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A PROOF OF THE MARKOV-KAKUTANI THEOREM ON NONCOMPACT SET VIA ZERMELO'S WELL-ORDERING THEOREM

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Abstract. The Markov-Kakutani fixed point theorem has been considered as one of the most remarkable theorems due to considerable diversity in its applications in the history of functional analysis. Different approaches have been investigated to prove this theorem; however, the condition of compactness of the underlying set is essentially used. In this paper, we develop a new method, based on Zermelo's well-ordering theorem, to weaken the compactness condition.
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One of the most celebrated results in the theory of common fixed points is a theorem proved independently by Markov [6] and Kakutani [5]. The classical Markov-Kakutani fixed point theorem states that every commuting family of continuous affine mappings on a compact convex set C in a Hausdorff topological vector space X into itself has a common fixed point. Several mathematicians have since tried to establish this theorem under weaker assumptions, such as that C is not necessarily compact. In this note, by appealing Zermelo's well-ordering theorem, we suggest a new approach to weaken the compactness condition in the Markov-Kakutani fixed point theorem, in context of separated locally convex spaces. Instead, we use a weaker condition that is called "property (C)" in the literature (see, e.g.,[8]). By giving an example, we show that our result does not remain valid for Hausdorff topological vector spaces.

We recall that a separated locally convex space X is a topological vector space whose topology is defined by a family of seminorms $\{p_\beta : \beta \in \Gamma\}$ such that

$$\bigcap_{\beta\in\Gamma}\{x \in X : p_\beta(x) = 0\} = \{0\}.$$

Hereafter, we suppose that X is a separated locally convex space. A subset C of X is said to have property (C), if every chain (by inclusion) of nonempty closed convex bounded subsets of C has a nonempty intersection. A set I with an order relation $\prec$ is said to be well-ordered if every nonempty subset of I has a smallest element.

Zermelo's well-ordering theorem states that if I is a nonempty set, then there exists an order relation on I under which I is well-ordered.

Main Theorem. *Let X be a separated locally convex space, C be a nonempty bounded closed convex subset of X that has property (C), and $\{T_\alpha:\ \alpha\in\Lambda\}$ be a commuting family of continuous affine mappings from C into C. Then, $\cap_{\alpha\in\Lambda}Fix(T_\alpha)\neq\emptyset$.*

Proof. We first prove that C has the fpp for continuous affine mappings. For this purpose, assume that $T: C\to C$ is a continuous affine mapping and fix some $x_0\in C$. Define $y_n=\frac{1}{n}(x_0+Tx_0+\cdots+T^nx_0)$. Suppose that the topology on X is defined by the family of seminorms $\{p_\beta:\ \beta\in\Gamma\}$. Take $m\in\mathbb{N}$, and define

$$K_m=\{x\in C:\ p_\beta(Tx-x)\leq\frac{1}{m},\ \forall\beta\in\Gamma\}.$$

We prove that $K_m\neq\emptyset$. Defining

$$K_{m,\beta}=\{x\in C:\ p_\beta(T^ix-T^{i-1}x)\leq\frac{1}{m},\forall i\in\mathbb{N}\},$$

for each $\beta\in\Gamma$, it is easy to check that $K_{m,\beta}$ is closed, convex and T-invariant. Moreover, $K_{m,\beta}\neq\emptyset$. Indeed, because C is bounded, for each $\beta\in\Gamma$, there exists a scaler r such that

$$C-C\subset\{z\in X: p_\beta(z)<r\}.$$

Then, choosing $n_0\in\mathbb{N}$ with $r/n_0<1/m$, we have

$$p_\beta(T^iy_{n_0}-T^{i-1}y_{n_0})=\frac{1}{n_0}p_\beta(T^{n_0+i}x_0-T^{i-1}x_0)<r/n_0<1/m,\ \forall i\in\mathbb{N}.$$

This implies that $y_0\in K_{m,\beta}$ and then $K_{m,\beta}\neq\emptyset$.

By Zermelo's well-ordering theorem, Γ can be well-ordered for some relation $\prec$. Let β_0 be the smallest element of Γ and, for each $\gamma\in\Gamma$, put

$$L_\gamma=\bigcap_{\beta_0\preceq\beta\preceq\gamma}K_{m,\beta}.$$

We show that, for each $\gamma\in\Gamma$, $L_\gamma\neq\emptyset$. Suppose, for contradiction, that

$$\Omega=\{\gamma\in\Gamma:\ L_\gamma=\emptyset\}\neq\emptyset.$$

Then, there exists a smallest element $\gamma_0\in\Omega$. Obviously, $\beta_0\prec\gamma_0$, and $L_\gamma\neq\emptyset$, for each $\gamma\prec\gamma_0$. Thus, by appealing the property (C),

$$\bigcap_{\beta_0\preceq\gamma\prec\gamma_0}L_\gamma\neq\emptyset.$$

The latter set is also closed, convex and T-invariant. Choosing some $x\in\bigcap_{\beta_0\preceq\gamma\prec\gamma_0}L_\gamma$, and taking $z_n=\frac{1}{n}(x+Tx+\cdots+T^nx)$, we have $z_n\in\bigcap_{\beta_0\preceq\gamma\prec\gamma_0}L_\gamma$ for each n, and

$$p_{\gamma_0}(T^iz_{n_0}-T^{i-1}z_{n_0})=\frac{1}{n_0}p_{\gamma_0}(T^{n_0+i}x-T^{i-1}x)<r/n_0<1/m,\ \forall i\in\mathbb{N}.$$

It yields that

$$z_{n_0}\in\bigcap_{\beta_0\preceq\gamma\preceq\gamma_0}L_\gamma=L_{\gamma_0}\neq\emptyset,$$

which is a contradiction. Hence, $\{L_\gamma : \gamma \in \Gamma\}$ is a chain of nonempty closed convex subsets of C. By applying the property (C), it follows that

$$K_m \supset \bigcap_{\beta \in \Gamma} K_{m,\beta} = \bigcap_{\gamma \in \Gamma} L_\gamma \neq \emptyset.$$

Therefore, $\{K_m\}_{m \in \mathbb{N}}$ is a descending sequence of nonempty closed convex subsets of C. Accordingly, using property (C), it follows

$$Fix(T) = \bigcap_m K_m \neq \emptyset.$$

In this stage, we will show that $\cap_{\alpha \in \Lambda} Fix(T_\alpha) \neq \emptyset$. According to Zermelo's well-ordering theorem, there is a relation $\prec$ under which Λ is well-ordered. Let α_0 be the smallest element of Λ and, for each $\alpha \in \Lambda$, let

$$M_\alpha = \bigcap_{\alpha_0 \preceq \beta \preceq \alpha} Fix(T_\beta).$$

Then, from the above assertion, $M_{\alpha_0} = Fix(T_{\alpha_0}) \neq \emptyset$. We show that $M_\alpha \neq \emptyset$, for each $\alpha \in \Lambda$. Define

$$\Delta = \{\alpha \in \Lambda : \ M_\alpha = \emptyset\}.$$

Suppose, for contradiction, that $\Delta \neq \emptyset$. Then, there exists a smallest element $\tau_0 \in \Delta$. Obviously, $\alpha_0 \prec \tau_0$ and $M_\alpha \neq \emptyset$, for each $\alpha \prec \tau_0$. Thus, $\{M_\alpha : \ \alpha_0 \preceq \alpha \prec \tau_0\}$ is a descending chain of nonempty closed convex subsets of C for which, by assumption,

$$\mathcal{M} = \bigcap_{\alpha_0 \preceq \alpha \prec \tau_0} M_\alpha \neq \emptyset.$$

On the other hand, $\mathcal{M}$ is T_{τ_0}-invariant. In fact, taking $x \in \mathcal{M}$, we obtain

$$T_\alpha(T_{\tau_0}(x)) = T_{\tau_0}(T_\alpha(x)) = T_{\tau_0}(x), \ \forall \alpha_0 \preceq \alpha \prec \tau_0.$$

Therefore,

$$T_{\tau_0}(x) \in \bigcap_{\alpha_0 \preceq \alpha \prec \tau_0} Fix(T_\alpha) = \mathcal{M}.$$

Hence, there exists some $y \in \mathcal{M}$ such that $T_{\tau_0}(y) = y$; i.e.,

$$y \in \mathcal{M} \cap Fix(T_{\tau_0}) = M_{\tau_0},$$

a contradiction. Therefore, $\Delta = \emptyset$, and so $\{M_\alpha\}_{\alpha \in \Lambda}$ is a descending chain of nonempty closed convex subsets of C, which it follows by property (C) that

$$\bigcap_{\alpha \in \Lambda} Fix(T_\alpha) = \bigcap_{\alpha \in \Lambda} M_\alpha \neq \emptyset.$$

This completes the proof. □

In the following, we show that our theorem can not be generalized to a Hausdorff topological vector space; however, the original Markov-Kakutani's theorem is valid for Hausdorff topological vector spaces, where the the underlying set is compact. First, recall that in a locally convex space, every weakly bounded set is originally bounded, and vice versa (see, e.g., Theorem 3.18 in [7]). Also, closed convex subsets are weakly closed. Thus, property (C) for a subset C is equivalent to say that every chain of nonempty weakly closed convex weakly bounded subsets of C has a nonempty intersection. Therefore our theorem may be stated as follows:

Let X be a separated locally convex space, C be a nonempty weakly bounded closed convex subset of X that has property (C), in the sense that every chain of nonempty weakly closed convex subsets of C has a nonempty intersection, and $\{T_\alpha : \alpha \in \Lambda\}$ be a commuting family of continuous affine mappings from C into C. Then, $\cap_{\alpha\in\Lambda} Fix(T_\alpha) \neq \emptyset$.

On the other hand, the topological vector space

$$X = L^p[0,1] = \{f : [0,1] \to \mathbb{R} :\ f \text{ is measurable and } \int_0^1 |f(t)|^p dt < \infty\},$$

with $0 < p < 1$, is not locally convex with the topology given by the complete metric

$$d(f,g) = \int_0^1 |f(t) - g(t)|^p dt.$$

In fact, it is known that the only convex open set in $L^p[0,1]$ is the whole space; also, It is well-known that $L^p[0,1]^* = \{0\}$ (see [4, 1]). Then, since $\sigma(X, X^*) = \{\emptyset, X\}$, X is weakly bounded and has property (C) in the sense that every chain of nonempty weakly closed convex subsets of X has a nonempty intersection. But, for any fixed $a \neq 0$ in X, $x \mapsto x + a$ is a continuous affine map without a fixed point.

It may be a question that whether the Markov-Kakutani theorem holds in a non-separated locally convex space. The answer to this question is negative. In fact, if $X = L^p[0,1]$, $0 < p < 1$, then, since $X^* = L^p[0,1]^* = \{0\}$, the weak topology $\sigma(X, X^*)$ is the trivial topology $\{\emptyset, X\}$. Accordingly, $(X, \sigma(X, X^*))$ is a non-separated locally convex space such that X is compact in the weak topology. However, for any fixed $a \neq 0$ in X, $x \mapsto x + a$ is a continuous affine map without a fixed point. That is, the Markov-Kakutani's theorem dose not hold for the non-separated case.

It is known that each commutative semigroup is amenable [3]; moreover, the conclusion of the Markov-Kakutani theorem holds when the family of maps is an amenable semigroup [2]. However, the following problem is open to us:

Can we replace commutativity with amenability in our result?

Finally, we remark that our approach is based on a method using Zermelo's well-ordering theorem, directly; howevere, applying Zorn's lemma, it is also possible to obtain the same result.

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