+\widehat{\phi}_b)\\
&-\int_0^b(b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,y_\eta+\widehat{\phi}_\eta)d\eta\\
&-\int_0^b(b-\eta)^{q-1}\mathscr{S}(b-\eta)f(\eta)d\eta\Bigg]ds\int_\delta^\infty\theta\varepsilon_q(\theta)d\theta\\
J_7 &\leq q\int_0^t\int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)f(s)d\theta ds\\
&\leq qM\int_0^t(t-s)^{q-1}l_r(s)ds\int_0^\delta\theta\varepsilon_q(\theta)d\theta\\
J_8 &\leq q\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)f(s)d\theta ds\\
&\leq qM\int_{t-\varepsilon}^t(t-s)^{q-1}l_r(s)ds\int_\delta^\infty\theta\varepsilon_q(\theta)d\theta,
\end{aligned}
$$

From J_1 to J_8, one can see that for each $y \in B_r$,

$$\|\overline{z}(t)-\overline{z}^{\varepsilon,\delta}(t)\| \to 0 \quad \text{as} \quad \varepsilon\to 0^+,\ \delta\to 0^+.$$

Therefore, there are pre-compact sets arbitrary close to the set $V(t)$, $t > 0$. Hence, the set $V(t)$, $t > 0$ is also pre-compact in X.

Step 5. Ψ has a closed graph.

Let $y_n \to y_*$ as $n \to \infty$, $\overline{z}_n \in \Psi y_n$ for each $y_n \in B_r$, and $\overline{z}_n \to \overline{z}_*$ as $n \to \infty$. We will show that $\overline{z}_* \in \Psi y_*$. Since $\overline{z}_n \in \Psi y_n$, there exists a $f_n \in S_{f,y_n}$ such that

$$\begin{aligned}\overline{z}_n(t) &= \mathscr{T}(t)g(0,\phi(0)) + g(t,(y_n)_t + \widehat{\phi}_t) + \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s)g(s,(y_n)_s + \widehat{\phi}_s)ds \\ &+ \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)BW^{-1}\Bigg[x_b - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] \\ &- g(b,(y_n)_b + \widehat{\phi}_b) - \int_0^b (b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,(y_n)_\eta + \widehat{\phi}_\eta)d\eta \\ &- \int_0^b (b-\eta)^{q-1}\mathscr{S}(b-\eta)f_n(\eta)d\eta\Bigg](s)ds + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f_n(s)ds,\ t \in J.\end{aligned}$$

We must prove that there exists $f_* \in S_{f,y_*}$ such that

$$\begin{aligned}\overline{z}_*(t) &= -\mathscr{T}(t)g(0,\phi(0)) + g(t,(y_*)_t + \widehat{\phi}_t) + \int_0^t (t-s)^{q-1} A\mathscr{S}(t-s)g(s,(y_*)_s + \widehat{\phi}_s)ds \\ &+ \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)BW^{-1}\Bigg[x_b - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] \\ &- g(b,(y_*)_b + \widehat{\phi}_b) - \int_0^b (b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,(y_*)_\eta + \widehat{\phi}_\eta)d\eta \\ &- \int_0^b (b-\eta)^{q-1}\mathscr{S}(b-\eta)f_*(\eta)d\eta\Bigg](s)ds \\ &+ \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f_*(s)ds,\ t \in J.\end{aligned}$$

Now, for every $t \in J$, since g is continuous, and from the definition of u^ε we get

$$\begin{aligned}&\Bigg\|\Bigg(\overline{z}_n(t) + \mathscr{T}(t)g(0,\phi(0)) - g(t,(y_n)_t + \widehat{\phi}_t) - \int_0^t (t-s)^{q-1}A\mathscr{S}(t-s)g(s,(y_n)_s + \widehat{\phi}_s)ds \\ &\quad - \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)BW^{-1}\Bigg[x_b - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] - g(b,(y_n)_b + \widehat{\phi}_b) \\ &\qquad - \int_0^b (b-\eta)^{q-1}A\mathscr{S}(b-\eta)g(\eta,(y_n)_\eta + \widehat{\phi}_\eta)d\eta\Bigg](s)ds\Bigg) \\ &-\Bigg(\overline{z}_*(t) + \mathscr{T}(t)g(0,\phi(0)) - g(t,(y_*)_t + \widehat{\phi}_t) - \int_0^t (t-s)^{q-1}A\mathscr{S}(t-s)g(s,(y_*)_s + \widehat{\phi}_s)ds\end{aligned}$$

$$- \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) B W^{-1} \Bigg[x_b - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] - g(b,(y_*)_b + \widehat{\phi}_b)$$

$$- \int_0^b (b-\eta)^{q-1} A \mathscr{S}(b-\eta) g(\eta,(y_*)_\eta + \widehat{\phi}_\eta) d\eta \Bigg](s) ds \Bigg) \Bigg\| \to 0 \text{ as } n \to \infty.$$

Consider the linear continuous operator $\Theta : L^1(J,X) \to C(J,X)$,

$$\begin{aligned}(\Theta f)(t) &= \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) f(s) ds \\ &\quad - \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) B W^{-1} \Big(\int_0^b (b-\tau)^{q-1} \mathscr{S}(b-\tau) f(\tau) d\tau \Big) ds.\end{aligned}$$

From Lemma 2.10, it follows that $\Theta \circ S_F$ is a closed graph operator. Also, from the definition of Θ, we have that

$$\Bigg(\overline{z}_n(t) + \mathscr{T}(t) g(0,\phi(0)) - g(t,(y_n)_t + \widehat{\phi}_t) - \int_0^t (t-s)^{q-1} A \mathscr{S}(t-s) g(s,(y_n)_s + \widehat{\phi}_s) ds$$

$$- \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) B W^{-1} \Bigg[x_b - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] - g(b,(y_n)_b + \widehat{\phi}_b)$$

$$- \int_0^b (b-\eta)^{q-1} A \mathscr{S}(b-\eta) g(\eta,(y_n)_\eta + \widehat{\phi}_\eta) d\eta \Bigg](s) ds \Bigg) \in \Theta(S_{F,y_n}).$$

Since $y_n \to y_*$, for some $y_* \in S_{F,y_*}$, it follows from 2.10 that

$$\Bigg(\overline{z}_*(t) + \mathscr{T}(t) g(0,\phi(0)) - g(t,(y_*)_t + \widehat{\phi}_t) - \int_0^t (t-s)^{q-1} A \mathscr{S}(t-s) g(s,(y_*)_s + \widehat{\phi}_s) ds$$

$$- \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) B W^{-1} \Bigg[x_b - \mathscr{T}(b)[\phi(0) - g(0,\phi(0))] - g(b,(y_*)_b + \widehat{\phi}_b)$$

$$- \int_0^b (b-\eta)^{q-1} A \mathscr{S}(b-\eta) g(\eta,(y_*)_\eta + \widehat{\phi}_\eta) d\eta \Bigg](s) ds \Bigg) \in \Theta(S_{F,y_*})$$

therefore Ψ has a closed graph.

As a consequence of **Step 1** to **Step 5** together with the Arzela-Ascoli theorem, we conclude that Ψ is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 2.15, we can deduce that Ψ has a fixed point x which is a mild solution of (1.1)-(1.2). Therefore, (1.1)-(1.2) is controllable on J. □

4. Fractional differential inclusions with nonlocal conditions

The study on nonlocal conditions are motivated by physical problems. For example, it is used to determine the unknown physical parameters in some inverse heat conduction problems [15]. The result concerning the existence and uniqueness of mild solutions to abstract Cauchy problems with nonlocal initial conditions was first formulated and proved by Byszewski, see [12]. Since the appearance of this paper, several papers have addressed the issue of existence and uniqueness results for various types of nonlinear differential equations [13, 14, 20, 36, 23, 33, 55]. Control problems for various types of differential systems and fractional differential systems with nonlocal initial conditions have been studied in [9, 26, 32, 33, 24, 45, 51].

Recently in [51], Wang et al. proved sufficient conditions for nonlocal controllability for fractional evolution systems by using Monch fixed point theorem and in [54] discussed the existence and controllability results for nonlocal fractional impulsive differential inclusions in Banach spaces by using theorem for contraction multivalued is proved by Covitz and Nadler. In [45] Vijayakumar et al. established the nonlocal controllability of mixed Volterra-Fredholm type fractional semilinear integro-differential inclusions in Banach spaces by using Bohnenblust-Karlin's fixed point theorem

Inspired by this consideration, we establishes a set of sufficient conditions for the controllability of fractional order functional neutral differential inclusions with infinite delay in Banach spaces with nonlocal condition of the form

$$
\begin{aligned}
{}^{C}D_t^q(x(t)+f(t,x_t)) &\in Ax(t)+g(t,x_t), \quad t\in[0,b] &(4.1)\\
x_0 &\in \phi+q(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_n})\in\mathcal{B}_h, &(4.2)
\end{aligned}
$$

where $0<t_1<t_2<t_3<\cdots<t_n\le b$, $q:\mathcal{B}_h^n\to\mathcal{B}_h$ is a given function which satisfies the following condition:

$\mathbf{H_7}$ $q:\mathcal{B}^n\to\mathcal{B}$ is continuous and exist positive constants $L_i(q)$ such that

$$\|q(\psi_1,\psi_2,\psi_3,\cdots,\psi_n)-q(\varphi_1,\varphi_2,\varphi_3,\cdots,\varphi_n)\|\le\sum_{i=1}^{n}L_i(q)\|\psi_i-\varphi_i\|_{\mathcal{B}},$$

for every $\psi_i,\varphi_i\in\mathcal{B}_h$ and assume

$$N_q=\sup\{\|q(\psi_1,\psi_{t_2},\psi_{t_3},\cdots,\psi_{t_n})\|:\psi_i\in\mathcal{B}_h\}.$$

Definition 4.1. A continuous function $x:(-\infty,b]\to X$ is said to be a mild solution of $(4.1)-(4.2)$ if $x_0=\phi\in\mathcal{B}_h$ on $(-\infty,0]$; the restriction of $x(\cdot)$ to the interval $[0,b]$ is continuous, for $s\in[0,t)$, the function $(t-s)^{q-1}A\mathscr{S}(t-s)g(s,x_s)$ is integrable such that

$$
\begin{aligned}
x(t)&=\mathscr{T}(t)[\phi(0)-g(0,\phi(0))+q(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_n})(0)]+g(t,x_t)\\
&+\int_0^t(t-s)^{q-1}A\mathscr{S}(t-s)g(s,x_s)ds+\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)f(s)ds\\
&+\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)Bu(s)ds,\ t\in J,
\end{aligned}
$$

where $\mathscr{T}(\cdot)$ and $\mathscr{S}(\cdot)$ are defined in Definition 2.11.

Theorem 4.2. *Suppose that the hypotheses* $\mathbf{H_1}$-$\mathbf{H_7}$ *are satisfied. Then* (4.1)-(4.2) *is controllable on* J *provided that*

$$\left(1+\frac{MM_2M_3b^q}{\Gamma(1+q)}\right)\left[\left(M_g\|A^{-\beta}\|+K(q,\beta)M_g\frac{b^{q\beta}}{q\beta}\right)l+\frac{qM}{\Gamma(1+q)}\gamma\right]<1. \tag{4.3}$$

Proof. The proof is similar to the proof of Theorem 3.1. We can omit the proof. □

5. An example

Consider a control system governed by the fractional order neutral functional differential inclusion of the form

$$\begin{aligned}{}^C D_t^q\left[z(t,\eta)+\int_{-\infty}^0 b(\theta,\eta)z(t,\theta)d\theta\right] &\in \frac{\partial^2}{\partial\eta^2}z(t,\eta)+\widehat{\mu}(t,\eta)\\ &+\mu\Big(t,\int_{-\infty}^t \mu_1(s-t)z(s,\eta)ds\Big),\ \eta\in[0,\pi],\ t\in[0,b], \qquad (5.1)\\ z(t,0)=&z(t,\pi)=0,\ t\ge 0, \qquad (5.2)\\ z(t,\eta)=&\psi(t,\eta),\quad 0\le\eta\le\pi,\quad t\in(-\infty,0], \qquad (5.3)\end{aligned}$$

where ${}^C D_t^q$ is a Caputo fractional partial derivative of order $0<q<1$, $\psi(t,\eta)$, μ and μ_1 are continuous.

To rewrite this system into the abstract form (1.1)-(1.2), Let $X=L^2(0,\pi)$ and let $A:X\to X$ be defined by $Ay=y''$, $y\in D(A)$, where $D(A)=\{y\in X: y,y'$ are absolutely continuous, $y(0)=y(1)=0\}$. Then A is the infinitesimal generator of an analytic semigroup $\{T(t),t\ge 0\}$ in X. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2$, $n=0,1,2,\cdots$ and corresponding normalized eigenfunctions are given by $z_n(\eta)=\sqrt{\frac{2}{\pi}}\sin(n\pi)$. We also use the following properties:

- **(i)** If $y\in D(A)$, then $Ay=\sum_{n=1}^{\infty} n^2\langle y,z_n\rangle z_n$.
- **(ii)** For each $y\in X$, $A^{-1/2}y=\sum_{n=1}^{\infty}\frac{1}{n}\langle y,z_n\rangle z_n$. In particular, $\|A^{-1/2}\|=1$.
- **(iii)** The operator $A^{1/2}$ is given by $A^{1/2}y=\sum_{n=1}^{\infty} n\langle y,z_n\rangle z_n$ on the space

$$D(A^{1/2})=\{y(\cdot)\in X,\sum_{n=1}^{\infty} n\langle y,z_n\rangle z_n\in X\}.$$

Now, we present a special phase space $\mathcal{B}_h$. Let $h(s)=e^{2s}$, $s<0$. Then

$$l=\int_{-\infty}^0 h(s)ds=\frac{1}{2}.$$

Let

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (\|\varphi(\theta)\|)^{\frac{1}{2}} ds.$$

Then $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.
For $(t,\varphi) \in [0,b] \times \mathcal{B}_h$, where $\varphi(\theta)(\eta) = \psi(\theta,\eta) \in (-\infty,0] \times [0,\pi]$, let $z(t) = z(t,\cdot)$, that is $z(t)(\eta) = z(t,\eta)$.
Define an infinite-dimensional space U by

$$U = \left\{ u | u = \sum_{n=2}^{\infty} u_n v_n, \text{ with } \sum_{n=2}^{\infty} U_n^2 < \infty \right\}$$

for each $v \in X$. The norm in U is defined by

$$\|u\|_U = \sum_{n=2}^{\infty} U_n.$$

Now, define a continuous linear mapping B from X into X as

$$Bu = u_2 v_1 + \sum_{n=2}^{\infty} u_n v_n \text{ for } \sum_{n=2}^{\infty} u_n v_n \in U.$$

Define the bounded linear operator $B : U \to X$ by $(Bu)(t)(\eta) = \widehat{\mu}(t,\eta)$, $0 \leq \eta \leq \pi$, $u \in U$, $g : J \times \mathcal{B}_h \to L^2([0,\pi])$ and $F : J \times \mathcal{B}_h \to L(L^2([0,\pi]), L^2([0,\pi]))$ by

$$g(t,\varphi)(\eta) = \int_{-\infty}^{0} b(\theta)\varphi(\theta)(\eta) d\eta,$$

$$f(t,\varphi) = \mu\Big(t, \int_{-\infty}^{t} \mu_1(\theta)\varphi(\theta) d\theta\Big).$$

On the other hand, the linear system corresponding to (5.1)-(5.3) is controllable. Thus, with the above choices, (5.1)-(5.3) can be written in the abstract form of (1.1)-(1.2) and all the conditions of Theorem 3.5 are satisfied. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 3.5, we can conclude that (5.1)-(5.3) is controllable on $[0,b]$.

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References

[1] N. Abada, M. Benchohra, H. Hammouche, *Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions*, J. Diff. Equ., **246**(2009), 3834-3863.

[2] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109**(2010), 973-1033.

[3] R.P. Agarwal, M. Belmekki, M. Benchohra, *A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative*, Adv. Diff. Equ., **2009** (2009), 1-47, Article ID 981728.

[4] K. Balachandran, E.R. Ananthi, *Controllability of neutral functional integrodifferential infinite delay systems in Banach spaces*, Taiwanese J. Math., **8**(2004), 689-702.

[5] K. Balachandran, A. Leelamani, J.H. Kim, *Controllability of neutral functional evolution integrodifferential systems with infinite delay*, IMA, J. Math. Cont. Inform., **25**(2008), 157-171.

[6] K. Balachandran, J.Y. Park, *Controllability of fractional integrodifferential systems in Banach spaces*, Nonlinear Anal. Hybrid Syst., **3**(2009), 363-367.

[7] D. Baleanu, Z.B. Gunvenc, J.A.T. Machdo, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, 2010.

[8] M. Benchohra, S.K. Ntouyas, *Controllability for functional differential and integrodifferential inclusions in Banach spaces*, J. Optim. Th. Appl., **113**(2002), 449-472.

[9] M. Benchohra, E.P. Gatsori, S.K. Ntouyas, *Controllability results for semilinear evolution inclusions with nonlocal conditions*, J. Optim. Th. Appl., **118**(2003), 493-513.

[10] M. Benchohra, A. Ouahab, *Controllability results for functional semilinear differential inclusions in Frechet spaces*, Nonlinear Anal., **61**(2005), 405-423.

[11] H.F. Bohnenblust, S. Karlin, *On a Theorem of Ville*, In: Contributions to the Theory of Games, vol. I, Princeton University Press, Princeton, NJ, 1950, 155-160.

[12] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., **162**(1991), 494-505.

[13] L. Byszewski, H. Akca, *On a mild solution of a semilinear functional-differential evolution nonlocal problem*, J. Appl. Math. Stoc. Anal., **10**(1997), no. 3, 265-271.

[14] L. Byszewskia, V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract cauchy problem in a Banach space*, Appl. Anal., **40**(1991), no. 1, 11-19.

[15] J. Cannon, *The one-dimensional heat equation*, In: Encyclopedia of Mathematics and its Applications, vol. 23, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[16] Y.K. Chang, W.T. Li, J.J. Nieto, *Controllability of evolution differential inclusions in Banach spaces*, Nonlinear Anal., **67**(2007), 623-632.

[17] Y.K. Chang, D.N. Chalishajar, *Controllability of mixed Volterra-Fredholm type integro-differential inclusions in Banach spaces*, J. Frank. Inst., **345**(2008), 499-507.

[18] Y.K. Chang, A. Anguraj, M. Mallika Arjunan, *Controllability of impulsive neutral functional differential inclusions with infinite delay in Banach spaces*, Chaos Solitons Fractals, **39**(2009), no. 4, 1864-1876.

[19] Y.K. Chang, *Controllability of impulsive functional differential systems with infinite delay in Banach spaces*, Chaos Solitons Fractals, **33**(2007), 1601-1609.

[20] Y.K. Chang, J.J. Nieto, *Existence of solutions for impulsive neutral integro-differential inclusions with nonlocal initial conditions via fractional operators*, Numer. Funct. Anal. Optim., **30**(2009), no. 3-4, 227-244.

[21] A. Debbouche, D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems*, Comput. Math. Appl., **62**(2011), no. 3, 1442-1450.

[22] K. Deimling, *Multivalued Differential Equations*, De Gruyter, Berlin, 1992.

[23] X. Fu, K. Ezzinbi, *Existence of solutions for neutral functional differential evolution equations with nonlocal conditions*, Nonlinear Anal., **54**(2003), 215-227.

[24] E.P. Gatsori, *Controllability results for nondensely defined evolution differential inclusions with nonlocal conditions*, J. Math. Anal. Appl., **297**(2004), 194-211.

[25] T. Guendouzi, L. Bousmaha, *Approximate controllability of fractional neutral stochastic functional integro-differential inclusions with infinite delay*, Qual. Th. Dyn. Syst., **13**(2014), 89-119.

[26] M. Guo, X. Xue, R. Li, *Controllability of impulsive evolution inclusions with nonlocal conditions*, J. Optim. Theory Appl. **120**(2004), no. 2, 355-374.

[27] E. Hernandez, D. O'Regan, *Controllability of Volterra-Fredholm type systems in Banach spaces*, J. Frank. Inst., **346**(2009), 95-101.

[28] S. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis (Theory)*, Kluwer Academic Publishers, Dordrecht Boston, London, 1997.

[29] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, In: North-Holland Mathematics Studies, vol. 204. Elsevier Science, Amsterdam, 2006.

[30] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[31] A. Lasota, Z. Opial, *An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations or noncompact acyclic-valued map*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys., **13**(1965), 781-786.

[32] G. Li, X. Xue, *Controllability of evolution inclusions with nonlocal conditions*, Appl. Math. Comput., **141**(2003), 375-384.

[33] H. Liu, J.C. Chang, *Existence for a class of partial differential equations with nonlocal conditions*, Nonlinear Anal., **70**(2009), 3076-3083.

[34] J.A. Machado, C. Ravichandran, M. Rivero, J.J Trujillo, *Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions*, Fixed Point Theory Appl., **66**(2013), 1-16.

[35] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, Wiley, New York, 1993.

[36] G.M. N'Guerekata, *A Cauchy problem for some fractional abstract differential equation with non local conditions*, Nonlinear Anal., **70**(2009), 1873-1876.

[37] J.Y. Park, K. Balachandran, G. Arthi, *Controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces*, Nonlinear Anal. Hybrid Syst., **3**(2009), 184-194.

[38] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, New York, Springer Verlag, 1983.

[39] I. Podlubny, *Fractional Differential eEquations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego, 1999.

[40] C. Ravichandran, J.J. Trujillo, *Controllability of impulsive fractional functional integro-differential equations in banach spaces*, J. Funct. Space. Appl., **2013**(2013), 1-8. Article ID-812501.

[41] C. Ravichandran, D. Baleanu, *Existence results for fractional neutral functional integro-differential evolution equations with infinite delay in Banach spaces*, Adv. Diff. Equ., **2013**(2013), no. 1, 1-12.

[42] R. Sakthivel, N.I. Mahmudov, J.J. Nieto, *Controllability for a class of fractional-order neutral evolution control systems*, Appl. Math. Comput., **218**(2012), 10334-10340.

[43] R. Sakthivel, R. Ganesh, S.M. Anthoni, *Approximate controllability of fractional nonlinear differential inclusions*, Appl. Math. Comput., **225**(2013), 708-717.

[44] V. Vijayakumar, C. Ravichandran, R. Murugesu, *Existence of mild solutions for nonlocal cauchy problem for fractional neutral evolution equations with infinite delay*, Surv. Math. Appl., **9**(2014), 117-129.

[45] V. Vijayakumar, C. Ravichandran, R. Murugesu, *Nonlocal controllability of mixed Volterra-Fredholm type fractional semilinear integro-differential inclusions in Banach spaces*, Dyn. Contin. Discrete Impuls. Syst., **20**(2013), no. 4, 485-502.

[46] V. Vijayakumar, C. Ravichandran, R. Murugesu, *Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay*, Nonlinear Stud., **20**(2013), no. 4, 511-530.

[47] V. Vijayakumar, C. Ravichandran, R. Murugesu, J.J. Trujillo, *Controllability results for a class of fractional semilinear integro-differential inclusions via resolvent operators*, Appl. Math. Comput., **247**(2014), 152-161.

[48] V. Vijayakumar, A. Selvakumar, R. Murugesu, *Controllability for a class of fractional neutral integro-differential equations with unbounded delay*, Appl. Math. Comput., **232**(2014), 303-312.

[49] J. Wang, Y. Zhou, *Existence and controllability results for fractional semilinear differential inclusions*, Nonlinear Anal. Real World Appl., **12**(2011), 3642-3653.

[50] Y. Zhou, V. Vijayakumar, R. Murugesu, *Controllability for fractional evolution inclusions without compactness*, Evolution Eq. Control Th., **4**(2015), 507-524.

[51] Y. Zhou, L. Peng, *On the time-fractional Navier-Stokes equations*, Computers Math. Appl. 73(2017), 874-891.

[52] Y. Zhou, L. Peng, *Weak solution of the time-fractional Navier-Stokes equations and optimal control*, Comput. Math. Appl. 73(2017),1016-1027.

[53] Y. Zhou, L. Zhang, *Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems*, Comput. Math. Appl. 73(6)(2017), 1325-1345.

[54] J. Wang, A.G. Ibrahim, *Existence and controllability results for nonlocal fractional impulsive differential inclusions in Banach spaces*, J. Funct. Space. Appl., **2013**(2013), 1-16. Article ID 518306.

[55] X. Xue, *Nonlinear differential equations with nonlocal conditions in Banach spaces*, Nonlinear Anal., **63**(2005), 575-586.

[56] B. Yan, *Boundary value problems on the half-line with impulses and infinite delay*, J. Math. Anal. Appl., **259**(2001), no. 1, 94-114.

[57] Z. Yan, *Controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay*, J. Frank. Inst., **348**(2011), 2156-2173.

[58] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

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