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COMMON FIXED POINT THEOREMS FOR GENERALIZED NON-EXPANSIVE SEMI-TOPOLOGICAL SEMIGROUPS IN LOCALLY CONVEX SPACES

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Abstract. In this paper, we shall be concerned with a special kind of equicontinuous semi-topological semigroups of self-mappings on a weakly compact convex subset of a separated locally convex space, namely, the generalized non-expansive mappings and we shall introduce some common fixed point results for this kind of semigroups. Also, we study a characterization of the existence of a left invariant mean on almost and weakly almost periodic functions on separable semi-topological semigroups. Our results extend the results due to Lau and Zhang [17] and Lau [13].

Key Words and Phrases: Fixed point property, non-expansive mapping, generalized non-expansive mapping, weakly compact convex set, weakly almost periodic, reversible semigroup, invariant mean.
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1. Introduction

A mapping T on a non-empty bounded closed convex subset C of a Banach space E is called a non-expansive mapping if

$$||Tx - Ty|| \leq ||x - y||, \tag{1.1}$$

and is called generalized non-expansive (see Suzuki [22]) if satisfy the following condition

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \quad \text{implies} \quad ||Tx - Ty|| \leq ||x - y||, \tag{1.2}$$

for all $x, y \in C$. We say that C has the fixed point property if every non-expansive mapping from C into C has a fixed point in C. The work of many researchers are concerned with fixed point property. For example, in 1965, Browder [4] proved that if E is uniformly convex Banach space, then every nonexpansive mapping on the bounded closed convex subset C of E into C has a fixed point in C. He also proved that a commuting family on nonexpansive mappings on C has a common fixed point in C. Since every uniformly convex Banach space has normal structure [20, Theorem 3.3.4, p. 148], then Kirk [12] extended the result due to Browder [4] by showing that if C is a weakly compact subset of E with normal structure, then C has the fixed point property. For more examples about fixed point property (see [2, 3, 8, 9, 10, 15, 21]).

Let Q be a (fixed) family of continuous semi-norms on a separated locally convex space E which determines the topology of E. We denote the space by (E, Q) or simply by E if there is no confusion. Then an action of S on a subset $K \subseteq E$ is Q-non-expansive if it satisfies the following condition:

$$\rho(s \cdot x - s \cdot y) \leq \rho(x - y) \; \forall \, s \in S, \;\; x, y \in K \text{ and } \rho \in Q. \tag{1.3}$$

In 1972, Holmes and Lau [11, Corollary 1] proved that if a semitopological semigroup S is left reversible (i.e., any two nonempty closed right ideals of S have nonvoid intersection; see [5, p. 34]), then S has the following property:
(D) For any separately continuous and Q-nonexpansive action of S on a compact subset C of a separated locally convex space, C has a common fixed point for S.
Also, this result was proved by Mitchell [19] for discrete left reversible semigroups, by De Marr in [7, p. 1139] for commuting semigroups and by W. Takahashi [23, p. 384] for discrete left amenable semigroups (i.e., the space of bounded real valued functions on the semigroup has a left invariant mean; see Day [6]).
In 1973, Lau [13] proved that $AP(S)$ (the space of continuous almost periodic functions on S) has LIM (a left invariant mean) if and only if the property (D) and the following property are hold.
(E) whenever S is a separately continuous and $Q-$nonexpansive action on a compact convex subset C of a separated locally convex space E, C has a common fixed point for S.

In 2008, Lau and Zhang [17] proved the following theorem which answered about the open question posed by Lau [14, 16].

Theorem 1.1. [17, Theorem 3.4] *Let S be a separable semitopological semigroup. Then $WAP(S)$ (the space of continuous weakly almost periodic functions on S) has a LIM if and only if*
(F) Whenever S acts on a weakly compact convex subset C of a separated locally convex space (E, Q) and the action is weakly separately continuous, weakly quasi-equicontinuous and $Q-$nonexpansive, then C contains a common fixed point for S.
Also, Lau and Zhang [17] *proved that if S is a semitopological semigroup, then $AP(S)$ has LIM if and only if S has the following fixed point property:*
(É) Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) as Q-nonexpansive mappings, if K has Q -normal structure and the action S is separately continuous and equicontinuous when K is equipped with the weak topology of (E, Q), then K contains a common fixed point for S.

In this paper, we shall define the concept of $Q-$generalized non-expansive of an action S and we prove that WAP(S) has a LIM if and only if every $Q-$generalized non-expansive action S which acts on a weakly compact convex subset C of a separated locally convex space (E, Q) has a common fixed point in C. Where the action is weakly separately continuous and weakly quasi-equicontinuous.

2. Preliminaries

Let S be a semitopological semigroup, i.e. S is a semigroup with Hausdorff topology such that for each $a \in S$, the mappings $s \mapsto sa$ and $s \mapsto as$ from S into S are continuous. S is called left reversible if any two closed right ideals of S have non-void intersection, i.e. $\overline{aS} \cap \overline{bS} \neq \phi$, for any $a, b \in S$.

Definition 2.1. [17] A semitopological semigroup S is said to be strongly left reversible if the family of countable subsemigroups $\{S_\alpha : \alpha \in I\}$ such that:
(1) $S = \bigcup_{\alpha \in I} S_\alpha$,
(2) $\overline{aS_\alpha} \cap \overline{bS_\alpha} \neq \emptyset$ for each $\alpha \in I$ and $a, b, \in S_\alpha$,
(3) for each pair $\alpha_1, \alpha_2 \in I$, there is $\alpha_3 \in I$ such that $S_{\alpha_1} \cup S_{\alpha_2} \subset S_{\alpha_3}$.

Definition 2.2. [17] Let S be a semitopological semigroup and let $l^\infty(S)$ be the commutative Banach algebra of all bounded complex-valued functions on S with supremum norm and pointwise multiplication. For each $s \in S$ and $f \in l^\infty(S)$ let $l_a f$ and $r_a f$ are the left and right translates of f by a respectivelly, which are defined as: $l_a f(s) = f(as)$ and $r_a f(s) = f(sa)$. Let X ba a closed subalgebra of $l^\infty(S)$ containing 1_S. An element μ in X^* is said to be mean on X if $\|\mu\| = \mu(1_S) = 1$. As is well known μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s).$$

The mean μ is said to be left (resp. right) invariant, denoted by LIM (resp. RIM), if $\mu(l_a f) = \mu(f)$ (reps. $\mu(r_a f) = \mu(f)$), for all $a \in S$ and $f \in X$. Let $C(S)$ denote the closed subalgebra of $l^\infty(S)$ consisting of all bounded continuous complex-valued functions on S. Denote by $AP(S)$ the space of all $f \in C(S)$ such that: $LO(f) = \{l_s f : s \in S\}$ is relatively compact in the norm topology of $C(S)$, and denote by $WAP(S)$ the space of all $f \in C(S)$ such that $LO(f)$ is relatively compact in the weak topology of $C(S)$. Functions in $AP(S)$ (resp. $WAP(S)$) are called almost periodic (resp. weakly almost periodic) functions on S.

Definition 2.3. [17] An action S on a topological space K is a mapping ψ from $S \times K$ into K, denoted by $\psi(s, x) = sx$ ($s \in S$ and $x \in K$), then we say that the action is jointly continuous at $(s_0, x_0) \in S \times K$ if for neighbourhood W of $\psi(s_0, x_0)$ there exists a product of open $U \times V \subseteq S \times K$ containing (s_0, x_0) such that $\psi(U \times V) \subseteq W$, and we say that the action is separately continuous if for each $s_0 \in S$ and $x_0 \in K$ the functions $x \to \psi(s_0, x)$ and $s \to \psi(s, x_0)$ are both continuous on K and S respectively. Thus it is clear that, joint continuity is a stronger condition then separate continuity. The action of a semitopological semigroup S on a Hausdorff space X is said to be quasi-equicontinuous is $\overline{S}^{\,p}$, the closure of S in the product space X^X (the space of all mapping from X into X), consists of only continuous mappings.

Definition 2.4. [17] The action S on a convex subset K of a linear topological space is said to be affine if for all $s \in S$, $x, y \in K$ and $\lambda \in [0, 1]$ then $s(\lambda x + (1 - \lambda)y) = \lambda sx + (1 - \lambda)sy$.

Definition 2.5. [17] Let E be separated locally convex linear topological space with the topology determined by the family of continuous semi-norms Q. For any $\rho \in Q$

and $A \subseteq E$, $\delta_\rho(A)$ will denote the ρ - diameter of A, which

$$\delta_\rho(A) = \sup\{\rho(x-y) : x, y \in A\}.$$

A closed convex subset C of E has normal structure if for each bounded closed subset D of C which contains more than one point, and $\rho \in Q$ there is a point $x \in D$ satisfy the following condition

$$r_\rho(D,x) < \delta_\rho(D)$$

where

$$r_\rho(D,x) = \sup\{\rho(x-y) : y \in D\}.$$

Lemma 2.1. [17, Lemma 3.1] *Let S be a semitopological semigroup that acts on a Hausdorff space X and the action is quasi-equicontinuous.*
(1) If S_0 is a subsemigroup of S, then the action of S_0 on X is also quasi-equicontinuous;
(2) If in addition, X is compact, then for each compact S-invariant subspace X_0 of X, the action of S on X_0 is quasi-equicontinuous.
Lemma 2.2. [17, Lemma 3.2] *Suppose that the action of S on a compact Hausdorff space X is separately continuous and quasi-equicontinuous. Then for each $x \in X$ and each $f \in C(X)$, we have $f_x \in WAP(S)$, where f_x is defined by*

$$f_x(s) = f(sx) \quad (s \in S). \tag{2.1}$$

Lemma 2.3. [17, Lemma 3.3] *Let S be a separable semitopological semigroup as Q-nonexpansive mappings weakly separately continuous that acts on a weakly compact convex subset K of a locally convex space (E,Q). Suppose that F is a minimal non-empty weakly compact S-invariant subset of K satisfying $sF = F$ $(s \in S)$. Then F is Q-compact.*
Lemma 2.4. [17, Lemma 5.3] *Suppose that S is a semitopological semigroup that acts on a compact Hausdorff space X and the action $S \times X \longrightarrow X$ is jointly continuous. If S contains a dense subset D such that $\overline{aS} \cap \overline{bS} \neq \emptyset$ for $a, b \in D$, then any minimal S-invariant non-empty compact subset K of X satisfies:*
(1) $\bar{S}x = K$ for all $x \in K$
(2) $sK = K$ for all $s \in S$.
Lemma 2.5. [11, Lemma 2] *If M is a non-empty compact subset of separated locally convex (E,Q), and $\rho \in Q$ such that $\delta_\rho > 0$ then there exists an element $u \in \overline{co}(M)$ (depending on ρ) such that*

$$\sup\{\rho(u-y) : y \in M\} < \delta_\rho(M),$$

where $\overline{co}(M)$ is the closed convex hull of M.
Lemma 2.6. [13, Lemma 3.1] *If the action S on a compact Hausdorff space Y is separately continuous and equicontinuous and $y \in Y$, then $T_y(C(Y)) \subseteq AP(S)$, where $T_y f(s) = f(s \cdot y)$ for all $s \in S$ and $f \in C(Y)$.*
Lemma 2.7. [17, Lemma 5.2] *A metrizable left reversible semitopological semigroup is strongly left reversible.*

3. Main results

In this section we introduce some results related to equicontinuous generalized nonexpansive semitopological semigroup on a weakly compact convex subset of a separated locally convex space (E, Q).

Definition 3.1. Let S be a semitopological semigroup and action on a subset $K \subseteq E$. Then S is Q-quasi-non-expansive if it satisfies the following condition:

$$\rho(s \cdot x - y) \leq \rho(x - y) \; \forall \; s \in S, \;\; x \in K, \; y \in F(s) \text{ and } \rho \in Q. \tag{3.1}$$

Where $F(s)$ denote by the fixed point set of s.

Definition 3.2. Let S be a semitopological semigroup and action on a subset $K \subseteq E$. Then S is Q-generalized non-expansive if it satisfies the following condition:

$$\frac{1}{2}\rho(x - s \cdot x) \leq \rho(x - y)$$

implies that

$$\rho(s \cdot x - s \cdot y) \leq \rho(x - y) \; \forall \; s \in S, \;\; x, y \in K \text{ and } \rho \in Q. \tag{3.2}$$

Proposition 3.1. [22] Q-non-expansive $\Longrightarrow$ Q-generalized non-expansive $\Longrightarrow$ Q-quasi-non-expansive.

Lemma 3.1. *Let S be a Q-generalized non-expansive semitopological semigroup and acts on a weakly compact convex subset K of a separated locally convex space (E, Q). Then for $x, y \in K$, the following hold:*
(i) $\rho(s \cdot x - s^2 \cdot y) \leq \rho(x - s \cdot y)$
(ii) Either $\frac{1}{2}\rho(x - s \cdot x) \leq (x - y)$ or $\frac{1}{2}\rho(s \cdot x - s^2 \cdot x) \leq \rho(s \cdot x - y)$ holds.
(iii) Either $\rho(s \cdot x - s \cdot y) \leq (x - y)$ or $\rho(s^2 \cdot x - s \cdot y) \leq \rho(s \cdot x - y)$ holds.

Proof. The proof similar to the proof of [22, Lemma 5] which follow by replacing the norm by ρ.

Lemma 3.2. *Let S be a separable continuous semitopological semigroup that acts on a weakly compact convex subset K of a locally convex space (E, Q) as weakly separately continuous and Q-generalized non-expansive mappings. Suppose that F is a minimal non-empty weakly compact S-invariant subset of K satisfying $sF = F$ $(s \in S)$. Then F is $Q-$compact.*

Proof. The idea of the proof is the same idea of proof [17, Lemma 3.3] which is based to show that F is $Q-$ totally bounded. By first paragraph of [17, Lemma 3.3] it follows that $\overline{co}^w(F)$ is closed and $Q-$separable.

Given a neighborhood N of 0 in (E, Q), then there are finite seminorms $\{p_1, ..., p_n\} \subset Q$ and $r, \epsilon > 0$ such that $U = \{x \in E : p_i(x) < r; i = 1, ..., n\}$ is a neighborhood of 0 contained in N. Then the same argument as in the proof [17, Lemma 3.3] leads to there is a weakly open neighborhood W of 0 and an element $w \in F$ such that $(w + W) \cap F \subset w + U$. Take another Q - open symmetrical neighborhood W_1 of 0 such that $W_1 + W_1 \subset W$, and finite seminorms $\{\rho_1, ..., \rho_m\} \subset Q$ and $r_0 > 0$ such that $H = \{x \in E : \rho_j(x) < r_0, j = 1, ..., m\} \subset W_1$. Therefore, due to the separability property, there is a sequence $\{y_n\} \subset F$ such that: $F \subset \cup_{n=1}^{\infty}\{y_n + H\}$. By the density of SF in F, there exits elements $a_1, a_2, ..., a_n$ in SF such that

$$\rho(y_n - a_n) < \epsilon, \;\; \forall \;\; \epsilon > 0, \forall \;\; \rho \in Q.$$

Let $z_n = y_n - a_n$ such that $F \subset \cup_{n=1}^{\infty}\{z_n + H\}$. Since F is non-empty minimal, then for all $a \in F, \overline{Sa}^W = F$ and $w \in \overline{Sa}^W$, then there is a sequence $\{s_n\} \subset S$ such that $s_1z_1 \in w + W_1, s_2s_1z_2 \in w + W_1, ..., s_ns_{n-1}...s_1z_n \in w + W_1 (n = 1, 2, ...)$. If $x \in (s_ns_{n-1}...s_1)(z_n + H) \cap F$ therefore $x \in F$ and $x \in (s_ns_{n-1}...s_1)(z_n + H)$. Then x can be written as: $x = s(z_n + h)$, for some $h \in H$ and $s = s_ns_{n-1}...s_1$. Since z_n converges to 0 in F, then $\rho(sz_n - s(0)) < \frac{\epsilon}{3}$ (by the continuity of s). Hence

$$\begin{aligned}\rho_j(z_n - sz_n) &\leq \rho_j(z_n) + \rho_j(sz_n - s(0)) + \rho_j(s(0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \rho_j(s(0)) \\ &= \frac{2\epsilon}{3} + \rho_j(z_k + h) < \frac{2\epsilon}{3} + \frac{\epsilon}{3} + \rho_i(h),\ \forall\, \epsilon > 0, j = 1, ..., m, \qquad (3.3)\end{aligned}$$

where $\rho_j(z_n) < \frac{\epsilon}{3}$ and $s(0) \in \cup_{n=1}^{\infty}\{z_n + H\}$, $s(0) = z_k + h$, $h \in H$ and for some k. Take $\epsilon \to 0$ in equation (3.2) and by $Q-$generalized nonexpansive, we obtain that

$$\rho_j(s(z_n + h) - sz_n) \leq \rho_j(h) < r_0. \qquad (3.4)$$

From (3.3) we get that $x \in (s_ns_{n-1}, ..., s_1)z_n + H$ and then

$$(s_ns_{n-1}...s_1)(z_n + H) \cap F \subseteq (s_ns_{n-1}, ..., s_1)z_n + H \subset w + W_1 + W_1 \subset w + W$$

Therefore, $\{(s_n...s_1)^{-1}(w + W)\}_{n=1}^{\infty}$ is weakly open cover of F. Therefore $F \subset \bigcup_{k=1}^{n}(s_ks_{k-1}...s_1)^{-1}(w + W)$ for some integer n. According to $F = (s_n...s_1)F$ then

$$F = \bigcup_{k=1}^{n}(s_n...s_{k+1})(w + W) \cap F \subseteq \bigcup_{k=1}^{n}(s_n...s_{k+1})(w + U) \cap F.$$

Let $x \in \bigcup_{k=1}^{n}(s_n...s_{k+1})(w + U) \cap F$. Hence $x \in F$ and $x \in \bigcup_{k=1}^{n}(s_n...s_{k+1})(w + U)$, for some $k = 1, ..., n$. By the density again of SF in F there exist an element c in SF such that $\rho(w - c) < \epsilon\ \forall\ \epsilon > 0,\ \rho \in Q$. Therefore x can be written as $x = \tilde{s}(z + u)$ such that $z = w - c$ for some $\tilde{s} = s_n...s_{k+1} \in S$ and $u \in U$. By argument of (3.2), (3.3) and from $Q-$generalized nonexpansive one can get

$$p_i(\tilde{s}(z + u) - \tilde{s}z) < r,\ i = 1, ..., n. \qquad (3.5)$$

which implies that

$$F \subset \bigcup_{k=1}^{n}(s_ns_{n-1}...s_{k+1})(w + U) \cap F \subset \bigcup_{k=1}^{n}(s_n.....s_{k+1}z + U)$$

Thus F is $Q-$compact.

Remark 3.1. Whenever S acts on a weakly compact convex subset K of a separated locally convex (E, Q), then the weak continuity implies weak quasi-equicontinuity if the action on K is affine and equicontinuous with respect to the topology determined by Q [17].

Consider the following generalized fixed point property.

(GF) Whenever the action S is Q-generalized non-expansive, weakly separately continuous and weakly quasi-equicontinuous and acts on a weakly compact convex subset K of a separated locally convex space (E, Q), then K contains a common fixed point for S.

Now, we are in the position to introduce our main theorem for this section.

Theorem 3.1. *Let S be a separable semitopological semigroup. Then $WAP(S)$ has a LIM if and only if S has the generalized fixed point property (GF).*
Proof. Suppose that $WAP(S)$ has a LIM. Let X be a non-empty minimal weakly compact convex subset of K that is invariant under S and let $F \subset X$ be a non-empty minimal weakly compact subset of X that is invariant under S. By the first paragraph of the proof of [17, Theorem 3.4], F is $Q-$compact. We now follow an idea similar to that in [7, Lemma 2], we show that F contains only one point. Suppose, to the contrary, that F has x_1 and x_2, $x_1 \neq x_2$, (since otherwise F has a common fixed point of s and the proof is finished), there exists a continuous seminorm ρ in Q such that $\rho(x_1 - x_2) = \epsilon > 0$. Let $\alpha = \lambda x_1 + (1-\lambda)x_2,\ \lambda \in [0, 1)$. Then $\alpha \in co(F)$. Moreover $\rho(\alpha - x) \leq \epsilon\ \forall\ x \in F$ such that

$$\epsilon_0 = \sup\{\rho(\alpha - x); x \in F\} < \epsilon.$$

Let

$$\Theta = \{x \in X : \rho(x - y) \leq \epsilon_0,\ \forall\ y \in F\}.$$

Then $\alpha \in \Theta$ and Θ is a nonempty weakly closed convex proper subset of X. Furthermore, if $x \in \Theta$, then $\rho(x - y) \leq \epsilon_0$, $y \in F$. Since S is $Q-$generalized nonexpansive, then by Lemma 3.1 (iii) one can obtain that

$$\rho(sx - sy) \leq \epsilon_0, \tag{3.6}$$

or

$$\rho(sx - s^2y) \leq \rho(x - sy). \tag{3.7}$$

Since $s \in S$ and $F = sF$, then $sy \in F$ and $s^2y = s(sy) \in F$. Hence (3.6) become

$$\rho(sx - s^2y) \leq \epsilon_0. \tag{3.8}$$

From (3.5) and (3.7), we get that $sx \in \Theta$ $(s \in S,\ x \in \Theta)$, which implies that Θ is $S-$invariant. This is a contradiction to the minimality of X. Consequently, F must consist of a single point, which, of course, must be a common fixed point for S.

Conversely if (GF) holds and let S acts linearly on $WAP(S)^*$ such as $s(\psi) = l_s^*\psi$ for all $s \in S$ and $\psi \in WAP(S)^*$, where l_s^* is the dual of the translation operator l_s. Hence $(s(\psi))(f) = (l_s^*\psi)(f) = \psi(l_sf)$ for all $f \in WAP(S)$. Let K be the set of all means on $WAP(S)$, then if m_1 and $m_2 \in K$ and $\lambda \in [0, 1]$, $(\lambda m_1 + (1 - \lambda)m_2)(1_S) = \lambda m_1(1_S) + (1-\lambda)m_2(1_S) = 1$, hence K is convex subset of $AWP(S)^*$. Define $Q = \{\rho_f : f \in WAP(S)\}$ where

$$\rho_f(\psi) = \sup_{s\in S}\{|\psi(l_sf)|, |\psi(f)|\} \qquad (\psi \in WAP(S)^*)$$

then ρ_f is a seminorm on $WAP(S)^*$. One can note that $(WAP(S)^*, Q)$ is separated locally convex space and therefore K is weakly compact convex subset of

$(WAP(S)^*, Q)$. Let ψ_1 and $\psi_2 \in WAP(S)^*$, hence

$$\begin{aligned}\rho_f(s\psi_1 - s\psi_2) &= \sup_{s\in S}\{|(l_s^*\psi_1 - l_s^*\psi_2)(l_s f)|, |(l_s^*\psi_1 - l_s^*\psi_2)(f)|\}\\ &= \sup_{s\in S}\{|(\psi_1 - \psi_2)(l_{s^2} f)|, |(\psi_1 - \psi_2)(l_s f)|\}\\ &= \sup_{s\in S}\{|(\psi_1 - \psi_2)(l_s f)|\}\\ &\le \sup_{s\in S}\{|(\psi_1 - \psi_2)(l_s f)|, |(\psi_1 - \psi_2)(f)|\} = \rho_f(\psi_1 - \psi_2). \qquad (3.9)\end{aligned}$$

and

$$\begin{aligned}\rho_f(s^2\psi_1 - s\psi_2) &= \sup_{s\in S}\{|(l_{s^2}^*\psi_1 - l_s^*\psi_2)(l_s f)|, |(l_{s^2}^*\psi_1 - l_s^*\psi_2)(f)|\}\\ &= \sup_{s\in S}\{|(l_s^*\psi_1 - \psi_2)(l_{s^2} f)|, |(l_s^*\psi_1 - \psi_2)(l_s f)|\}\\ &= \sup_{s\in S}\{|(l_s^*\psi_1 - \psi_2)(l_s f)|\}\\ &\le \sup_{s\in S}\{|(l_s^*\psi_1 - \psi_2)(l_s f)|, |(l_s^*\psi_1 - \psi_2)(f)|\} = \rho_f(s\psi_1 - \psi_2) \qquad (3.10)\end{aligned}$$

Therefore the action on $WAP(S)^*$ (and therefore on K) is Q - generalized nonexpansive. Since for all $m \in K$ and $f \in WAP(S)$, $LO(f)$ is relatively compact in the norm of weak topology of $C(S)$, and since the norm topology in $LO(f)$ is the same a the topology of point wise convergence. since the action $(sf)(t) = (l_s f)(t) = f(st)$ is continuous for each $s \in S$, the map $s(f) = l_s f$ is a continuous map $s \to (LO(f)$, weak norm). Hence the action $s(m) = l_s^*(m)$ is continuous on S into (K, weak *). Also it is clearly the action S on $WAP(S)^*$ (and therefore on K) is separately continuous and weakly separated continuous. Since for m_1 and $m_2 \in K$ and $\lambda \in [0, 1]$

$$\begin{aligned}s(\lambda m_1 + (1-\lambda)m_2) &= l_s^*(\lambda m_1 + (1-\lambda)m_2) = \lambda(l_s^* m_1) + (1-\lambda)(l_s^* m_2)\\ &= \lambda(sm_1) + (1-\lambda)(sm_2), \qquad (3.11)\end{aligned}$$

then the action S on K is affine and hence is weak quasi-equicontinuous on K. Since the property (GF) hold, then the action has a fixed point in K for S (let it is m) then $sm = l_s^* m = m$ and since $(l_S^*(m)f) = m(l_s) = m(f)$ for all $f \in WAP(S)$ then m is LIM of $WAP(S)$.

Example 3.1. Let S be a semigroup and acts on $[0, \frac{5}{2}]$ equipped with the seminorm $\rho(x-y) = |x-y|$ such that for each $s \in S$, $sx = 0$ at $x \neq \frac{5}{2}$ and $sx = 1$ at $x = \frac{5}{2}$. Then S is Q-generalized nonexpansive semigroup but not Q-nonexpansive semigroup.

Proof. If $x < y$ and $x \in [0, 2] \cup \{\frac{5}{2}\}$ and $y \in [0, \frac{5}{2})$, then for each $s \in S$, $\rho(s(x) - s(y)) \le \rho(x-y)$ holds. If $x \in (2, \frac{5}{2})$ and $y = \frac{5}{2}$, then

$$\frac{1}{2}\rho(x - s(x)) = \frac{x}{2} > \rho(x-y) \text{ and } \frac{1}{2}\rho(y - s(y)) = \frac{15}{20} > \rho(x-y).$$

For all $s \in S$.

Thus S is Q-generalized nonexpansive semigroup. However, since S is not continuous, therefore it is not Q-nonexpansive semigroup.

Example 3.2. Let S be a semigroup and acts on $[0, \frac{7}{2}]$ equipped with the seminorm $\rho(x-y) = |x-y|$ such that

$$s(x) = \begin{cases} 0, & \text{if } x \in [0,3], \\ 4x-12, & \text{if } x \in [3, \frac{13}{4}], \\ -4x+14, & \text{if } x \in [\frac{13}{4}, \frac{7}{2}] \end{cases}$$

for all $s \in S$.
Then S is continuous and Q-generalized nonexpansive semigroup. However, S is not Q-nonexpansive semigroup.
Proof. See example 2.2 due to Abkar and Eslamian [1].
Remark 3.2. Theorem 3.1 extending result of Lemma 3.13 and Theorem 3.14 due to Lau and Zhang [18].
Theorem 3.2. *Let S be a separable semitopological semigroup. If $AP(S)$ has a LIM, then the fixed point property (GE) holds.*
(GE) *Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) as Q - generalized non-expansive self mappings and, the action is separately continuous and equicontinuous when K is equipped with the weak topology of (E, Q) then K contains a common fixed point for S.*

The proof is similar to Theorem 3.1 and [13, Theorem 3.2].
Theorem 3.3. *Let S be a semitopological semigroup, then $AP(S)$ has LIM if and only if S has the following fixed point property $(G\acute{E})$*
$(G\acute{E})$ *Whenever S acts on a weakly compact convex space (E, Q) as Q - generalized non-expansive mappings, if K has Q - normal structure and the S-action is separately continuous and equicontinuous when K is equipped with the weak topology of (E, Q), then K contains a common fixed point for S.*
Proof. Let $AP(S)$ has a LIM ψ, and X be a non-empty minimal weakly compact convex subset of K that is invariant under S action. Consider $F \subset X$ be a non-empty minimal weakly compact subset of X that is invariant under S. Since the action on X is separately continuous and equicontinuous, f_y for each $f \in C(F)$ and $y \in F$. Hence μ defined by $\mu(f) = \psi(f_y)$ is a mean on $C(F)$. By the same steps in the proof of Theorem 3.1, we get F is Q-compact and Q-bounded. Let F has x_1, x_2 such that $x_1 \neq x_2$ and by taking $\alpha = \lambda x_1 + (1-\lambda)x_2$ where $\lambda \in [0,1]$ and $\rho \in Q$, by normal structure of K

$$r_0 = \sup\{\rho(\alpha - x) : x \in F\} < \delta_r(F)$$

Then by the same argument as in the proof of Theorem 3.1 lead to contradiction, consequently, F must consist of single point and this point is a common fixed point for S. Conversely, let $(G\acute{E})$ holds. By replace E by $AP(S)^*$ with the topology determined by the family of continuous semi-norm $Q = \{\rho_f : f \in WAP(S)\}$ where

$$\rho_f(\psi) = \sup_{s \in S}\{|\psi(l_s f)|, |\psi(f)|\} \qquad (\psi \in AP(S)^*).$$

Define the action of S on $AP(S)^*$ by $s(\psi) = l_s^* \psi$ for all $s \in S$ and $\psi \in AP(S)^*$. It is clear that, the semigroup S acts linearly on $AP(S)^*$ by $s \mapsto l_s^*$. Let K be the set of all means on $AP(S)$, therefore K is compact closed subset of $AP(S)^*$. Since from Lemma 2.5, a compact subset of separated locally convex space has normal structure, K has

$Q-$ normal structure. Following the same argument as in the proof of Theorem 3.1, it is clear the action of S on $AP(S)^*$ (and therefore on K) is separately continuous and equicontinuous with respect to the topology determined by Q, and Q -generalized nonexpansive. Since property $(G\acute{E})$ hold. Then K has a common fixed point for S, which is a LIM on $AP(S)$.

Theorem 3.4. *Suppose that S is a separable semitopological semigroup. Then $WAP(S) \cap LUC(S)$ has a LIM if and only if fixed point property (GF^*) holds.*
(GF^*) Whenever S acts on a weakly compact subset K of a separated locally convex space (E, Q) and the action is weakly joint continuous, weakly quasi- equicontinuous and Q -generalized non-expansive, then K contains a common fixed point for S.
Where $LUC(S)$ denoted the space of left uniformly continuous functions on S.

Proof. Let $WAP(S) \cap LUC(S)$ has LIM and K be a weakly compact convex subset of E. If the action of S on K is weakly joint continuous and weakly quasi-equicontinuous, then we get F non-empty minimal weakly compact $S-$ invariant subset of K. Hence according to joint continuity and the compactness of the action of S and by Theorem 3.1 we get that S has a common fixed. Conversely, define the action of S on $(WAP(S) \cap LUC(S))^*$ by $s(\psi) = l^*\psi$ for all $s \in S$ and $\psi \in (WAP(S) \cap LUC(S))^*$ and let K be the set of all means on $WAP(S) \cap LUC(S)$. The action S on $(WAP(S) \cap LUC(S))^*$ (and therefore on K) is weakly jointly continuous. By replace $WAP(S)$ by $WAP(S) \cap LUC(S)$ in the steps of the proof of Theorem 3.1, we get that the action of S on (K, weak*) has a common fixed point which is a left invariant mean on $WAP(S) \cap LUC(S)$.

Theorem 3.5. *Let S be left reversible and metrizable semitopological semigroup. Then S has a fixed point property (GR)*
(GR) Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) and the action is weak joint continuous and Q -generalized non-expansive, then K contains a common fixed point for S.

Proof. Since S is metrizable left reversible semigroup, from Lemma 2.7 S is strongly left reversible. If $\{S_\alpha : \alpha \in I\}$ be the family of countable subsemigroups of S and S_α is strong left reversible for all $\alpha \in I$. Therefore for all $\alpha \in I$, $\overline{S}_\alpha$ is separable. Hence $S = \cup_\alpha S_\alpha$. If K is weakly compact convex subset of E. From Lemmas 2.4 and 3.2, for all $\alpha \in I$, every non-empty minimal $\overline{S_\alpha}$ invariant and weakly compact subset F of K is Q- compact and hence is singleton. Therefore K contains a common fixed point for $\overline{S_\alpha}$. By take $F_\alpha = \{k \in K : S_\alpha k = k\}$, then $\cap_{\alpha \in I} S_\alpha \neq 0$. Therefore, there exists an element $k \in \cap_{\alpha \in I} F_\alpha$ such that $S_\alpha k = k$ for all $\alpha \in I$. Hence k is an common fixed point for S. Conversely, define the action of S on $(WAP(S) \cap LUC(S))^*$ by $s(\psi) = l^*\psi$ for all $s \in S$ and $\psi \in (WAP(S) \cap LUC(S))^*$ and let K be the set of all means on $WAP(S) \cap LUC(S)$. It is clear that the action $s \mapsto l_s^*$ is weakly jointly continuous and $Q-$generalized non-expansive. Following the same argument as in the proof of Theorem (3.1) and (3.4), we get S has a common fixed point in K, which is a LIM on $WAP(S) \cap LUC(S)$.

Remark 3.3. It will be interesting to establish Theorem 3.1 of a $Q-$quasi-non-expansive semitopological semigroup S.

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