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MULTIVALUED $\mathcal{R}_{\psi,\phi}$-WEAKLY CONTRACTIVE MAPPINGS IN ORDERED CONE METRIC SPACES WITH APPLICATIONS

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Abstract. In this article we define multivalued $\mathcal{R}$-weakly contractive multi-valued mappings in ordered cone metric spaces without assumption of normality on cone, and generalize many results existing in the literature. We provide applications to solutions of integral inclusions and give non-trivial examples to support our main theorem.
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1. Introduction

The order-theoretic fixed point theory has its applications to a wide range of various fields such as, e.g., ordinary differential equations, integral equations, (single and multi-valued) non-local discontinuous partial differential equations of various types, mathematical economics, game theory and so on. In all these topics we are faced with the central problem of handling the loss of continuity of mappings on their underlying domain of definition. It is significant that, in particular, for proving the existence of certain optimal strategies in game theory, there is a need for order-related fixed point results in partially ordered sets (po-sets) that are neither convex nor they have lattice structure, and in which the fixed point operator may not be continuous.

The existence of fixed points in partially ordered sets has been considered in [18], where some applications to matrix equations are presented. Many generalizations have been made in ordered structure of metric spaces. The weak contractions in ordered metric spaces were introduced by Turinici in [21], and after that many researchers invested their efforts to generalize and extend fixed and coincidence points results by using weak contractions.

In [13] the author established some fixed point results in ordered complete metric spaces for set-valued operators. Some results of [13] have been generalized in [1, 12]. In their setting the multivalued operators are compact valued. Many results for fixed and coincidence points of multivalued mappings are available in the literature, and in most of them the distance function $\delta(A, B) = \mathrm{Dist}\,(A, B)$ is used to obtain distance between two sets A and B of the metric space X. Some fixed point results by using the Hausdorff metric on $\mathsf{C}(X)$ (compact subsets of the metric space X under consideration) are also available in the literature to deal with weak contractions (see [2, 9]).

The cone metric spaces were properly introduced in [14] by replacing the set of real numbers by an ordered Banach space in which the convergence and order are defined. Later on it was discovered that to work with normal cones is redundant and the real generalization is to work in cone metric spaces with non-normal cones. Many researchers generalized and extended a variety of results in cone metric space endowed with a partial order on X. A number of these results are concerning single valued mappings and with normal and non-normal cones (see [4, 5, 6, 11, 15, 19]).

The Hausdorff distance function for a cone metric space was invented by authors in [10]. Clearly, this Hausdorff distance function is a generalization of the Hausdorff metric $H(\cdot,\cdot)$ on $\mathsf{CB}(X)$ defined in [17]. In [7, 16, 20] the authors have used this Hausdorff distance function to generalize the results for set-valued mappings in cone metric spaces. We use the Hausdorff distance function for multivalued mappings in ordered cone metric spaces.

In this article we establish some multivalued fixed point theorems with weak contractions using the Hausdorff distance function on the closed and bounded subsets of a given cone metric space. We define approximative valued mappings and provide their applications in ordered cone metric spaces. We deduce a few corollaries and generalize many results in the literature. To prove the validity and novelty of our main result, we have given a non-trivial example. As an application we provide a theorem for the existence of a certain type of differential inclusion using our main result.

2. Preliminaries

Let $\mathbb{E}$ be a real Banach space with its zero element θ. A non-empty subset K of $\mathbb{E}$ is called a *cone* if

(a) K is non-empty closed and $K \neq \{\theta\}$;

(b) $K \cap (-K) = \{\theta\}$;

(c) if α, β are nonnegative real numbers and $x, y \in K$, then $\alpha x + \beta y \in K$.

For a given cone $K \subseteq \mathbb{E}$ we define a partial ordering $\preccurlyeq$ with respect to K by $x \preccurlyeq y$ if and only if $y - x \in K$; $x \prec y$ stands for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \mathrm{int}K$, where $\mathrm{int}K$ denotes the interior of K. The cone K is said to be *solid* if it has non-empty interior.

The following definitions and lemmas will be used to prove our main results.

Definition 2.1. ([14]) Let X be a non-empty set. A vector-valued function $d : X \times X \to \mathbb{E}$ is said to be a *cone metric* if the following conditions hold:
$(CM1)$ $\theta \preccurlyeq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;
$(CM2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$;
$(CM3)$ $d(x,z) \preccurlyeq d(x,y) + d(y,z)$ for all $x, y, z \in X$.
The pair (X,d) is then called a *cone metric space.*

Definition 2.2. ([14]) Let(X,d) be a cone metric space, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then:
(i) $\{x_n\}$ *converges to* x, denoted $\lim_{n\to\infty} x_n = x$, if for every $\varepsilon \in \mathbb{E}$ with $\theta \ll \varepsilon$ there is a natural number n_0 such that $d(x_n, x) \ll \varepsilon$ for all $n \geq n_0$. A set $A \subset (X,d)$ is called *closed* if for any sequence $\{x_n\} \subset A$ converging to x we have $x \in A$.
(ii) $\{x_n\}$ is a *Cauchy sequence* if for every $\varepsilon \in \mathbb{E}$ with $\theta \ll \varepsilon$ there is a natural number n_0 such that $d(x_n, x_m) \ll \varepsilon$ for all $n, m \geq n_0$;
(iii) (X,d) is *complete* if every Cauchy sequence in X is convergent.

Remark 2.3. ([15]) *The cone metric is not continuous in the general case, i.e. from $x_n \to x$, $y_n \to y$ it need not follow that $d(x_n, y_n) \to d(x,y)$.*

Let (X,d) be a cone metric space. The following properties will be used very often (see [20]):
$(P1)$ If $u \preccurlyeq v$ and $v \ll w$, then $u \ll w$;
$(P2)$ If $c \in \text{int}K$, $a_n \in \mathbb{E}$ and $a_n \to \theta$, then there exists an n_0 such that, for all $n > n_0$, we have $a_n \ll c$.

A partial order on a set X will be denoted by $\mathcal{R}$, and elements x and y in X are said to be *comparable*, denoted by $x \asymp y$, if either $x\mathcal{R}y$ or $y\mathcal{R}x$.

Let $\mathsf{C}(X)$ denotes the family of nonempty closed subsets of X.

We denote for $p \in \mathbb{E}$:

$$s(p) = \{q \in \mathbb{E} : p \preccurlyeq q\}$$

and

$$s(a,B) = \bigcup_{b\in B} s(d(a,b)) = \bigcup_{b\in B} \{x \in \mathbb{E} : \ d(a,b) \preccurlyeq x\} \text{ for } a \in X \text{ and } B \in \mathsf{C}(X).$$

For $A, B \in \mathsf{C}(X)$ we denote

$$s(A,B) = \left(\bigcap_{a\in A} s(a,B)\right) \bigcap \left(\bigcap_{b\in B} s(b,A)\right).$$

3. Main results

Consider the following classes of functions:

- Ψ denotes the set of all functions $\psi : K \longrightarrow K$ satisfying

(1) ψ is continuous and strongly monotone;
(2) $\psi(t) = \theta$ if and only if $t = \theta$;
(3) $t - \psi(t) \in \text{int}K$ for $t \in \text{int}K$;
(4) For a non-increasing sequence $\{s_n\}$ in K, $\lim_{n\to\infty} \psi(s_n)$ exists in K.

- Φ denotes the set of all functions $\phi : K \longrightarrow K$ such that:

(1) ϕ is continuous and $\phi(\theta) = \theta$.

Definition 3.1. Let X be a non-empty set. Then $(X, \mathcal{R}, d)$ is called an *ordered cone metric space* if d is a cone metric on X and $\mathcal{R}$ is a partial order on X.

The following two definitions are generalizations to ordered cone metric spaces of the corresponding notions for ordered metric spaces (see, for instance, [1]).

Definition 3.2. An ordered cone metric space is said to have the *sequential limit comparison property* if for every non decreasing sequence $\{x_n\}$ in X with $x_n \to x$, we have $(x_n, x) \in \mathcal{R}$.

Definition 3.3. An ordered cone metric space is said to have the *subsequential limit comparison property* if for every non decreasing sequence $\{x_n\}$ in X with $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in \mathcal{R}$.

In [13], Hong introduced several notions which were very useful for obtaining some fixed point results for multivalued mappings of ordered metric spaces. Following Hong's idea and the notion of approximative sets in cone metric spaces we introduce similar notions for ordered cone metric spaces (see Definition 3.5).

Definition 3.4. Let X be a cone metric space. A subset W of X is said to be *approximative* if

$$\mathcal{R}_W(x) := \{y \in W : s(d(x,y)) = s(x, W)\}, \ x \in X,$$

is non-empty for all $x \in X$.

Definition 3.5. Let $(X, \mathcal{R}, d)$ be an ordered cone metric space. A multivalued mapping $T : X \to 2^X$ is said to have:

(i) *approximative values*, (AV for short), if Tx is approximative for each $x \in X$;

(ii) *Comparable approximative values*, CAV for short, if T is approximative and for all $w, x \in X$ there exists $y \in \mathcal{R}_{Tw}(x)$ such that y is comparable to w;

(iii) *Upper comparable approximative values*, UCAV for short (resp. *lower comparable approximative values*, LCAV for short) if T is approximative and for each $w \in X$, there exists $y \in \mathcal{R}_{Tw}(x)$ such that $(w, y) \in \mathcal{R}$ (resp. $(y, w) \in \mathcal{R}$).

Definition 3.6. Let (X, d) be a cone metric space endowed with a partial order $\mathcal{R}$ on X. A multivalued mapping $T : X \to \mathsf{C}(X)$ is said to be a *multi-valued $\mathcal{R}$-contractive mapping* if there exists $\lambda \in (0, 1)$ such that T is a UCAV mapping and

$$\lambda d(x, y) \in s(Tx, Ty) \ for \ (x, y) \in \mathcal{R}.$$

Definition 3.7. Let (X, d) be a cone metric space endowed with a partial order $\mathcal{R}$. A multivalued mapping $T : X \to \mathsf{C}(X)$ is said to be $\mathcal{R}_{\psi,\phi}$ *weakly L-contractive* (resp. $\mathcal{R}_{\psi,\phi}$ *weakly U-contractive*) if T is LCAV (resp. UCAV) and

$$\psi(d(x, y)) - \phi(d(x, y)) \in s(Tx, Ty) \tag{3.1}$$

for some $\psi \in \Psi$, $\phi \in \Phi$ and for all comparable x and y in X.

Observe that the weakly contractive (single-valued) mappings have been introduced and studied in [3], and in recent years extended and generalized in various directions.

Definition 3.8. For two subsets A and B of a cone metric space X we denote $A\mathcal{R}B$ if for each $a \in A$ and $b \in B$ we have $(a, b) \in \mathcal{R}$. A multivalued mapping T is said to be nondecreasing (nonincreasing) if $(x, y) \in \mathcal{R}$ implies that $Tx\mathcal{R}Ty$ ($Ty\mathcal{R}Tx$).

Now we prove our main theorem.

Theorem 3.9. *Let $(X, \mathcal{R})$ be an ordered complete cone metric space. Let $T : X \longrightarrow \mathsf{C}(X)$ be a $\mathcal{R}_{\psi,\phi}$ weakly U-contractive mapping. If X has sequential limit comparison property, then T has a fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary but fixed point. If $x_0 \in Tx_0$ then we are done. Otherwise, as T is UCAV, there exists some $x_1 \in \mathcal{R}_{Tx_0}(x_0)$ with $(x_0, x_1) \in \mathcal{R}$, such that

$$s(d(x_0, x_1)) = s(x_0, Tx_0).$$

From (3.1) we have

$$\psi\left(d\left(x_0, x_1\right)\right) - \phi\left(d\left(x_0, x_1\right)\right) \in s\left(Tx_0, Tx_1\right)$$

which implies

$$\psi\left(d\left(x_0, x_1\right)\right) - \phi\left(d\left(x_0, x_1\right)\right) \in s\left(x_1, Tx_1\right).$$

By a similar argumentation we find a point $x_2 \in Tx_1$ with $(x_1, x_2) \in \mathcal{R}$ such that

$$\psi\left(d\left(x_1, x_2\right)\right) - \phi\left(d\left(x_1, x_2\right)\right) \in s(x_2, Tx_2).$$

Continuing in the same way we will get the sequence $\{x_n\}$ in X so that

$$(x_n, x_{n+1}) \in \mathcal{R}, \text{ and } x_{n+1} \in Tx_n \text{ for } n \geq 0.$$

Further, we have

$$\psi\left(d\left(x_{n-1}, x_n\right)\right) - \phi\left(d\left(x_{n-1}, x_n\right)\right) \in s\left(d(x_n, x_{n+1})\right),$$

which implies

$$d(x_n, x_{n+1}) \preccurlyeq \psi\left(d\left(x_{n-1}, x_n\right)\right) - \phi\left(d\left(x_{n-1}, x_n\right)\right), \tag{a}$$

and

$$\begin{aligned} d(x_n, x_{n+1}) &\preccurlyeq \psi\left(d\left(x_{n-1}, x_n\right)\right), \\ d(x_n, x_{n+1}) &\preccurlyeq d\left(x_{n-1}, x_n\right). \end{aligned}$$

Thus $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a nonincreasing sequence in K so that, by definition of ψ,

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = r, \text{ for some } r \in \operatorname{int}K. \tag{b}$$

If $r = \theta$, then $d(x_n, x_{n+1}) \to \theta$, as $n \to \infty$, so that supposing that $r \gg \theta$ and using (a) and properties of ψ we have

$$\psi(d(x_n, x_{n+1})) \ll d(x_n, x_{n+1}) \preccurlyeq \psi\left(d\left(x_{n-1}, x_n\right)\right) - \phi\left(d\left(x_{n-1}, x_n\right)\right).$$

From (b) we have

$$r \ll r - \lim_{n\to\infty} \phi\left(d\left(x_{n-1}, x_n\right)\right),$$

which implies

$$\lim_{n\to\infty} \phi\left(d\left(x_{n-1}, x_n\right)\right) \ll \theta.$$

Using definition of ϕ we have

$$d\left(x_{n-1}, x_n\right) \to \theta,\ as\ n \to \infty. \tag{c}$$

Choose $m(j) < n(j)$. Using transitivity of $\mathcal{R}$ we have $\left(x_{m(j)}, x_{n(j)}\right) \in \mathcal{R}$ so that we get

$$\psi(d(x_{m(j)}, x_{n(j)})) - \phi(d(x_{m(j)}, x_{n(j)})) \in s(Tx_{m(j)}, Tx_{n(j)}).$$

Since $x_{n(j)+1} \in Tx_{n(j)}$ and T is UCAV, we can find some $x_{m(j)+1} \in Tx_{m(j)}$, such that

$$d\left(x_{m(j)+1}, x_{n(j)+1}\right) \preccurlyeq \psi\left(d\left(x_{m(j)}, x_{n(j)}\right)\right) - \phi\left(d\left(x_{m(j)}, x_{n(j)}\right)\right), \tag{d}$$

and by virtue of transitivity of $\mathcal{R}$ we have $(x_{m(j)+1}, x_{n(j)+1}) \in \mathcal{R}$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. Suppose that $\{x_n\}$ is not a Cauchy sequence; then there exists an $\varepsilon \gg 0$ for which we can find two sequences of positive integers (say) $\{m(j)\}$ and $\{n(j)\}$ such that for all positive integers j, $n(j) > m(j) > j$ and $d(x_{m(j)}, x_{n(j)}) \succcurlyeq \varepsilon$. Assume that $n(j)$ is the smallest positive integer such that

$$d(x_{m(j)}, x_{n(j)}) \succcurlyeq \varepsilon \quad \text{and} \quad d(x_{m(j)}, x_{n(j)-1}) \ll \varepsilon.$$

Now

$$\varepsilon \preccurlyeq d\left(x_{m(j)}, x_{n(j)}\right) \preccurlyeq d\left(x_{m(j)}, x_{n(j)-1}\right) + d\left(x_{n(j)-1}, x_{n(j)}\right) \ll \varepsilon + d\left(x_{n(j)-1}, x_{n(j)}\right).$$

Letting $j \to \infty$ in the above inequality and using (c), we get

$$\lim_{j\to\infty} d(x_{m(j)}, x_{n(j)}) = \varepsilon. \tag{e}$$

Consider, by using the triangular property of cone metric space,

$$d(x_{m(j)+1}, x_{n(j)+1}) \preccurlyeq d(x_{m(j)+1}, x_{m(j)}) + d(x_{m(j)}, x_{n(j)}) + d(x_{n(j)}, x_{n(j)+1})$$

and

$$d(x_{m(j)}, x_{n(j)}) \preccurlyeq d(x_{m(j)}, x_{m(j)+1}) + d(x_{m(j)+1}, x_{n(j)+1}) + d(x_{n(j)+1}, x_{n(j)}).$$

Letting $j \to \infty$ in the above inequalities and using (c) and (e), we get

$$\lim_{j\to\infty} d(x_{m(j)+1}, x_{n(j)+1}) = \varepsilon. \tag{f}$$

Consider form (d):

$$\begin{aligned} \psi\left(d\left(x_{m(j)+1}, x_{n(j)+1}\right)\right) &\ll d\left(x_{m(j)+1}, x_{n(j)+1}\right) \\ &\preccurlyeq \psi\left(d\left(x_{m(j)}, x_{n(j)}\right)\right) - \phi\left(d\left(x_{m(j)}, x_{n(j)}\right)\right). \end{aligned}$$

Letting $j \to \infty$ in the above inequalities, using (b), (e) and the continuity of ϕ, we obtain

$$\begin{gathered} r \ll r - \lim_{j\to\infty} \phi\left(d\left(x_{n(j)}, x_{m(j)}\right)\right) \\ \phi(\lim_{j\to\infty} d\left(x_{n(j)}, x_{m(j)}\right)) \ll \theta\ , \\ \phi(\varepsilon) \ll \theta. \end{gathered}$$

A contradiction, so by definition of ϕ we have

$$\lim_{j\to\infty} d\left(x_{n(j)}, x_{m(j)}\right) = \theta.$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Therefore, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Using $(P1)$ and $(P2)$ choose a natural number k_1 such that $d(x_n, x) \ll \frac{c}{2}$ for all $n \geq k_1$.

By sequential limit comparison property of X we have, $(x_n, x) \in \mathcal{R}$ for all $n \in \mathbb{N}$. Now consider

$$\psi\left(d\left(x_n, x\right)\right) - \phi\left(d\left(x_n, x\right)\right) \in s\left(Tx_n, Tx\right).$$

For $x_{n+1} \in Tx_n$, as T is $\mathcal{R}_{\psi,\phi}$ weakly U-contractive mapping, we can choose $z_n \in Tx$ such that $(x_{n+1}, z_n) \in \mathcal{R}$, and

$$\psi\left(d\left(x_n, x\right)\right) - \phi\left(d\left(x_n, x\right)\right) \in s\left(d(x_{n+1}, z_n)\right),$$

which implies

$$d(x_{n+1}, z_n) \preccurlyeq \psi\left(d\left(x_n, x\right)\right) - \phi\left(d\left(x_n, x\right)\right) \preccurlyeq \psi\left(d\left(x_n, x\right)\right) \prec d\left(x_n, x\right).$$

Next

$$d(x, z_n) \preccurlyeq d(x, x_{n+1}) + d(x_{n+1}, z_n) \prec d(x, x_{n+1}) + d\left(x_n, x\right) \ll c$$

for all $n \geq k_1$.

This implies $z_n \to x$, and since Tx is closed we have $x \in Tx$. This completes the proof.

The following theorem is to find a fixed point of the $\mathcal{R}_{\psi,\phi}$ weakly L-contractive mapping and can be proved by using the similar steps of the above theorem.

Theorem 3.10. *Let $(X, \mathcal{R})$ be a complete cone metric space endowed with a partial order $\mathcal{R}$ on X. Let $T : X \longrightarrow \mathsf{C}(X)$ be $\mathcal{R}_{\psi,\phi}$ weakly L-contractive mapping. If X has sequential limit comparison property, then T has a fixed point in X.*

For approximative multivalued mappings we have the following theorem.

Theorem 3.11. *Let $(X, \mathcal{R})$ be a complete cone metric space endowed with a partial order $\mathcal{R}$ on X. Let $T : X \longrightarrow \mathsf{C}(X)$ be an* AV *multivalued nondecreasing mapping satisfying* (3.1)*, and let X have sequential limit comparison property. If there exists $x_0 \in X$ such that $\{x_0\}\,\mathcal{R}Tx_0$, then T has a fixed point in X.*

Proof. Let $x_0 \in X$. If $x_0 \in Tx_0$, then there is nothing to prove. If not, then choose some $x_1 \in Tx_0$ with $(x_0, x_1) \in \mathcal{R}$. Working as in the proof of Theorem 3.9 and using the fact that T is AV and nondecreasing (instead of the fact from Theorem 3.9 that T is UCAV) we find a point $x_2 \in Tx_1$ with $(x_1, x_2) \in \mathcal{R}$ such that

$$d\left(x_1, x_2\right) \preccurlyeq \psi\left(d\left(x_0, x_1\right)\right) - \phi\left(d\left(x_0, x_1\right)\right).$$

The remaining portion of the proof coincides with the proof of Theorem 3.9.

An other interesting way to generalize the results under assumption that the consecutive terms in the constructed sequence are comparable, we have the following theorem.

Theorem 3.12. *Let $(X, \mathcal{R})$ be a complete cone metric space endowed with a partial order $\mathcal{R}$ on X. Let $T : X \longrightarrow \mathsf{C}(X)$ be* CAV *satisfying* (3.1)*. If $\{x_n\}$ is a sequence whose consecutive terms are comparable and $x_n \to x$ as $n \to \infty$, and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every its term is comparable to the limit u, then T has a fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary but fixed point. If $x_0 \in Tx_0$, then we are done. If not, then (as Tx_0 is CAV) there exists some $x_1 \in \mathcal{R}_{Tx_0}(x_0)$ with $x_0 \asymp x_1$, such that

$$s(d(x_0, x_1)) = s(x_0, Tx_0).$$

From (3.1) we have

$$\psi\left(d\left(x_0, x_1\right)\right) - \phi\left(d\left(x_0, x_1\right)\right) \in s\left(Tx_0, Tx_1\right).$$

As $x_1 \in Tx_0$, then

$$\psi\left(d\left(x_0, x_1\right)\right) - \phi\left(d\left(x_0, x_1\right)\right) \in s\left(x_1, Tx_1\right).$$

Using the fact that T is CAV there exists some $x_2 \in Tx_1$ with $x_1 \asymp x_2$ such that

$$s(d(x_1, x_2)) = s(x_1, Tx_1).$$

So we have

$$\psi\left(d\left(x_0, x_1\right)\right) - \phi\left(d\left(x_0, x_1\right)\right) \in s\left(d\left(x_1, x_2\right)\right),$$

which implies

$$d\left(x_1, x_2\right) \preccurlyeq \psi\left(d\left(x_0, x_1\right)\right) - \phi\left(d\left(x_0, x_1\right)\right).$$

Similarly, for $x_2 \in Tx_1$, there exists some $x_3 \in Tx_2$ with $x_2 \asymp x_3$ such that

$$\psi\left(d\left(x_1, x_2\right)\right) - \phi\left(d\left(x_1, x_2\right)\right) \in s\left(d(x_2, x_3)\right).$$

Continuing in the same way we will get the sequence $\{x_n\}$ in X such that

$$x_n \asymp x_{n+1}, \text{ such that } x_{n+1} \in Tx_n \text{ for } n \geq 0.$$

By a similar procedure of Theorem 3.9, we can construct a Cauchy sequence $\{x_n\}$ in X, such that $x_n \to x$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$.

By assumptions we have $\{x_{n_k}\}$, a subsequence of $\{x_n\}$, with $x_{n_k} \asymp x$ for all $k \in \mathbb{N}$. Choose a natural number k_1 such that $d\left(x_{n_k}, x\right) \ll \frac{c}{2}$ for all $n \geq k_1$. Now consider

$$\psi\left(d\left(x_{n_k}, x\right)\right) - \phi\left(d\left(x_{n_k}, x\right)\right) \in s\left(Tx_{n_k}, Tx\right).$$

For $x_{n_{k+1}} \in Tx_{n_k}$, as T is a CAV mapping, we can choose $z_{n_k} \in Tx$ such that $x_{n_k} \asymp z_{n_k}$. We have

$$\psi\left(d\left(x_{n_k}, x\right)\right) - \phi\left(d\left(x_{n_k}, x\right)\right) \in s\left(d(x_{n_{k+1}}, z_{n_k})\right),$$

which implies

$$\begin{aligned} d(x_{n_{k+1}}, z_{n_k}) &\preccurlyeq \psi\left(d\left(x_{n_k}, x\right)\right) - \phi\left(d\left(x_{n_k}, x\right)\right) \\ &\preccurlyeq \psi\left(d\left(x_{n_k}, x\right)\right) \prec d\left(x_{n_k}, x\right). \end{aligned}$$

Now consider

$$\begin{aligned} d(x, z_{n_k}) &\preccurlyeq d(x, x_{n_{k+1}}) + d(x_{n_{k+1}}, z_{n_k}) \\ &\prec d(x, x_{n_{k+1}}) + d\left(x_{n_k}, x\right) \\ &\ll c \text{ for all } k_1(c) \geq n. \end{aligned}$$

Hence $z_{n_k} \to x$, and since Tx is closed, $x \in Tx$.

Example 3.13. Let $X = [0,1]$ be the closed unit interval with the usual partial order $\leq$, and $\mathbb{E}$ be the set of all real valued functions on X which also have continuous derivatives on X. Then $\mathbb{E}$ is a real vector space under the following operations:

$$(x+y)(t) = x(t) + y(t), \quad (\alpha x)(t) = \alpha x(t),$$

for all $x, y \in \mathbb{E}$, $\alpha \in \mathbb{R}$. That is, $E = C_R^1[0,1]$ with norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and

$$K = \{x \in \mathbb{E} : \theta \preceq x\}, \text{ where } \theta(t) = 0 \text{ for all } t \in X,$$

is a non-normal cone in $\mathbb{E}$. Define $d : X \times X \to \mathbb{E}$ as follows:

$$(d(x,y))(t) = |x-y| e^t.$$

Then (X,d) is a complete cone metric space.

Let $T : X \to \mathsf{CB}(X)$ be such that

$$Tx = \left[0, \frac{x}{10\pi}\right];$$

then we have for $x \leq y$

$$s(Tx,Ty) = s\left(\left|\frac{x}{10\pi} - \frac{y}{10\pi}\right| e^t\right).$$

Since

$$\left|\frac{x}{10\pi} - \frac{y}{10\pi}\right| e^t \leq \frac{1}{2\pi}|x-y| e^t - \frac{1}{4\pi}|x-y| e^t,$$

we have

$$\frac{1}{2\pi}|x-y| e^t - \frac{1}{4\pi}|x-y| e^t \in s\left(\left|\frac{x}{10\pi} - \frac{y}{10\pi}\right| e^t\right).$$

Thus for $\psi(t) = \frac{t}{2\pi}$ and $\phi(t) = \frac{t}{4\pi}$ we have

$$\psi(d(x,y)) - \phi(d(x,y)) \in s(Tx,Ty).$$

All conditions of our main theorem are satisfied, so T has a fixed point.

If we take $\phi(t) = \theta$ for all $t \in K$, then we have the following corollary:

Corollary 3.14. *Let $(X,\mathcal{R})$ be a complete cone metric space endowed with partial order $\mathcal{R}$ on X. Let $T : X \longrightarrow \mathsf{C}(X)$ be an* AV *multivalued nondecreasing mapping satisfying*

$$\psi(d(x,y)) \in s(Tx,Ty).$$

If X has sequential limit comparison property, and there exists $x_0 \in X$ such that $\{x_0\}\mathcal{R}Tx_0$, then T has a fixed point in X.

If we take $\phi(t) = \theta$ for all $t \in K$ and $\psi(t) = kt$, for $k \in [0,1)$, then we have the following corollary:

Corollary 3.15. *Let $(X,\mathcal{R})$ be a complete cone metric space endowed with a partial order $\mathcal{R}$ on X. Let $T : X \longrightarrow \mathsf{C}(X)$ be a Nadler type $\mathcal{R}$-contractive mapping. If X has sequential limit comparison property, then T has a fixed point in X.*

Remark 3.16. ([10]) *Let (X,d) be a cone metric space. If $\mathbb{E} = R$ and $K = [0,+\infty)$, then (X,d) is a metric space. Moreover, for $A, B \in \mathsf{CB}(X)$, $H(A,B) = \inf s(A,B)$ is* *the Hausdorff distance induced by* d*, and* $\inf s(a,B) = d(a,B)$*. Also,* $s(\{x\},\{y\}) = s(d(x,y))$ *for all* $x, y \in X$*.*

Using the above remark all the related terms and definitions can be defined in metric spaces and we have the following corollaries:

Corollary 3.17. *Let* $(X, \mathcal{R})$ *be a complete metric space endowed with a partial order* $\mathcal{R}$ *on* X*. Let* $T : X \longrightarrow \mathsf{C}(X)$ *be a* $\mathcal{R}_{\psi,\phi}$ *weakly* U*-contractive(resp.* L*-contractive) mapping. If* X *has sequential limit comparison property, then* T *has a fixed point in* X*.*

Corollary 3.18. *Let* $(X, \mathcal{R})$ *be a complete metric space endowed with a partial order* $\mathcal{R}$ *on* X*. Let* $T : X \longrightarrow \mathsf{C}(X)$ *be an* AV *multivalued nondecreasing mapping satisfying;*

$$\psi(d(x,y)) \in s(Tx, Ty).$$

If X *has sequential limit comparison property, and there exists* $x_0 \in X$ *such that* $\{x_0\}\mathcal{R}Tx_0$*, then* T *has a fixed point in* X*.*

Corollary 3.19. *Let* $(X, \mathcal{R})$ *be a complete metric space endowed with a partial order* $\mathcal{R}$ *on* X*. Let* $T : X \longrightarrow \mathsf{C}(X)$ *be an* AV *multivalued nondecreasing mapping and there exists* $k \in [0, 1)$ *such that*

$$kd(x,y) \in s(Tx, Ty).$$

If X *has sequential limit comparison property, and there exists* $x_0 \in X$*, such that* $\{x_0\}\mathcal{R}Tx_0$*, then* T *has a fixed point in* X*.*

If T is a single valued mapping we have the following results:

Theorem 3.20. *Let* $(X, \mathcal{R})$ *be a complete cone metric space endowed with a partial order* $\mathcal{R}$ *on* X*. Let* $T : X \longrightarrow X$ *be a mapping satisfying;*

$$d(Tx, Ty) \preccurlyeq \psi(d(x,y)) - \phi(d(x,y)).$$

If X *has sequential limit comparison property, and one of the following conditions is satisfied:*

(*a*) $x \preccurlyeq Tx$ *for all* $x \in X$*;*

(*b*) $Tx \preccurlyeq x$ *for all* $x \in X$*;*

(*c*) T *is increasing and there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$*;*

(*d*) Tx *is comparable with each* $x \in X$ *(*X *satisfies subsequential limit comparison property instead of sequential limit comparison property),*

then T *has a fixed point in* X*.*

Proof. Case (a): It is easy to construct a sequence $\{x_n\}$ in X such that $x_n \preccurlyeq Tx_n = x_{n+1}$ for $n = 0, 1, 2, 3 \cdots$ Also

$$d(x_n, x_{n-1}) \to \theta, \text{ as } n \to \infty. \tag{g}$$

We need to show that $\{x_n\}$ is a Cauchy sequence. Suppose contrary that $\{x_n\}$ is not a Cauchy sequence; then there exists an $\varepsilon \gg 0$ for which we can find two sequences of positive integers (say) $\{m(j)\}$ and $\{n(j)\}$ such that for all positive integers j,

$n(j) > m(j) > j$ and $d(x_{m(j)}, x_{n(j)}) \succcurlyeq \varepsilon$. Assuming that $n(j)$ is the smallest such positive integer, we get

$$d(x_{m(j)}, x_{n(j)}) \succcurlyeq \varepsilon$$

and

$$d(x_{m(j)}, x_{n(j)-1}) \ll \varepsilon.$$

Now

$$\varepsilon \preccurlyeq d\left(x_{m(j)}, x_{n(j)}\right) \preccurlyeq d\left(x_{m(j)}, x_{n(j)-1}\right) + d\left(x_{n(j)-1}, x_{n(j)}\right) \ll \varepsilon + d\left(x_{n(j)-1}, x_{n(j)}\right).$$

Letting $j \to \infty$ in the above inequality and using (g), we get

$$\lim_{j\to\infty} d(x_{m(j)}, x_{n(j)}) = \varepsilon. \tag{h}$$

Consider, by using the triangular property of cone metric space,

$$d(x_{m(j)+1}, x_{n(j)+1}) \preccurlyeq d(x_{m(j)+1}, x_{m(j)}) + d(x_{m(j)}, x_{n(j)}) + d(x_{n(j)}, x_{n(j)+1})$$

and

$$d(x_{m(j)}, x_{n(j)}) \preccurlyeq d(x_{m(j)}, x_{m(j)+1}) + d(x_{m(j)+1}, x_{n(j)+1}) + d(x_{n(j)+1}, x_{n(j)}).$$

Letting $j \to \infty$ in the above inequalities and using (g) and (h), we get

$$\lim_{j\to\infty} d(x_{m(j)+1}, x_{n(j)+1}) = \varepsilon. \tag{i}$$

By transitivity of $\mathcal{R}$ we have $x_{m(j)+1} \preccurlyeq x_{n(j)+1}$, hence we have

$$\begin{aligned} \psi\left(d\left(x_{m(j)+1}, x_{n(j)+1}\right)\right) &\ll d\left(x_{m(j)+1}, x_{n(j)+1}\right) \\ &\preccurlyeq \psi\left(d\left(x_{m(j)}, x_{n(j)}\right)\right) - \phi\left(d\left(x_{m(j)}, x_{n(j)}\right)\right). \end{aligned}$$

Letting $j \to \infty$ in the above inequalities, using (b), (h) and the continuity of ϕ, we obtain

$$\begin{aligned} r &\ll r - \lim_{j\to\infty} \phi\left(d\left(x_{n(j)}, x_{m(j)}\right)\right) \\ \theta &\gg \phi(\lim_{j\to\infty} d\left(x_{n(j)}, x_{m(j)}\right)), \\ \phi(\varepsilon) &\ll \theta \end{aligned}$$

which is a contradiction. By definition of ϕ we have

$$\lim_{j\to\infty} d\left(x_{n(j)}, x_{m(j)}\right) = \theta.$$

Thus $\{x_n\}$ is a Cauchy sequence. By completeness of X there exists $z \in X$ such that $x_n \to z$. Choose $k(c) \in \mathbb{N}$ such that for $\theta \ll c$, we have $d(x_n, z) \ll \frac{c}{2}$ for all $n \geq k(c)$. By virtue of sequential limit comparison property of X we have $x_n \preccurlyeq z$ for all $n \in \mathbb{N}$. Consider

$$\begin{aligned} d(x_{n+1}, Tz) &= d(Tx_n, Tz) \preccurlyeq \psi(d(x_n, z)) - \phi(d(x_n, z)) \\ &\preccurlyeq \psi(d(x_n, z)) - \phi(d(x_n, z)) \\ &\preccurlyeq \psi(d(x_n, z)) \ll d(x_n, z). \end{aligned} \tag{3.2}$$

Now

$$d(z, Tz) \preccurlyeq d(z, x_{n+1}) + d(x_{n+1}, Tz) \ll d(z, x_{n+1}) + d(x_n, z) \ \ \text{(using (3.2))} \ \ll c$$

for all $n \geq k(c)$.

Thus $z = Tz$ is the fixed point of T.

Case (b): As $Tx \preccurlyeq x$, then form a sequence $x_{n+1} = Tx_n \preccurlyeq x_n$ for $n = 0, 1, 2, 3 \cdots$ Also $d(x_n, x_{n-1}) \to \theta$, as $n \to \infty$. The remaining proof is similar to the above part.

Case (c): As $x_0 \preccurlyeq Tx_0 = x_1$ and T is increasing, one can form a sequence $x_n \preccurlyeq x_{n+1}$ and the proof coincides with proof of Case (a).

Case (d): As $x \asymp Tx$ we can construct a sequence $x_{n+1} \asymp x_n$ for $n = 0, 1, 2, \cdots$ It is easy to prove that $\{x_n\}$ is a Cauchy sequence, so that $x_n \to z \in X$. By subsequential limit property of X we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ consisting of terms comparable to z. Therefore for each k

$$d(x_{n_{k+1}}, Tz) = d(Tx_{n_k}, Tz) \preccurlyeq \psi\left(d\left(x_{n_k}, z\right)\right) - \phi\left(d\left(x_{n_k}, z\right)\right) \ll d\left(x_{n_k}, z\right).$$

Now we have

$$d(Tz, z) \preccurlyeq d(Tz, x_{n_{k+1}}) + d(x_{n_{k+1}}, z) \ll c$$

for all $n \geq k_1$. This completes the proof.

4. Applications

Consider the following differential inclusion:

$$\begin{cases} \frac{\partial^3 u(t,x,y)}{\partial t \partial x \partial y} \in F(t, x, y, u(t, x, y)) \text{ for } (t, x, y) \in \Omega_a \times \Omega_b \times \Omega_c := \Omega_{a,b,c} \text{ with} \\ u(t, x, 0) = \eta(t, x), \ u(t, o, y) = \zeta(t, y), \ u(0, x, y) = \vartheta(x, y), \ (t, x, y) \in \Omega_{a,b,c}, \end{cases} \tag{4.1}$$

where $\Omega_a = [0, a]$, $\Omega_b = [0, b]$ and $\Omega_c = [0, c]$.

Let $F : \Omega_{a,b,c} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multivalued mapping and $C(\Omega_{a,b,c}, \mathbb{R}^n)$ be a complete metric space with the metric

$$(d(u, v))(t, x, y) = \sup\left\{|u(t, x, y) - v(t, x, y)| : (t, x, y) \in \Omega_{a,b,c}\right\}$$

for $u, v \in C(\Omega_{a,b,c}, \mathbb{R}^n)$.

For $u, v \in C(\Omega_{a,b,c}, \mathbb{R}^n)$ we define the partial order $u\mathcal{R}v$ if and only if $u(t, x, y) \leq v(t, x, y)$ for each $(t, x, y) \in \Omega_{a,b,c}$.

Let

$$K = \{u(t, x, y) : u(t, x, y) \in C(\Omega_{a,b,c}, \mathbb{R}^n), \ u(t, x, y) \geq 0, \ \text{ for each } (t, x, y) \in \Omega_{a,b,c}\}$$

be the cone of $C(\Omega_{a,b,c}, \mathbb{R}^n)$.

Let $L^1(\Omega_{a,b,c}, \mathbb{R}^n)$ be the Banach space of all measurable functions from $\Omega_{a,b,c}$ to $\mathbb{R}^n$ which are Lebesgue integrable with norm

$$\|u\|_L = \int_0^a \int_0^b \int_0^c |u(t, x, y)| \, dydxdt \ \text{ for } u \in L^1(\Omega_{a,b,c}, \mathbb{R}^n).$$

Let $F : \Omega_{a,b,c} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multivalued mapping having non-empty values. For each $u \in C(\Omega_{a,b,c}, \mathbb{R}^n)$ denote and define the set of selections of F by

$$\mathcal{S}_{F,u} = \left\{v \in L^1(\Omega_{a,b,c}, \mathbb{R}^n) : v \in F(t, x, y, u(t, x, y)) \ a.e \ (t, x, y) \in \Omega_{a,b,c}\right\}$$

and assign to F the multivalued operator $N : C(\Omega_{a,b,c},\mathbb{R}^n) \to 2^{L^1(\Omega_{a,b,c},\mathbb{R}^n)}$ by letting

$$N(u) = \left\{w \in L^1(\Omega_{a,b,c},\mathbb{R}^n) : w(t,x,y) \in F(t,x,y,u(t,x,y)),\ (t,x,y) \in \Omega_{a,b,c}\right\},$$

where N is the Niemytsky operator associated with F.

In order to state and verify our theorem, we need the continuous mapping $\mathcal{L} : L^1(\Omega_{a,b,c},\mathbb{R}^n) \to C(\Omega_{a,b,c},\mathbb{R}^n)$ defined by

$$\mathcal{L}(u(t,x,y)) = \int_0^t\int_0^x\int_0^y u(\nu,s,\tau)d\nu ds d\tau.$$

Theorem 4.1. *Suppose the multivalued mapping $F : \Omega_{a,b,c} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ satisfies the following conditions:*

L1: $F(t,x,u)$ is compact subset for all $(t,x,y,u) \in \Omega_{a,b,c} \times C(\Omega_{a,b,c},\mathbb{R}^n)$. Moreover, $\mathcal{S}_{F,u}$ is non-empty for each $u \in C(\Omega_{a,b,c},\mathbb{R}^n)$;

L2: For any $u, v \in C(\Omega_{a,b,c},\mathbb{R}^n)$, if $u\mathcal{R}v$ then for each $u_1 \in F(t,x,y,u(t,x,y)$ there exists $u_2 \in F(t,x,y,v(t,x,y))$ such that

$$|u_1(t,x,y) - u_2(t,x,y)| \le l(t,x,y)\ln(|u(t,x,y) - v(t,x,y)| + e)$$

for a.e $(t,x,y) \in \Omega_{a,b,c}$, where $l \in L^1(\Omega_{a,b,c},\mathbb{R}^n)$ with $\|l\|_L \le 1$;

L3: For each $u \in C(\Omega_{a,b,c},\mathbb{R}^n)$ implies

$$\mathcal{L}\circ v(t,x,y)+\vartheta(x,y)+\eta(t,x)-\eta(0,x)+\zeta(t,y)-\zeta(0,y)-\zeta(t,0)+\zeta(0,0) \in s(u(t,x,y))$$

for $(t,x,y) \in \Omega_{a,b,c}$ and $v \in \mathcal{S}_{F,u}$.

Then the problem (4.1) *has a solution $u^* \in C(\Omega_{a,b,c},\mathbb{R}^n)$.*

Proof. $\mathbb{E} = C(\Omega_{a,b,c},\mathbb{R}^n)$ is a complete metric space with metric

$$(d(u,v))(t,x) = \sup\{|u(t,x,y) - v(t,x,y)| : (t,x,y) \in \Omega_{a,b,c}\}$$

and satisfies the sequential limit comparison property. The problem (4.1) is equivalent to the following integral inclusion,

$$u(t,x,y) \in \left\{ \begin{array}{c} h \in \mathbb{E} : h(t,x,y) = \vartheta(x,y) + \eta(t,x) - \eta(0,x) + \zeta(t,y) - \zeta(0,y) \\ -\zeta(t,0) + \zeta(0,0) + \int_0^t\int_0^x\int_0^y v(r,s,\tau)drdsd\tau \text{ for } v \in \mathcal{S}_{F,u} \end{array} \right\}.$$

Define $T : \mathbb{E} \to 2^{\mathbb{E}}$ by

$$(Tu)(t,x,y) = \left\{ \begin{array}{l} h \in \mathbb{E} : h(t,x,y) = \vartheta(x,y) + \eta(t,x) - \eta(0,x) + \zeta(t,y) \\ -\zeta(0,y) - \zeta(t,0) + \zeta(0,0) + \mathcal{L}\circ v(t,x,y) \text{ for } v \in \mathcal{S}_{F,u} \end{array} \right\}.$$

Prove $(Tu)(t,x,y)$ is compact for each $u \in \mathbb{E}$ and $v \in \mathcal{S}_{F,u}$. It is enough to show that $\mathcal{L} \circ \mathcal{S}_{F,u}$ is compact. For this suppose $u \in \mathbb{E}$ and $\{u_n\}$ is a sequence in $\mathcal{S}_{F,u}$. Then by definition of $\mathcal{S}_{F,u}$ we get $u_n \in F(t,x,y,u(t,x,y))$ a.e for all $(t,x,y) \in \Omega_{a,b,c}$. So $u_n(t,x,y) \to v(t,x,y)$ for some $v(t,x,y) \in F(t,x,y,u(t,x,y))$, as $F(t,x,y,u(t,x,y))$ is compact. By continuity of $\mathcal{L}$ we get $\mathcal{L}\circ u_n(t,x,y) \to \mathcal{L}\circ u(t,x,y) \in \mathcal{L} \circ \mathcal{S}_{F,u}$ a.e for all $(t,x,y) \in \Omega_{a,b,c}$, as required. Now let $u, v \in K$ with $u\mathcal{R}v$ and let $h_1 \in Tu$; then there exists $v_1 \in \mathcal{S}_{F,u}$ such that

$$h_1(t,x,y) = \vartheta(x,y) + \eta(t,x) - \eta(0,x) + \zeta(t,y) - \zeta(0,y) - \zeta(t,0)$$

$$+\zeta(0,0)+\int_0^t\int_0^x\int_0^y v_1(r,s,\tau)drdsd\tau \text{ for } (t,x,y)\in\Omega_{a,b,y}.$$

There exists $w\in F(t,x,y,v(t,x,y))$ with

$$|v_1(t,x,y)-w|\leq l(t,x,y)\ln(|v(t,x,y)-u(t,x,y)|+e).$$

Define the multivalued mapping

$$U(t,x,y)=\{w\in\mathbb{R}^n:|v_1(t,x,y)-w|\leq l(t,x,y)\ln(|v(t,x,y)-u(t,x,y)|+e)\};$$

then the multivalued mapping

$$V(t,x,y)=U(t,x,y)\cap\mathcal{S}_{F,u}$$

has non-empty values and is a measurable selection [8]. So there exists $v_2\in V$ with $v_2\in F(t,x,y,v(t,x,y))$ for all $(t,x,y)\in\Omega_{a,b,c}$, satisfying

$$|v_1(t,x,y)-v_2(t,x,y)|\leq l(t,x,y)\ln(|v(t,x,y)-u(t,x,y)|+e).$$

Define for each $(t,x,y)\in\Omega_{a,b,c}$,

$$h_2(t,x)=\vartheta(x,y)+\eta(t,x)-\eta(0,x)+\zeta(t,y)-\zeta(0,y)-\zeta(t,0)$$
$$+\zeta(0,0)+\int_0^t\int_0^x\int_0^y v_2(r,s,\tau)drdsd\tau\in(Tv)(t,x,y).$$

Then

$$\begin{aligned}|h_2(t,x)-h_1(t,x)| &\leq \int_0^t\int_0^x\int_0^y|v_2(r,s,\tau)-v_1(r,s,\tau)|\,drdsd\tau\\ &\leq \|l\|_L\ln(d(u,v)+e)\\ &= \|l\|_L\,[d(u,v)-(d(u,v)-\ln(d(u,v)+e)]\\ &= kd(u,v)-k(d(u,v)-\ln(d(u,v)+e),\ \text{where } \|l\|_L=k.\end{aligned}$$

Thus

$$kd(u,v)-k(d(u,v)-\ln(d(u,v)+e)\in s(Tu,Tv).$$

Taking $\psi(t)=kt$, $\phi(t)=kt-k\ln(t+e)$ and $f=I$, we have

$$\psi(d(fu,fv))-\phi(d(fu,fv))\in s(Tu,Tv).$$

Thus all the conditions of Theorem 3.9 are satisfied to obtain $u^*\in\mathbb{E}$, such that $u^*\in Tu^*$, so a solution of (4.1) exists.

Example 4.2. Let $E\subset K$ be the set of all functions satisfying $7\pi u(t,x,y)\leq\mathcal{L}\circ u(t,x,y)$. Then E is a complete metric space with metric

$$(d(u,v))(t,x)=\sup\{|u(t,x,y)-v(t,x,y)|:(t,x,y)\in\Omega_{a,b,c}\}$$

and satisfies the condition $(L1)$ of the above theorem. Let

$$F(t,x,y,u)=[\frac{1}{7}u(t,x,y)e^{-(t+x+y)}-t-x-y,\frac{1}{5}u\,(t,x,y)\,e^{-(t+x+y)}-t-x-y$$

for $u \in E$. Then F satisfies the condition $(L2)$ of the above theorem.
Take $v \in \mathcal{S}_{F,u}$ then $\frac{1}{7}u(t,x,y)e^{-(t+x+y)} - t - x - y \leq v$. So we have

$$\begin{aligned} u\,(t,x,y) - t - x - y &\leq \frac{1}{7\pi}\mathcal{L}\circ u(t,x,y) - t - x - y \\ &\leq \frac{1}{7}\int_0^t\int_0^x\int_0^y e^{-(\tau+\eta+r)}u(\tau,\eta,r)d\tau d\eta dr \\ &- \frac{xy}{2}t^2 - \frac{ty}{2}x^2 - \frac{tx}{2}y^2 \leq \mathcal{L}\circ v(t,x,y). \end{aligned}$$

Hence F satisfies the condition $(L3)$ of the above theorem.

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