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REMARKS ON PASICKI'S ABSTRACT METRIC SPACES AND LIFSHITS'S CONSTANT

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Abstract. In attempting to formulate a more axiomatic approach to metric fixed point theory, some recent concepts due to Lech Pasicki give rise to a number of intriguing problems, especially regarding the Lifshits constant of such spaces. We review these results here, with an emphasis on a number of questions that seem to remain open. For the convenience of the reader certain proofs of known results are also included. Some new observations are also included.

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1. Discus spaces

In the following we use $B(x;t)$ to denote the closed balls centered at x with radius $t>0$. In these remarks we consider only complete metric spaces. The following definition was introduced by L. Pasicki in 2006 (although it was originally formulated in terms of open balls, the formulation given here is equivalent).

Definition 1.2. [19] A complete metric space (X,d) is a *discus space* if there exists a mapping $\rho:[0,\infty)\times(0,\infty)\to[0,\infty)$ such that
(1) $\rho(\beta,r)<\rho(0,r)=r$ for all $\beta,r>0$,
(2) $\rho(\cdot,r)$ is nonincreasing for all $r>0$,
(3) $\rho(\varepsilon,\cdot)$ is upper semicontinuous for all $\varepsilon>0$,
(4) for each $x,y\in X$ and r, $\epsilon>0$ there exists $z\in X$ such that

$$B(x;r)\cap B(y;r)\subset B(z;\rho(d(x,y),r)+\epsilon).$$

Recall that a function $\varphi:M\to\mathbb{R}$ is upper semicontinuous if given any sequence $\{x_n\}$ in M, the conditions $\lim_n x_n=x$ and $\lim_n\varphi(x_n)=r\Rightarrow\varphi(x)\geq r$. This is equivalent to saying that the set $\{u\in M:\varphi(u)\geq r\}$ is closed.

Example 1.2. [19] Let $(Y,(\cdot,\cdot))$ be a Hilbert space with $\|x\|=\sqrt{(x,x)}$. Then

$$\|x+h\|^2+\|x-h\|^2=2\left(\|x\|^2+\|h\|^2\right)$$

and hence

$$\|h\|^2 = \frac{1}{2}\left(\|x+h\|^2 + \|x-h\|^2 - 2\|x\|^2\right).$$

Now let $x, y \in Y$ and $r > 0$. By translation we may assume $y = -x$. Set $2\|x\| = \varepsilon$ and choose h so that $\|x+h\| = \|x-h\| = r$ (assuming $\varepsilon \leq 2r$). Define

$$\rho(\delta, r) = \|h\| = \begin{cases} \sqrt{r^2 - \delta^2/4} & \text{if } \delta \in [0, 2r] \\ 0 & \text{if } \delta > 2r. \end{cases}$$

Then if $\|x-u\| \leq r$ and $\|x+u\| \leq r$ it must be the case that $\|u\| \leq \|h\| = \rho(2\|x\|, r)$ and (4) is satisfied for $z = 0$. Thus each nonempty convex set $X \subset Y$ is a discus space for the given ρ.

If a Banach space E is uniformly convex there exists a strictly increasing surjection $\delta : [0,2] \to [0,1]$ such that $\|x\| \leq R$, $\|y\| \leq R$ and $\|x-y\| \geq \varepsilon$ implies $\|(x+y)/2\| \leq R(1 - \delta(\varepsilon/R))$. Here δ us the usual modulus of convexity, which is known to be continuous (see [8], pp. 52-55).

Example 1.3. [19] Let $(Y, \|\cdot\|)$ be a uniformly convex Banach space. By choosing $x, y \in Y$, with $y = -x$, and suppose $u \in B(x; r) \cap B(-x; r)$. Then $\|(u-x)\| \leq r$, and $\|(u+x)\| \leq r$. Hence $\left\|\frac{(u-x)+(u+x)}{2}\right\| = \|u\| \leq r(1 - \delta(\varepsilon/r))$ where $\varepsilon = 2\|x\|$. Thus

$$u \in B(x; r) \cap B(-x; r) \Rightarrow u \in B(0; (1 - \delta(\varepsilon/r))r).$$

Therefore (4) is satisfied if we define

$$\rho(\varepsilon, r) = \begin{cases} r(1 - \delta(\varepsilon/r)) & \text{if } \varepsilon \in [0, 2r] \\ 0 & \text{if } \varepsilon > 2r. \end{cases}$$

By now, the role of CAT(0) spaces in metric fixed point theory is well understood. These spaces are geodesically connected, and every geodesic triangle in such a space is at least as 'thin' as its comparison triangle in the Euclidean plane. See, e.g., [3], [14] for a discussion. The relevant property is that if x_1, x_2, x_3 are three points of a CAT(0) space and if $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are three points of $\mathbb{R}^2$ for which $d(x_i, x_j) = d(\bar{x}_i, \bar{x}_j)$, $i, j = 1, 2, 3$, then points of the sides of the triangle $\Delta(x_1, x_2, x_3)$ are no longer than the distance between their corresponding counterparts on the triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$.

Example 1.4. A CAT(0) space (X, d) is a discus space. Define

$$\rho(\delta, r) = \begin{cases} \sqrt{r^2 - \delta^2/4} & \text{if } \delta \in [0, 2r] \\ 0 & \text{if } \delta > 2r. \end{cases}$$

For each $x, y \in X$, let z be the midpoint of the segment $[x, y]$. Suppose $d(x, u) \leq r$ and $d(y, u) \leq r$. Choose a comparison triangle $\Delta(\bar{x}, \bar{y}, \bar{u})$ for $\Delta(x, y, u)$ in $\mathbb{R}^2$ so that $\bar{y} = -\bar{x}$. Then $d(z, u) \leq d_{\mathbb{R}^2}(0, \bar{u}) = \rho(d(\bar{x}, \bar{y})) = \rho(d(x, y))$.

Because many theorems in metric fixed point theory, especially those involving nonexpansive mappings, hold in both uniformly convex Banach spaces and also in complete CAT(0) spaces, the notion of a complete discus space would appear to be a unifying concept, despite the fact that these spaces are quite different. In fact the *only* Banach spaces that are also complete CAT(0) spaces are the Hilbert spaces.

2. Complete discus spaces

Theorem 2.1. [19] *Let (X, d) be a (complete) discus space and let $A \subset X$ be nonempty and bounded. Then the Chebyshev center of A relative to X is a singleton.*

Discus spaces are defined solely in terms of balls and a related a priori parameter. This gives rise to our first question.

Question 2.2. Let (X, d) be a discus space and let B be a closed ball in X. Suppose $T : B \to X$ is nonexpansive and satisfies

$$\inf \{d(x, T(x)) : x \in B\} = 0.$$

Then does T have a fixed point? In particular, we note that the answer is affirmative if X is a CAT(0) space (see [10, Theorem 21]).

A metric space (X, d) is said to be *metrically convex* ([18]) if for any $x, y \in X$ with $x \neq y$ there exists $z \in X$ with $x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Menger has shown [18] (also see, e.g. [2]) that in a complete metrically convex space each two point are the endpoints of at least one metric segment (i.e., an isometric image of a real line interval).

Pasicki introduces a 'seemingly' more general class of spaces, called bead spaces, in [20].

3. Bead spaces

In 2009 Pasicki introduced the following definition.

Definition 3.1. [20] A metric space (X, ρ) is a *bead space* if the following is satisfied:

$$\begin{aligned} \forall R, \beta &> 0 \; \exists \, \delta = \delta(R, \beta) > 0 \text{ such that } \forall x, y \in X \text{ with } \rho(x, y) \geq \beta \\ \exists \, z &\in X \text{ such that } B(x; R + \delta) \cap B(y; R + \delta) \subset B(z; R - \delta). \end{aligned} \tag{3.1}$$

In a rather remarkable subsequent development, Pasicki proved the following.

Theorem 3.2. [21] *Any metric space is a bead space if and only if it is a discus space.*

Lemma 3.3. [20] *Let (X, ρ) be a bead space and let $\emptyset \neq A \subset X$ be bounded. Then the Chebyshev center of A consists of at most one point. If, in addition, X is complete, the Chebyshev center of A is a singleton.*

Proof. Let (R_n) decrease to $R = R(A)$ while $A \subset B(x_n; R_n)$. Suppose (x_n) is not a Cauchy sequence. Thus for some $\beta > 0$, $\rho(x_n, x_k) \geq \beta$ for infinitely many pairs (n, k) with $n > k$. Let δ be the constant associated with R and β according to (3.1). Then for $R_k < R_n < \delta$ we have

$$A \subset B(x_n; R_n) \cap B(x_k; R_k) \subset B(x_n; R + \delta) \cap B(x_k; R + \delta) \subset B(z; R - \delta).$$

This implies $R(A) \leq R - \delta$, which is a contradiction. Since X is complete we conclude that (x_n) converges, say to $x \in X$. Hence for any $\xi > 0$ there exists $n_0 \in \mathbb{N}$ such that $A \subset B(x_n; R_n) \subset B(x; R + \xi)$ for all $n \geq n_0$. This in turn implies $A \subset B(x; R)$.

Remark 3.4. A discus space not necessarily metrically convex in the sense of Menger: see Example 3 of [22].

4. The Lifshits constant

Definition 4.1. The *characteristic of convexity* of a Banach space X is defined to be the number

$$\varepsilon_0(X) := \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\},$$

where δ_X is the usual modulus of convexity of X.

(Since δ_X is continuous, $\delta_X(\varepsilon_0(X)) = 0$.)

Observe that if γ satisfies

$$\gamma(1 - \delta_X(1/\gamma)) = 1$$

then $\gamma \geq 1$. Moreover $\gamma > 1 \Leftrightarrow \varepsilon_0(X) < 1$, in which case $1 < \gamma < 1/\varepsilon_0(X)$.

Theorem 4.2. [7], [9] *Let X be a Banach space with $\varepsilon_0(X) < 1$, and let $\gamma > 1$ satisfy $\gamma(1 - \delta_X(1/\gamma)) = 1$. If K is a nonempty closed, bounded convex subset of a Banach space, and if $T : K \to K$ is k-uniformly lipschitzian for $k < \gamma$, then T has a fixed point.*

Definition 4.3. The *Lifshits constant* $\kappa(M)$ of a metric space $(M;\rho)$ is defined as follows:

$$\kappa(M) := \sup\{\beta > 0 : \exists\ \alpha > 1 \text{ such that } \forall x, y \in M \text{ and } r > 0,$$

$$\rho(x,y) > r \Rightarrow \exists\ z \in M \text{ such that } B(x;\beta r) \cap B(y;\alpha r) \subset B(z;r)\}.$$

It is clear that $\kappa(M) \geq 1$ for any metric space. (If $\beta < 1$ then obviously $B(x;\beta r) \cap B(y;\alpha r) \subset B(x;r)$ for any $\alpha > 1$ so one may take $z = x$.Thus $\kappa(M) \geq \beta$ for any $\beta < 1$.)

Theorem 4.4. (Lifshits [16]) *If (M,ρ) is a bounded complete metric space and if $T : M \to M$ is uniformly k-lipschitzian for $k < \kappa(M)$, then T has a fixed point.*

A more general version of Lifshits's theorem is proved in [13] A mapping $f : X \to X$ is said to be *eventually k-lipschitzian* if there exists $n_0 \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \leq kd(x,y)$ for all $x, y \in X$ and $n \geq n_0$. The Lifshits character is fundamental in metric fixed point theory because of the following result.

Theorem 4.5. *Let (X,d) be a complete metric space. Then every eventually k-lipschitzian mapping $T : X \to X$ with $k < \kappa(X)$ has a fixed point if it has a bounded orbit.*

Now let X be a Banach space and let $\kappa_0(X)$ be the infimum of $\kappa(C)$ where C ranges over all nonempty bounded closed convex subsets of X. Lifshits proved that $\kappa_0(X) \geq \sqrt{2}$ if X is a Hilbert space. It is noted in [9] that in Hilbert space $\gamma = \sqrt{5}/2$ is the solution to $\gamma(1 - \delta_X(1/\gamma)) = 1$. Therefore for a Hilbert space Lifshits's estimate on k is better than that given in Theorem 4.2.

Theorem 4.6. [6] *Let X be a Banach space and assume $\gamma > 1$ satisfies*

$$\gamma(1 - \delta_X(1/\gamma)) = 1.$$

Then $\gamma \leq \kappa_0(X)$.

In [6] Downing and Turett attribute the following lemma to Lifshits. For convenience we give a proof here.

Lemma 4.7. *Let X be a normed linear space. Then*

$$\begin{aligned} \kappa_0(X) &\geq \sup\{\beta > 0 : \textit{for some } \alpha > 1 \textit{ and all } y \in X \textit{ with } \|y\| > 1, \\ \textit{there exists } t &\in [0,1] \textit{ with } B(0;\beta) \cap B(y;\alpha) \subset B(ty;1)\}. \end{aligned}$$

Proof. Let C be a bounded closed convex subset of X. Set

$$\begin{aligned} \kappa_1 &= \{\beta > 0 : \text{for some } \alpha > 1 \text{ and all } y \in X \text{ with } \|y\| > 1, \\ \exists\, t &\in [0,1] \text{ such that } B(0;\beta) \cap B(y;\alpha) \subset B(ty;1)\}. \end{aligned}$$

If

$$\begin{aligned} \kappa &= \{\beta > 0 : \text{for some } \alpha > 1 \text{ and all } x, y \in C \text{ and all } r > 0, \|x-y\| > r \\ &\Rightarrow \exists\, z \in C \text{ such that } B(x;\beta r) \cap B(y;\alpha r) \subset B(z;r)\}, \end{aligned}$$

then

$$\begin{aligned} \kappa &= \{\beta > 0 : \text{for some } \alpha > 1 \text{ and all } r > 0,\ x, y \in \frac{1}{r}C \text{ with } \|x-y\| > 1 \\ &\Rightarrow \exists\, z \in C \text{ such that } B(x;\beta) \cap B(y;\alpha) \subset B(z;1)\}. \end{aligned}$$

Without loss of generality (by translation) we may assume $x = 0$. Thus

$$\begin{aligned} \kappa &= \{\beta > 0 : \text{for some } \alpha > 1 \text{ and all } r > 0\ y \in \frac{1}{r}C \text{ with } \|y\| > 1 \\ &\Rightarrow \exists\, z \in C \text{ such that } B(0;\beta) \cap B(y;\alpha) \subset B(z;1)\}. \end{aligned}$$

Now we see that if $u \in \kappa_1$ then $u \in \kappa$. Hence $\sup \kappa \geq \sup \kappa_1$. It follows that $\kappa(C) \geq \sup \kappa_1$, hence $\kappa_0(X) \geq \sup \kappa_1$.

Question 4.8. If $\varepsilon_0(X) < 1$ then $1 < \gamma < 1/\varepsilon_0(X)$. Is there a relation between $\kappa_0(X)$ and $1/\varepsilon_0(X)$?

The next fact shows that while Lifshits's result gives a better estimate for k, qualitatively Theorems 4.2 and 4.4 are equivalent.

Theorem 4.9. [6] *Let X be a Banach space. Then $\varepsilon_0(X) < 1$ if and only if $\kappa_0(X) > 1$.*

It follows that for a Banach space X, $\kappa_0(X) > 1 \Rightarrow X$ has normal structure. Does this fact carry over to metric spaces? Specifically recall that the *admissible* subsets of a metric space are precisely those sets that are intersections of closed balls.

Question 4.10. If M is a complete metric space for which $\kappa(M) > 1$, are the admissible subsets of M normal?

Now suppose $\kappa(M) > 1$ for a metric space M. Choose $1 < \beta < \kappa(M)$. Then there exists $\alpha > 1$ such that for any $r > 0$ and $x, y \in M$, $\rho(x,y) > r \Rightarrow \exists z \in M$ such that $B(x;\beta r) \cap B(y;\alpha r) \subset B(z;r)$. Taking $\lambda = \min\{\beta, \alpha\}$ we conclude that there exist $\lambda > 1$ such that for any $r > 0$ and $x, y \in M$, $\rho(x,y) > r \Rightarrow \exists\, z \in M$ such that $B(x;\lambda r) \cap B(y;\lambda r) \subset B(z;r)$. Therefore we have the following qualitative fact.

Proposition 4.11. *For a metric space M, $\kappa(M) > 1 \Leftrightarrow \exists\, \lambda > 1$ such that for any $r > 0$ and $x, y \in M$, $\rho(x,y) > r \Rightarrow \exists\, z \in M$ such that $B(x;\lambda r) \cap B(y;\lambda r) \subset B(z;r)$. Moreover, $\kappa(M) \geq \lambda$.*

Let $\Lambda(M) = \sup\{\lambda :$ for any $r > 0$ and $x, y \in M$,
$\rho(x,y) > r \Rightarrow \exists\, z \in M$ such that $B(x;\lambda r) \cap B(y;\lambda r) \subset B(z;r)\}$.

Clearly $\kappa(M) \geq \Lambda(M)$.

Now let X be a Banach space and define $\Lambda_0(X)$ to be the infimum of $\Lambda(C)$ where C ranges over all bounded convex subsets of X.

Question 4.12. How is $\Lambda(M)$ related to the characteristic of convexity? Is this a metric space analog of the Banach space characteristic of convexity?

Question 4.13. In a Banach space does $\varepsilon_0(X) = 1/\Lambda_0(X)$?

Lemma 4.14. *Let X be a normed linear space. Then*
$\Lambda_0(X) \geq \sup\{\beta > 0$ *: for all $y \in X$ with $\|y\| > 1$, there exists $t \in [0,1]$ with* $B(0;\beta) \cap B(y;\beta) \subset B(ty;1)\}$.

Proof. Let C be a bounded closed convex subset of X. Set

$$\begin{aligned} \Lambda_1 &= \{\beta > 0 : \text{for all } y \in X \text{ with } \|y\| > 1, \exists\, t \in [0,1] \\ \text{such that } B(0;\beta) \cap B(y;\beta) &\subset B(ty;1)\}. \end{aligned}$$

If

$$\begin{aligned} \Lambda &= \{\beta > 0 : \text{for all } x, y \in C \text{ and all } r > 0, \|x-y\| > r \\ &\Rightarrow \exists\, z \in C \text{ such that } B(x;\beta r) \cap B(y;\beta) \subset B(z;r)\}, \end{aligned}$$

then

$$\Lambda = \{\beta > 0 : \text{for all } r > 0 \text{ if } x, y \in \frac{1}{r}C$$

with $\|x-y\| > 1. \Rightarrow$

$$\exists\, z \in C \text{ such that } B(x;\beta) \cap B(y;\beta) \subset B(z;1)\}.$$

Without loss of generality (by translation) we may assume $x = 0$. Thus

$$\Lambda = \{\beta > 0 : \text{for all } r > 0 \text{ if } y \in \frac{1}{r}C$$

with $\|y\| > 1 \Rightarrow \exists\, z \in C$ such that

$$B(0;\beta) \cap B(y;\alpha) \subset B(z;1)\}.$$

Now we see that if $u \in \Lambda_1$ then $u \in \Lambda$. Hence $\sup \Lambda \geq \sup \Lambda_1$. It follows that $\Lambda(C) \geq \sup \Lambda_1$; hence $\Lambda_0(X) \geq \sup \Lambda_1$.

Theorem 4.15. *Let X be a Banach space and assume $\gamma > 1$ satisfies*

$$\gamma(1 - \delta_X(1/\gamma)) = 1.$$

Then $\gamma \leq \Lambda_0(X)$.

5. More on Lifshits's theorem

Now let (M, ρ) be a complete metric space. The balls in M are said to be *c-regular* if for each $k < c\ \exists\ \mu, \alpha \in (0,1)$ such that $\forall\ x, y \in M$ and $r > 0$

The balls in M are always 1-regular. To see this suppose $k < 1$. Then it is possible to choose μ so near 0 that

$$k(1+\mu) := \alpha < 1.$$

In which case $\forall\ x, y \in M$ and $r > 0$,

$$B(x;(1+\mu)r) \cap B(y;k(1+\mu)r) \subset B(y;\alpha r).$$

With this notation, the *Lifshits constant* of M is the number

$$\kappa(M) = \sup\{c \geq 1 : \text{ the balls in } M \text{ are } c\text{-regular}\}.$$

To see that the above definition is equivalent to the one given earlier, suppose $k < c$ and choose μ, α so that $k\left(\dfrac{1+\mu}{\alpha}\right) < c$. Let $r' = \alpha r$, $\alpha' = \dfrac{1+\mu}{\alpha}$, $\beta' = k\left(\dfrac{1+\mu}{\alpha}\right)$. Upon interchanging the roles of x and y, the above becomes: $\exists\ \alpha' > 1$ such that $\forall$ $x, y \in M$

$\rho(x,y) \geq \dfrac{1-\mu}{\alpha} \Rightarrow \exists\ z \in M$ such that

$$B(x;\beta' r') \cap B(y;\alpha' r') \subset B(z;r').$$

The number $\sup \beta'$ for which the above holds is equivalent to $\sup c$ such that M is c-regular.

We return to Pasicki's concept. Again suppose $\Lambda(M) > 1$, and let $1 < \lambda < \Lambda(M)$. Taking $\alpha = \lambda^{-1}$ we see that $\forall r > 0$ and $\forall\ x, y \in M$, $\rho(x,y) > \alpha r \Rightarrow \exists\ z \in M$ such that

$$B(x;r) \cap B(y;r) \subset B(z;\alpha r).$$

The idea now is to replace αr in the above with $\alpha(r)$ where $\alpha(r) < r$. What properties does α need to assure the Lifshits argument will carry over?

It is shown in [4] that the Lifshits constant of a CAT(0) space is $\sqrt{2}$ (the same as that of Hilbert space) and the Lifshits constant of an $\mathbb{R}$-tree is 2. It was conjecture there that the Lifshits constant for a CAT(κ) space, $\kappa < 0$, is a continuous decreasing function of κ which takes values in the interval $(\sqrt{2}, 2)$. (We have since learned that this is true only in spaces of constant curvature [5].)

6. A modification

Let (M, ρ) be a complete metric space. The balls in X are said to be *weakly c-regular* if for each $k < c\ \exists\ \mu \in (0,1)$ and $\alpha_k : \mathbb{R}^+ \to (0,1)$ such that $\forall\ x, y \in M$ and $r > 0$

$\rho(x,y) \geq (1-\mu)r \Rightarrow \exists\ z \in M$ such that

$$B(x;(1+\mu)r) \cap B(y;k(1+\mu)r) \subset B(z;\alpha_k(r)).$$

Now set

$$\kappa_w(M) = \sup\{c \geq 1 : \text{ the balls in } M \text{ are weakly } c\text{-regular}\}.$$

The condition $\alpha_k^n(r) \to 0$ as $n \to \infty$ for each $r > 0$ will imply that every uniformly k-lipschitzian mapping will have approximate fixed points.

The balls in X are said to be *c-regular* if for each $k < c\ \exists\ \mu, \alpha \in (0,1)$ such that $\forall$ $x, y \in X$ and $r > 0$

$\rho(x,y) \geq (1-\mu)r \Rightarrow \exists\ z \in X$ such that

$$B(x;(1+\mu)r) \cap B(y;k(1+\mu)r) \subset B(z;\alpha r).$$

To show that a bead space has Lifshits character greater than 1 it must be shown that the balls of X are c-regular for some $c > 1$. We now present a criterion which implies this.

Theorem 6.1. *Let (X, ρ) be a bead space for which*

$$\inf_{R,\beta} \frac{\delta(R,\beta)}{R} > 0.$$

Then X has Lifshits character greater than 1.

Proof. Choose $\mu \in (0,1)$ so that

$$\mu \leq \frac{\delta(R,\beta)}{R}$$

and choose $k > 1$ so that

$$\alpha := k(1-\mu) < 1.$$

If $r = \dfrac{R}{k}$, we now have

$$1 + \frac{\delta}{R} \geq 1 + \mu \Rightarrow R + \delta \geq R(1+\mu) = k(1+\mu)\frac{R}{k} = k(1+\mu)r.$$

Also

$$1 - \frac{\delta}{R} \leq 1 - \mu \Rightarrow R - \delta \leq (1-\mu)R = k(1-\mu)\frac{R}{k} = \alpha\frac{R}{k} = \alpha r.$$

Letting $\beta = (1-\mu)r$, for $x, y \in X$ such that $\rho(x,y) \geq \beta$ there exists $z \in X$ such that

$$\begin{aligned} B(x;(1+\mu)r) \cap B(y;k(1+\mu)r) &\subset B(x;k(1+\mu)r) \cap B(y;k(1+\mu)r) \\ &\subset B(x;R+\delta) \cap B(y;R+\delta) \\ &\subset B(z;R-\delta) \\ &\subset B(z;\alpha r) \end{aligned}$$

Question 6.2. Do all bead spaces have Lifshits character greater than 1?

7. Δ-convergence in metric spaces

We now turn to some observations of Kirk and Panyanak [12]. Let X be a complete CAT(0) space, let (x_n) be a bounded sequence in X and for $x \in X$ set

$$r(x,(x_n)) = \limsup_{n\to\infty} d(x, x_n).$$

Recall that the *asymptotic radius* $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x,(x_n)) : x \in X\}.$$

The *asymptotic center* $A((x_n))$ of (x_n) is the set

$$A((x_n)) = \{x \in X : r(x,(x_n)) = r((x_n))\}.$$

It is known (see, e.g., [4], Proposition 7) that in a CAT(0) space, $A((x_n))$ consists of exactly one point.

These observations suggest a notion of convergence for the CAT(0) spaces which actually coincides with weak convergence in the special case that the space is a Hilbert space. This notion of convergence was first introduced in metric spaces by T. C. Lim [17], who called it Δ-convergence. (T. Kuczumow [15] introduced a similar notion of convergence in Banach spaces which he called 'almost convergence'.)

Definition 7.1. A sequence (x_n) in X is said to *converge weakly* to $x \in X$ if x is the unique asymptotic center of (u_n) for every subsequence (u_n) of (x_n). In this case we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$.

Next recall that a bounded sequence (x_n) in X is said to be *regular* if $r((x_n)) = r((u_n))$ for every subsequence (u_n) of (x_n). It is known that every bounded sequence in a Banach space has a regular subsequence (see, e.g., [8], p. 166). The proof is metric in nature and carries over to the present setting without change. Since every regular sequence converges weakly, we see immediately that the statement *every bounded sequence in X has a weakly convergent subsequence* is equivalent to the statement *every bounded sequence has a regular subsequence.*

Notice that given $(x_n) \subset X$ such that (x_n) converges weakly to x and given $y \in X$ with $y \neq x$,

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y).$$

Thus X satisfies a condition which is known in Banach space theory as *the Opial property.*

Remark 7.2. Every bounded closed convex subset K of X is weakly closed in the sense that it contains the limits of all of its asymptotcally convergent sequences. To see this suppose (x_n) converges weakly to $x \in X$. Let $P : X \to K$ be the nearest point projection of X onto K. Then P is nonexpansive ([3], p. 177). If $x \notin K$ then $r(P(x), (x_n)) < r(x, (x_n))$, a contradiction.

As a consequence of the preceding observation,

$$x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, \cdots\},$$

where $\overline{conv}(A) = \bigcap \{B : B \supseteq A \text{ and } B \text{ is closed and convex}\}$.

The preceding ideas readily extend to nets. We define the asymptotic radius and asymptotic center for nets analogous to the way they are defined for sequences.

Definition 7.3. A net (x_α) in X is said to Δ*-converge to* $x \in X$ if x is the unique asymptotic center of (u_ξ) for every subnet (u_ξ) of (x_α).

Proposition 7.4. ([11], Proposition 4) *A bounded ultranet is* Δ*-convergent.*

Since every net has a subnet which is an ultranet, we immediately have the following.

Proposition 7.5. *Every bounded net in a complete CAT*(0) *space has a* Δ*-convergent subnet.*

The preceding fact can be reformulated as follows. (Cf., Theorem 3 of [17].)

Proposition 7.6. *Every bounded closed convex set in a complete CAT*(0) *space is* Δ*-compact.*

This brings us to our final question.

Question 7.7. Is there a corresponding notion of Δ-convergence in the spaces described above?

Perhaps remarkably, the answer is 'yes'. We turn now to a another observation of Pasicki [22].

Lemma 7.8. [22] *Let* (X, d) *be a bead space. Then if a bounded sequence* (x_n) *is regular, it is* Δ*-convergent.*

Proof. Suppose for some subsequence (x_{n_k}) of (x_n), $A(x_{n_k}) = \{x\} \neq \{y\} = A(x_n)$. If $x \neq y$ it follows that $r(x_n) > 0$ (otherwise (x_n) would converge to y.) Now we have $x_{n_k} \in B(x; r+\delta) \cap B(y; r+\delta) \subset B(z; r = \delta)$ for some $\delta > 0$ and large n. Hence it follows that $r(x_{n_k}) = r \leq r - \delta$ – a contradiction.

This quickly gives rise to the following result.

Theorem 7.9. [22] *Let (X, d) be a bead space and $F : X \to 2^X$ a mapping. Assume that (x_n) is a regular sequence in X, let $Y := \{x_n : n \in \mathbb{N}\}$, and let G be a selection for $F_{|Y}$ such that*

$$\lim_{n\to\infty} d(x_n, G(x_n)) = 0.$$

If $F(x)$ is compact and $G(y) \subset B(F(x), d(x, y))$ for all $y \in Y$, then $x \in F(x)$.

We refer to [22] for related results.

We close by stating the central result of [23] This is an extension of Theorem 11 of [19], which in turn can be viewed as an extension of the well-known Browder-Göhde-Kirk theorem for nonexpansive mappings in Hilbert space.

Theorem 7.10. *Let (X, d) be a metric space and let $f : X \to X$ be a mapping. Assume that a nonempty subset Y in X is such that $f|_Y : Y \to Y$ and $A(f|_Y) = \{x\}$ (a singleton). Suppose also that*

$$d(f(x), f(y)) \leq d(x, y) \text{ for all } y \in Y.$$

Then x is a fixed point of F.

In connection with the discussion in this section, it is appropriate to call attention to a recent paper by Ahmadi Kakavandi [1]. See Chapter 9 of [14] for more details.

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