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FIXED POINTS, NORMAL STRUCTURE AND SLICES OF BANACH SPACES

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Abstract. In this paper, we first study the fixed point property for nonexpansive mappings of a Banach space and some existing result in [7] is extended. We secondly study the relationship between uniform normal structure and slices and some results in [11] are improved too.
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1. Introduction

Let X be a Banach space with the dual space X^*. Let $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \leq 1\}$ be the unit sphere and the closed unit ball of X, respectively.

Definition 1.1. ([2]) A nonempty bounded and convex subset K of a Banach space X is said to have *normal structure* if for every convex subset H of K that contains more than one point there is a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \operatorname{diam} H,$$

where $\operatorname{diam} H = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have *normal structure* if every bounded convex subset of X has normal structure. A Banach space X is said to have *weak normal structure* if for each weakly compact convex set K of X that contains more than one point has normal structure. We also say that X have *uniform normal structure* if there exists $0 < c < 1$ such that for any subset K as above, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \cdot \operatorname{diam} K.$$

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Remark 1.2. For a reflexive Banach space, normal structure and weak normal structure coincide. Moreover, if a space have uniform normal structure, then it is reflexive.

Definition 1.3. A Banach space X is said to have the *fixed point property* if every *nonexpansive* mapping $T : C \to C$, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, always has a fixed point whenever C is a bounded closed convex subset C of X. If "closed convex" above is replaced by "weakly compact", then we say that X has the *weak fixed point property*.

In [8], Kirk proved that if a Banach space has weak normal structure, then it has the weak fixed point property. Since then many geometric properties guaranteeing the existence of a fixed point of nonexpansive mappings have been widely investigated.

The purpose of this paper is two-fold: (1) to extend Jimenez-Melado's result and (2) to improve a sufficient condition for uniform normal structure in terms of the slices of X.

2. Extension of Jimenez-Melado's result

Let us recall the concept of ultraproduct of a Banach space X which has been viewed as a standard tool in this area (for more detail, we refer to [3], [12]).

Let $\mathbb{N}$ be the set of natural numbers, and $\mathcal{U}$ be a ultrafilter on $\mathbb{N}$. We say that a sequence $\{x_n\}$ in X *converges to* $x \in X$ *with respect to* $\mathcal{U}$ if $\{n \in \mathbb{N} : \|x_n - x\| < \varepsilon\} \in \mathcal{U}$ for every $\varepsilon > 0$. In this case, we write $x = \lim_{\mathcal{U}} x_n$. Let $\ell_\infty(X)$ denotes the space of bounded sequences $\{x_n\}$ in X equipped with the norm $\|\{x_n\}\| := \sup\{\|x_n\| : n \in \mathbb{N}\}$. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ let $N_{\mathcal{U}} = \{\{x_n\} \in \ell_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$. The *ultraproduct* of X is the quotient space $X_{\mathcal{U}} := l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm $|\cdot|$. We write $[z_n]$ for the equivalence class of $\{z_n\} \in \ell_\infty(X)$ and it is not difficult to see that $|[z_n]| = \lim_{\mathcal{U}} \|z_n\|$. We can also consider $x \in X$ as an element in $X_{\mathcal{U}}$ by identifying x with the equivalence class of a constant sequence $\{x, x, \dots\}$. Moreover, for a subset C of X and a nonexpansive mapping $T : C \to C$, we also write $[C] := \{[x_n] : x_n \in C \text{ for all } n\}$ and let $[T]$ be the mapping on $[C]$ defined by $[T]([x_n]) := [Tx_n]$. In this case, $|[T]([x_n]) - [T]([y_n])| = \lim_{\mathcal{U}} \|Tx_n - Ty_n\| \leq \lim_{\mathcal{U}} \|x_n - y_n\| = |[x_n] - [y_n]|$ for all $[x_n], [y_n] \in [C]$, that is, $[T]$ is also nonexpansive.

The following two lemmas are known as Goebel–Karlovitz's lemma (see [6]) and Lin's lemma [9], respectively.

Lemma 2.1. (Goebel–Karlovitz) *Let X be a Banach space and let K be a minimal weakly compact convex subset of X whcih is invariant under a nonexpansive mapping T. If $\{x_n\}$ is a sequence in K such that* $\lim_n \|x_n - Tx_n\| = 0$, *then*

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam} K \quad \forall x \in K.$$

Lemma 2.2. (Lin) *Let X be a Banach space and let K be a minimal weakly compact convex subset of X which is invariant under a nonexpansive mapping T. If W is a nonempty closed convex subset of $[K]$ which is invariant under $[T]$, then*

$$\sup\{|[w_n] - x| : [w_n] \in W\} = \operatorname{diam} K \quad \forall x \in K.$$

In [13], Sims introduced the following parameter to Banach spaces:

$$w(X) = \sup\{\lambda > 0 : \lambda \liminf_{n\to\infty} \|x_n + x\| \leq \liminf_{n\to\infty} \|x_n - x\|\}$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in X and for all elements $x \in X$. Clearly, $\frac{1}{3} \leq w(X) \leq 1$ for all Banach spaces X. Moreover, a Banach space X has WORTH property if and only if $w(X) = 1$. We prefer to use the reciprocal $\mu(X)$ of $w(X)$, namely $\mu(X) = \frac{1}{w(X)}$.

We use $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively. By modification a proof given in [7], we obtain the following result:

Theorem 2.3. *A Banach space X with $B(X^*)$ is weak* sequentially compact has the weak fixed point property whenever*

$$\left(1 - \frac{1}{2\mu(X)}\right)\left(1 - \delta\left(\frac{1}{1 - \frac{1}{2\mu(X)}}\right)\right) + \frac{\mu(X)}{2} < 1.$$

Proof. Suppose that X fails the weak fixed point property. Then by a classical argument there exist a nonempty convex and weakly compact subset K of X and a nonexpansive mapping $T : K \to K$ such that T is fixed-point free and K is minimal for T. By dilation, we may assume that $\operatorname{diam}(K) = 1$. Let $\{x_n\}$ be an approximate fixed point sequence for T, that is, $x_n - Tx_n \to 0$. We may also assume that $x_n \rightharpoonup 0 \in K$ by translation. Applying the Goebel–Karlovitz's lemma [6], we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - y\| = 1 \quad \forall y \in K.$$

Then

$$\lim_{n\to\infty} \|x_n + x_{n+1}\| \leq \mu := \mu(X) \text{ and } \lim_{n\to\infty} \|x_n + y\| \leq \mu \quad \forall y \in K.$$

In particular, we have $\lim_{n\to\infty} \|x_n\| = 1$.

Claim 2.4.

$$\lim_{n\to\infty} \left\| \frac{1}{2\mu} x_n - \left(1 - \frac{1}{2\mu}\right) x_{n+1} \right\| = \lim_{n\to\infty} \left\| \frac{1}{2\mu} x_{n+1} - \left(1 - \frac{1}{2\mu}\right) x_n \right\| = 1 - \frac{1}{2\mu}.$$

We first observe that

$$\begin{aligned}
&\limsup_{n\to\infty} \left\| \frac{1}{2\mu} x_n - \left(1 - \frac{1}{2\mu}\right) x_{n+1} \right\| \\
&= \limsup_{n\to\infty} \left\| \frac{1}{2\mu} (x_n - x_{n+1}) - \left(1 - \frac{1}{\mu}\right) x_{n+1} \right\| \\
&\leq \frac{1}{2\mu} \limsup_{n\to\infty} \|x_n - x_{n+1}\| + \left(1 - \frac{1}{\mu}\right) \limsup_{n\to\infty} \|x_{n+1}\| = 1 - \frac{1}{2\mu}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}&\liminf_{n\to\infty}\left\|\frac{1}{2\mu}x_n-\left(1-\frac{1}{2\mu}\right)x_{n+1}\right\|\\&\geq\liminf_{n\to\infty}\left\langle\frac{1}{2\mu}x_n-\left(1-\frac{1}{2\mu}\right)x_{n+1},-f_{n+1}\right\rangle=1-\frac{1}{2\mu}.\end{aligned}$$

The second assertion follows similarly, so the proof is omitted. Hence the claim is proved.

Define

$$\begin{aligned}W:=\{[z_n]\in[K]:|[z_n]-[x_n]|\leq 1-\frac{1}{2\mu},\\ |[z_n]-[x_{n+1}]|\leq 1-\frac{1}{2\mu},|[z_n]-x|\leq\frac{1}{2\mu}\text{ for some }x\in K\}.\end{aligned}$$

Note that

$$\left[\frac{x_n+x_{n+1}}{2\mu}\right]\in W\neq\varnothing.$$

Moreover, W is closed, convex and $[T]$-invariant. It follows from Lin's lemma [1] and [9] that

$$\sup\{|[z_n]|:[z_n]\in W\}=1.$$

We are going to find a contradiction. Let $[z_n]\in W$. This implies that there exists an $x\in K$ such that

$$|[z_n]-[x_n]|\leq 1-\frac{1}{2\mu},\quad |[z_n]-[x_{n+1}]|\leq 1-\frac{1}{2\mu},\quad |[z_n]-x|\leq\frac{1}{2\mu}.$$

Put

$$\widetilde{x_1}=\frac{[z_n]-[x_n]}{1-\frac{1}{2\mu}},\quad \widetilde{x_2}=\frac{[z_n]-[x_{n+1}]}{1-\frac{1}{2\mu}},\quad \widetilde{x_3}=\frac{[z_n]-x}{1-\frac{1}{2\mu}}.$$

Then $|\widetilde{x_i}|\leq 1$ for all $i=1,2,3$. Notice that

$$|\widetilde{x_1}-\widetilde{x_2}|=|\widetilde{x_2}-\widetilde{x_3}|=|\widetilde{x_1}-\widetilde{x_3}|=\frac{1}{1-\frac{1}{2\mu}}.$$

Next, we consider the following estimates:

$$\begin{aligned}&|[z_n]-\frac{1}{3}([x_n]+[x_{n+1}]+x)|\\&=\frac{1-\frac{1}{2\mu}}{3}|\widetilde{x_1}+\widetilde{x_2}+\widetilde{x_3}|\\&\leq\left(1-\frac{1}{2\mu}\right)\left(1-\delta\left(\frac{1}{1-\frac{1}{2\mu}}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{3}|[x_n]+[x_{n+1}]+x| &= \frac{1}{6}|([x_n]+[x_{n+1}])+([x_n]+x)+([x_{n+1}]+x)| \\ &\le \frac{1}{6}(|[x_n]+[x_{n+1}]|+|[x_n]+x|+|[x_{n+1}]+x|) \\ &\le \frac{\mu}{6}(|[x_n]-[x_{n+1}]|+|[x_n]-x|+|[x_{n+1}]-x|) = \frac{\mu}{2}.\end{aligned}$$

Then

$$\begin{aligned}|[z_n]| &\le |[z_n]-\frac{1}{3}([x_n]+[x_{n+1}]+x)|+\frac{1}{3}|[x_n]+[x_{n+1}]+x| \\ &\le \left(1-\frac{1}{2\mu}\right)\left(1-\delta\left(\frac{1}{1-\frac{1}{2\mu}}\right)\right)+\frac{\mu}{2}.\end{aligned}$$

It follows from Lin's lemma [1] and [9] that $|[z_n]|$ can be arbitrarily closed to one and this leads to a contradiction if

$$\left(1-\frac{1}{2\mu}\right)\left(1-\delta\left(\frac{1}{1-\frac{1}{2\mu}}\right)\right)+\frac{\mu}{2} < 1.$$

Remark 2.5. Very recently H. Fetter and B. Gamboa de Buen [5] proved that every reflexive Banach space with WORTH property, that is, $\mu(X)=1$ enjoy the fixed point property. Hence Jimenez-Melado's result [7] holds without the presence of $\varepsilon_0(X)<2$. Our result gives some more information when dealing with spaces in the absence of the WORTH property.

3. Slices and Normal structure

Definition 3.1. ([4]) Let D be a bounded subset of a Banach space X and suppose that $0\neq f\in X^*$. Let

$$M(D,f) := \sup\{\langle x,f\rangle : x\in D\}.$$

If $\alpha>0$, then the set

$$S(D,f,\alpha) := \{x\in D : \langle x,f\rangle > M(D,f)-\alpha\}$$

is called the *slice of D determined by f and α*, or more briefly, a slice of D.

Recently, we introduced the following concepts in a Banach space X:

Definition 3.2. ([11]) Let X be a Banach space. Let $f\in S(X^*)$ and $\varepsilon>0$. Define

$$sl(X,\varepsilon) := \sup\{\operatorname{diam}(S(B(X),f,\varepsilon)) : f\in S(X^*)\},$$

and

$$sl_0(X) := \lim_{\varepsilon\to 0^+} sl(X,\varepsilon).$$

We also proved the following results:

Theorem 3.3. ([11]) *Let X be a Banach space.*

- *If $sl(X,0)<2$, then X is uniformly nonsquare, therefore X is super-reflexive and then reflexive.*
- *If $sl(X,0)<1$, then X has uniform normal structure.*
- *If $sl(X,\frac{2}{3})<2$, then X has uniform normal structure.*

We are going to include Theorem 3.3 into the following more general result.

Theorem 3.4. *If a Banach space X satisfies the following condition:*

$$sl\left(X, \frac{2t}{1+t}\right) < \frac{1}{1-t} \quad \textit{for some } t \in \left(0, \frac{1}{2}\right],$$

then X has uniform normal structure.

We need the following lemma from [10].

Lemma 3.5. *Let X be a super-reflexive Banach space. If X does not have normal structure, then for each ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there exist $y_1, y_2 \in S(X_{\mathcal{U}})$ and $g_1, g_2 \in S((X_{\mathcal{U}}))^*)$ such that the following conditions hold true:*

- $|y_1 - y_2| = 1$;
- $\langle y_i, g_i \rangle = 1$ *for all* $i = 1, 2$;
- $\langle y_j, g_i \rangle = 0$ *for all* $i \neq j$.

Proof of Theorem 3.4. Assume that $sl(X, \frac{2t}{1+t}) < \frac{1}{1-t}$ for some $t \in \left(0, \frac{1}{2}\right]$. It follows then that X is super-reflexive. Suppose that X does not have normal structure. Let $\mathcal{U}$ be a given ultrafilter in $\mathbb{N}$. Then there exist $y_1, y_2 \in S(X_{\mathcal{U}})$ and $g_1, g_2 \in S((X_{\mathcal{U}})^*)$ such that all the conditions in Lemma 3.5 are satisfied. Let $Y = \operatorname{span}\{-ty_1 + y_2\}$. Then Y is a closed subspace of $X_{\mathcal{U}}$. Moreover, it is not hard to see that $y_1 \notin Y$. By Hahn–Banach Extension Theorem, there exists a bounded linear functional f on $X_{\mathcal{U}}$ such that

- $|f| = 1$;
- $\langle y, f \rangle = 0$ for all $y \in Y$;
- $\langle y_1, f \rangle = \inf\{|y_1 - y| : y \in Y\}$.

For any scalar $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} |y_1 - \alpha(-ty_1 + y_2)| &= |(1 + \alpha t)y_1 - \alpha y_2| \\ &\geq \max\{|\langle (1 + \alpha t)y_1 - \alpha y_2, g_1\rangle|, |\langle (1 + \alpha t)y_1 - \alpha y_2, g_2\rangle|\} \\ &= \max\{|1 + \alpha t|, |\alpha|\}. \end{aligned}$$

It follows from some calculation that

$$\langle y_1, f \rangle \geq \inf\{\max\{|1 + \alpha t|, |\alpha|\} : \alpha \in \mathbb{R}\} = \frac{1}{1+t}.$$

Put

$$x_1 = y_1 - y_2 \quad \text{and} \quad x_2 = \frac{1-2t}{1-t} y_1 + \frac{t}{1-t} y_2.$$

It is clear that x_1 and x_2 belong to the unit ball of $X_{\mathcal{U}}$. Moreover, it follows that

$$x_1 - x_2 = \frac{t}{1-t} y_1 - \frac{1}{1-t} y_2 \in Y.$$

Hence $\langle x_1, f \rangle = \langle x_2, f \rangle$. Since

$$tx_1 + (1-t)x_2 = (1-t)y_1,$$

we can conclude that

$$\langle x_1, f \rangle = \langle x_2, f \rangle = (1-t)\langle y_1, f \rangle \geq \frac{1-t}{1+t} = 1 - \frac{2t}{1-t}.$$

Moreover, we also have

$$|x_1 - x_2| = \frac{1}{1-t}|ty_1 - y_2| \geq \frac{1}{1-t}\langle ty_1 - y_2, -g_2\rangle = \frac{1}{1-t}.$$

Consequently,

$$sl\left(X, \frac{2t}{1+t}\right) = sl\left(X_{\mathcal{U}}, \frac{2t}{1+t}\right) \geq \operatorname{diam} sl\left(B_{X_{\mathcal{U}}}, f, \frac{2t}{1+t}\right) \geq \frac{1}{1-t}.$$

This is a contradiction. To conclude the uniform normal structure of the space, we just invoke again the fact that our condition is closed under the taking of ultrapower.

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