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# ON THE HYPERSTABILITY OF $(m, n)$-DERIVATIONS

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**Abstract.** Let $\mathcal{A}$ be a unital algebra, let $\mathcal{X}$ be a unital $\mathcal{A}$-module for which $\mathcal{X}_\rho$ is a $\rho$–complete modular space and let $f : \mathcal{A} \to \mathcal{X}_\rho$ be a mapping. We present some observations concerning hyperstability of the following functional equations

$$\mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) = f(\mu x), \qquad (m+n)f(xy) = 2mx \cdot f(y) + 2ny \cdot f(x)$$

for all $x, y \in \mathcal{A}$ and all $\mu \in \mathbb{T}_{1/n_0} = \{e^{i\theta};\ 0 \le \theta \le 2\pi/n_0\}$, where $m, n \ge 0$ with $m + n \ne 0$ are fixed integers.

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## 1. Introduction and Preliminaries

Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbb{F}$, let $\mathcal{X}$ be a left $\mathcal{A}$–module and let $m, n \ge 0$ with $m + n \ne 0$ be some fixed integers. Then an additive mapping $d : \mathcal{A} \to \mathcal{X}$ is called a module left $(m, n)$-derivation if

$$(m + n)d(xy) = 2mx \cdot d(y) + 2ny \cdot d(x)$$

for all $x, y \in \mathcal{A}$. Clearly, module left $(m, n)$-derivations are one of the natural generalizations of module left derivations (the case $m = n$). In the last few decades a lot of work has been done in the field of left derivations (see, for example [33, 34] and the references therein). Recently also $(m, n)$-derivations were defined and investigated [1, 7, 8, 35, 36].

That notion of stability, for functional equations, has arisen in connection with a problem of Ulam [32] and a solution to it published by Hyers [10]. This work started an avalanche in the theory of stability theory of functional equations, and since then many results have been obtained in this field, studying the Ulam-Hyers stability of differential and integral equations, etc. We should mention also the work of Rassias, who generalized this notion of stability in [28], proving the Ulam-Hyers-Rassias stability of the Cauchy additive functional equation. A very good and deep insight to this theory can be found in [5, 11, 12]. Let us mention that a functional equation is called *hyperstable* if every approximately solution is an exact solution of it. It seems that the first well known hyperstability result appeared in [3] and concerned some ring homomorphisms.

The stability result concerning derivations between operator algebras was first obtained by Šemrl [31]. Badora [2] and Miura et al. [21] considered the Bourgin-type hyperstability of ring derivations on Banach algebras. Also, Park et al. [6, 9, 15, 18, 23, 26, 27] studied the stability and hyperstability of linear derivations and Lie derivations. In [13] Jung examined the stability and hyperstability of module left derivations. Recently, Fošner studied the stability of a functional inequality associated with module left $(m, n)$-derivations [7]. Also, Sadeghi et al. [4, 29, 30] studied the stability of some functional equations in modular spaces.

In this paper, we study the stability and hyperstability of linear module $(m, n)$-derivations from a unital algebra to a unital module by using the Khamsi fixed point theorem in modular spaces [14].

The notion of modular spaces, as a generalization of that of metric spaces, was introduced by Nakano in 1950 [24] and was intensively developed by Luxemburg [19], Koshi and Shimogaki [16] and Yamamuro [37] and their collaborators. Moreover, the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [25] and interpolation theory [17, 20], which in their turn have broad applications [22].

**Definition 1.1.** Let $\mathcal{X}$ be a real (or complex) vector space. A functional $\rho : \mathcal{X} \to [0, \infty]$ is called a modular if for every $x, y \in \mathcal{X}$, the following hold:

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every scaler $\alpha$ with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ provided that $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

If we replace (iii) by

(iii)$'$ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

then the modular $\rho$ is called a convex modular.

**Remark 1.2.** If $a$ and $b$ are positive real numbers with $a \leq b$, then property (iii) shows that

$$\rho(ax) = \rho\left(\frac{a}{b}bx\right) = \rho\left(\frac{a}{b}bx + (1 - \frac{a}{b})0\right) \leq \rho(bx) + \rho(0) = \rho(bx)$$

for all $x \in \mathcal{X}$. If $\alpha_1, \ldots, \alpha_n$ are nonnegative numbers with

$$\sum_{i=1}^{n} \alpha_i = 1,$$

then for all $x_1, \ldots, x_n \in \mathcal{X}$,

$$\rho\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \rho(x_i).$$

The vector space $\mathcal{X}_\rho$ given by $\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \;\text{ as }\; \lambda \to 0\}$ is called a modular space. Generally, the modular $\rho$ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space $\mathcal{X}_\rho$ can be equipped with an $F$–norm defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \quad \rho\left(\frac{x}{\lambda}\right) \le \lambda\right\}.$$

If $\rho$ is convex modular, then

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \quad \rho\left(\frac{x}{\lambda}\right) \le 1\right\}$$

defines a norm on the modular space $\mathcal{X}_\rho$ and is called the Luxemburg norm.

**Definition 1.3.** A function modular is said to satisfy the $\Delta_2$–condition if there exists $M > 0$ such that $\rho(2x) \le M\rho(x)$ for all $x \in \mathcal{X}_\rho$.

If $\rho$ is a convex modular on $\mathcal{X}$ and $|\alpha| \le 1$, then $\rho(\alpha x) \le \alpha\rho(x)$ and also

$$\rho(x) \le \frac{1}{2}\rho(2x) \le \frac{M}{2}\rho(x)$$

if $\rho$ satisfy the $\Delta_2$–condition for all $x \in \mathcal{X}$.

**Definition 1.4.** Let $\mathcal{X}_\rho$ be a modular space.

(i) A sequence $\{x_n\}$ in $\mathcal{X}_\rho$ is said to be

(1) $\rho$–convergent to $x \in \mathcal{X}_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$ (denoted by $\rho - \lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{\rho} x$, the $x_n$ is $\rho$–convergent to $x$),

(2) $\rho$–Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(ii) $\mathcal{X}_\rho$ is $\rho$–complete if any $\rho$–Cauchy sequence is $\rho$–convergent.

(iii) Say that $\rho$ has the Fatou property if $\rho(x) \le \liminf_{n\to\infty} \rho(x_n)$ whenever $x_n \xrightarrow{\rho} x$.

**Example 1.5.** Let $\mathcal{X}_\rho$ be a modular space, then the function $d_\rho$ defined on $\mathcal{X}_\rho \times \mathcal{X}_\rho$ by

$$d_\rho(x,y) = \begin{cases} 0 & x = y, \\ \rho(x) + \rho(y) & x \ne y, \end{cases}$$

is a generalized metric and $(\mathcal{X}_\rho, d_\rho)$ is a generalized metric space.

**Example 1.6.** Let $\varphi$ be a convex, nondecreasing and continuous function defined on the interval such that $\varphi(0) = 0$, $\varphi(\alpha) > 0$ for $\alpha > 0$, $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$. The function $\varphi$ is called an Orlicz function. The Orlicz function $\varphi$ satisfies the $\Delta_2$–condition if there exists $M > 0$ such that $\varphi(2\alpha) \le M\varphi(\alpha)$ for all $\alpha > 0$. Let $(\Omega, \Sigma, \mu)$ be a measure space. Suppose $L^0(\mu)$ is the space of all measurable real-valued (or complex-valued) functions on $\Omega$. Define for every $f \in L^0(\mu)$ the Orlicz modular $\rho_\varphi(f)$ as

$$\rho_\varphi(f) = \int_\Omega \varphi(|f|)d\mu.$$

The associated modular function space with respect to this modular is called an Orlicz space, and will be denoted by $L^\varphi(\Omega,\mu)$ or briefly $L^\varphi$. In other words,

$$L^\varphi = \{f \in L^0(\mu) : \quad \rho_\varphi(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

It is known that the Orlicz space $L^\varphi$ is $\rho_\varphi$–complete. Moreover, $(L^\varphi, \|.\|_{\rho_\varphi})$ is a Banach space, where the Luxemburg norm $\|.\|_{\rho_\varphi}$ is defined as follows

$$\|f\|_{\rho_\varphi} = \inf\left\{\lambda > 0 : \quad \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) d\mu \le 1\right\}.$$

The following fixed point theorem will play an important role in proving our main theorems.

**Theorem 1.7.** ([14]) *Let $\mathcal{C}$ be a $\rho$–complete nonempty subset of $\mathcal{X}_\rho$ and let $T : \mathcal{C} \to \mathcal{C}$ be a quasi-contraction, that is, there exists $K < 1$ such that*

$$\rho(T(x)-T(y)) \leq K \max\{\rho(x-y), \rho(x-T(x)), \rho(y-T(y)), \rho(x-T(y)), \rho(y-T(x))\}.$$

*Let $x \in \mathcal{C}$ such that*

$$\delta_\rho(x) := \sup\{\rho(T^n(x) - T^m(x)) : \ m, n \in \mathbb{N}\} < \infty.$$

*Then $\{T^n(x)\}$ $\rho$–converges to a point $\omega \in \mathcal{C}$. Moreover, if $\rho(\omega - T(\omega)) < \infty$ and $\rho(x - T(\omega)) < \infty$, then the $\rho$–limit of $T^n(x)$ is a fixed point of $T$. Furthermore, if $\omega^*$ is any fixed point of $T$ in $\mathcal{C}$ such that $\rho(\omega - \omega^*) < \infty$, then one has $\omega = \omega^*$.*

## 2. Approximately linear module left $(m, n)$-derivations

In the rest of this paper, unless otherwise explicitly stated, we will assume that $\mathcal{A}$ is an algebra, $\mathcal{X}$ is a $\mathcal{A}$-module for which $\mathcal{X}_\rho$ is a $\rho$–complete modular space, $m$ and $n$ are nonnegative integers with $m + n \neq 0$ and $n_0 \in \mathbb{N}$ is a positive integer and suppose that $\mathbb{T}_{1/n_0} := \{e^{i\theta};\ 0 \leq \theta \leq 2\pi/n_0\}$ and the convex modular $\rho$ has the Fatou property such that it satisfies the $\Delta_2$–condition with $0 < M \leq 2$. For convenience, we use the following abbreviations for a given mapping $f : \mathcal{A} \to \mathcal{X}_\rho$,

$$\Delta_\mu f(x, y) := \mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) - f(\mu x),$$

$$\Delta_{m,n} f(x, y) := (m+n) f(xy) - 2mx \cdot f(y) - 2ny \cdot f(x)$$

for all $x, y \in \mathcal{A}$ and all $\mu \in \mathbb{T}_{1/n_0}$.

**Theorem 2.1.** *Let $f : \mathcal{A} \to \mathcal{X}_\rho$ be a mapping for which there are functions $\phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\rho(\Delta_\mu f(x, y)) \leq \phi(x, y), \tag{2.1}$$

$$\rho(\Delta_{m,n} f(x, y)) \leq \psi(x, y), \tag{2.2}$$

$$\lim_{k\to\infty} \frac{1}{2^k}\phi(2^k x, 2^k y) = 0, \qquad \lim_{k\to\infty} \frac{1}{4^k}\psi(2^k x, 2^k y) = 0 \tag{2.3}$$

*for all $x, y \in \mathcal{A}$ and all $\mu \in \mathbb{T}_{1/n_0}$. If there exists $0 < L < 1$ such that*

$$\phi(2x, 0) \leq 2L\phi(x, 0)$$

*for all $x \in \mathcal{A}$, then there exists a unique linear module left $(m, n)$-derivation $\mathcal{D} : \mathcal{A} \to \mathcal{X}_\rho$ such that*

$$\rho(f(x) - \mathcal{D}(x)) \leq \frac{L}{1-L}\phi(x, 0) \tag{2.4}$$

*for all $x \in \mathcal{A}$.*

*Proof.* Consider the set $\mathfrak{W} := \{g : \mathcal{A} \to \mathcal{X}_\rho\}$ and introduce the mapping $\widetilde{\rho}$ on $\mathfrak{W}$ as follows,

$$\widetilde{\rho}(g) = \inf\{c > 0 : \rho(g(x)) \leq c\phi(x, 0)\}.$$

By the same method as in the proof of Theorem 2.1 in [29], we conclude that $\widetilde{\rho}$ is convex modular and satisfies the $\Delta_2$–condition with $0 < M < 2$. Also, $\mathfrak{W}_{\widetilde{\rho}}$ is $\widetilde{\rho}$–complete.

Now, we define the mapping $\Lambda : \mathfrak{W}_{\widetilde{\rho}} \to \mathfrak{W}_{\widetilde{\rho}}$ as follows

$$(\Lambda g)(x) := 2g\big(\frac{x}{2}\big), \qquad \textit{for all } g \in \mathfrak{W}_{\widetilde{\rho}} \textit{ and } x \in \mathcal{A}.$$

Let $g, h \in \mathfrak{W}_{\widetilde{\rho}}$ and let $c \in [0, \infty]$ be an arbitrary constant with $\widetilde{\rho}(g-h) \le c$. We obtain

$$\rho(g(x) - h(x)) \le c\phi(x, 0)$$

for all $x \in \mathcal{A}$. By the assumption and the last inequality, we get

$$\rho\left(\frac{g(2x)}{2} - \frac{h(2x)}{2}\right) \le \frac{1}{2}\rho(g(2x) - h(2x)) \le \frac{1}{2}c\phi(2x, 0) \le Lc\phi(x, 0)$$

for all $x \in \mathcal{A}$. Hence, $\widetilde{\rho}(\Lambda g - \Lambda h) \le L\widetilde{\rho}(g-h)$ for all $g, h \in \mathfrak{W}_{\widetilde{\rho}}$, so $\Lambda$ is a $\widetilde{\rho}$–strict contraction.

Substituting $y = 0$ and $\mu = 1$ in (2.1), we obtain

$$\rho\left(2f\left(\frac{x}{2}\right) - f(x)\right) \le \phi(x, 0) \tag{2.5}$$

for all $x \in \mathcal{A}$. Letting $x = 2x$ in (2.5), we get

$$\rho\left(2f(x) - f(2x)\right) \le \phi(2x, 0)$$

for all $x \in \mathcal{A}$. Since $\rho$ is convex modular, we obtain

$$\rho\left(f(x) - \frac{f(2x)}{2}\right) \le \frac{1}{2}\rho\left(2f(x) - f(2x)\right) \le \frac{1}{2}\phi(2x, 0) \le L\phi(x, 0) \tag{2.6}$$

for all $x \in \mathcal{A}$. Let $x = 2x$ in (2.6) and then divide both sides by 2 to yield

$$\rho\left(\frac{f(2x)}{2} - \frac{f\left(2^2x\right)}{2^2}\right) \le \frac{1}{2}L\phi(2x, 0) \le L^2\phi(x, 0) \tag{2.7}$$

for all $x \in \mathcal{A}$. It follows from (2.6) and (2.7) that

$$\begin{aligned}\rho\left(f(x) - \frac{f\left(2^2x\right)}{2^2}\right) &\le \frac{1}{2}\rho\left(2f(x) - f(2x)\right) + \frac{1}{2}\rho\left(f(2x) - \frac{1}{2}f\left(2^2x\right)\right)\\ &\le L\phi(x, 0) + L^2\phi(x, 0)\end{aligned}$$

for all $x \in \mathcal{A}$. By induction we obtain

$$\rho\left(f(x) - \frac{f(2^kx)}{2^k}\right) \le \sum_{i=1}^{k} L^i\phi(x, 0) \le \frac{L}{1-L}\phi(x, 0) \tag{2.8}$$

for all $x \in \mathcal{A}$. Now we assert that

$$\delta_{\widetilde{\rho}}(f) = \sup\left\{\widetilde{\rho}\left(\Lambda^k f - \Lambda^\ell f\right); k, \ell \in \mathbb{N}\right\} < \infty.$$

Since $\rho$ is convex modular and satisfies the $\Delta_2$–condition, it follows from (2.8) that

$$\begin{aligned}\rho\left(\frac{f(2^kx)}{2^k}-\frac{f(2^\ell x)}{2^\ell}\right)&\le\frac{1}{2}\rho\left(2f(x)-2\frac{f(2^kx)}{2^k}\right)+\frac{1}{2}\rho\left(2f(x)-2\frac{f(2^\ell x)}{2^\ell}\right)\\&\le\frac{M}{2}\rho\left(f(x)-\frac{f(2^kx)}{2^k}\right)+\frac{M}{2}\rho\left(f(x)-\frac{f(2^\ell x)}{2^\ell}\right)\\&\le\frac{2L}{1-L}\phi(x,0)\end{aligned}$$

for all $x\in\mathcal{A}$ and $k,\ell\in\mathbb{N}$, which implies that $\widetilde{\rho}\left(\Lambda^kf-\Lambda^\ell f\right)\le\frac{2L}{1-L}$ for all $k,\ell\in\mathbb{N}$. Thus, $\delta_{\widetilde{\rho}}(f)<\infty$ and $\{\Lambda^kf\}$ is $\widetilde{\rho}$–converges to $\mathcal{D}\in\mathfrak{W}_{\widetilde{\rho}}$. Since $\rho$ has the Fatou property, (2.8) gives $\widetilde{\rho}(\Lambda\mathcal{D}-f)<\infty$.

Let $x=2^kx$ in (2.6) and then divide both sides by $2^k$ to yield

$$\begin{aligned}\rho\left(\frac{f(2^kx)}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}}\right)&\le\frac{1}{2^k}\rho\left(f\left(2^kx\right)-\frac{1}{2}f\left(2^{k+1}x\right)\right)\\&\le\frac{L}{2^k}\phi(2^kx,0)\le\frac{L}{2^k}(2L)^k\phi(x,0)\le L^{k+1}\varphi(x,0)\\&\le\phi(x,0)\end{aligned}$$

for all $x\in\mathcal{A}$. So, $\widetilde{\rho}(\Lambda\mathcal{D}-\mathcal{D})<\infty$. It follows from Theorem 1.7 that $\widetilde{\rho}$–limit of $\{\Lambda^kf\}$ is fixed point of map $\Lambda$.

It follows from (2.1) that

$$\rho\left(\frac{1}{2^k}\Delta_\mu f\left(2^kx,2^ky\right)\right)\le\frac{1}{2^k}\rho\left(\Delta_\mu f\left(2^kx,2^ky\right)\right)\le\frac{1}{2^k}\phi(2^kx,2^ky)$$

for all $x,y\in\mathcal{A}$ and all $\mu\in\mathbb{T}_{1/n_0}$. Using (2.3) we see that the limit of the right hand side of the above inequality is zero when $k\to\infty$. So, $\Delta_\mu\mathcal{D}\left(x,y\right)=0$ for all $x,y\in\mathcal{A}$ and all $\mu\in\mathbb{T}_{1/n_0}$. Putting $\mu=1$ in $\Delta_\mu\mathcal{D}\left(x,y\right)=0$, we have

$$\mathcal{D}\left(\frac{x+y}{2}\right)+\mathcal{D}\left(\frac{x-y}{2}\right)=\mathcal{D}(x),$$

for all $x,y\in\mathcal{A}$. Setting $x=x+y$ and $y=x-y$ in the last equality gives $\mathcal{D}(x+y)=\mathcal{D}(x)+\mathcal{D}(y)$ for all $x,y\in\mathcal{A}$, that is, $\mathcal{D}$ is additive. So by $\Delta_\mu\mathcal{D}\left(x,y\right)=0$, we can get

$$\mathcal{D}(\mu x)=\frac{1}{2}\mu\mathcal{D}\left(x+y\right)+\frac{1}{2}\mu\mathcal{D}\left(x-y\right)$$

for all $x,y\in\mathcal{A}$ and all $\mu\in\mathbb{T}_{1/n_0}$. Putting $y=0$ in the last equality gives

$$\mathcal{D}(\mu x)=\mu\mathcal{D}\left(x\right)$$

for all $x\in\mathcal{A}$ and all $\mu\in\mathbb{T}_{1/n_0}$. Now, let $\mu=e^{i\theta}\in\mathbb{T}_1$ (i.e., $n_0=1$). We set $\nu=e^{i\theta/n_0}$, thus $\nu\in\mathbb{T}_{1/n_0}$ and

$$\mathcal{D}(\mu x)=\mathcal{D}(\nu^{n_0}x)=\nu^{n_0}\mathcal{D}(x)=\mu\mathcal{D}(x)$$

for all $x\in\mathcal{A}$ and all $\mu\in\mathbb{T}_1$. If $\mu\in j\mathbb{T}_1:=\{j\lambda:\lambda\in\mathbb{T}_1\}$, then by additivity of $\mathcal{D}$, $\mathcal{D}(\mu x)=\mu\mathcal{D}(x)$ for all $x\in\mathcal{A}$ and all $\mu\in j\mathbb{T}_1$. If $\alpha\in(0,\infty)$, then by archimedean property there exists a natural number $j$ such that the point $(\alpha, 0)$ lies in the interior of circle with center at origin and radius $j$. Let

$$\beta = \alpha + \sqrt{j^2 - \alpha^2}\, i$$

and

$$\gamma = \alpha - \sqrt{j^2 - \alpha^2}\, i.$$

Then $\beta, \gamma \in j\mathbb{T}_1$ and $\alpha = \frac{\beta+\gamma}{2}$. Thus

$$\mathcal{D}(\alpha x) = \mathcal{D}(\frac{\beta+\gamma}{2}x) = \frac{\beta+\gamma}{2}\mathcal{D}(x) = \alpha\mathcal{D}(x)$$

for all $x \in \mathcal{A}$ and all $\alpha \in (0, \infty)$. Now, if $\mu \in \mathbb{C}$, then $\mu = |\mu|e^{i\theta}$ and so

$$\mathcal{D}(\mu x) = \mathcal{D}(|\mu|e^{i\theta}x) = |\mu|e^{i\theta}\mathcal{D}(x) = \mu\mathcal{D}(x)$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{C}$. So, the mapping $\mathcal{D}$ is $\mathbb{C}$–linear.

It follows from (2.2) that

$$\rho\left(\frac{1}{2^{2k}}\Delta_{m,n}f\left(2^k x, 2^k y\right)\right) \leq \frac{1}{2^{2k}}\rho\left(\Delta_{m,n}f\left(2^k x, 2^k y\right)\right) \leq \frac{1}{2^{2k}}\psi(2^k x, 2^k y)$$

for all $x, y \in \mathcal{A}$. Using (2.3) we see that the limit of the right hand side of the above inequality is zero when $k \to \infty$. So, $\Delta_{m,n}\mathcal{D}\,(x, y) = 0$, that is, $\mathcal{D}$ is a linear module left $(m, n)$-derivation.

It follows from (2.8) that $\widetilde{\rho}(f - \mathcal{D}) \leq \frac{L}{1-L}$. i.e., the inequality (2.4) holds true for all $x \in \mathcal{A}$.

Also, if $\mathcal{G}$ is another fixed point of $\Lambda$, then

$$\begin{aligned}\widetilde{\rho}(\mathcal{D} - \mathcal{G}) &\leq \frac{1}{2}\widetilde{\rho}\big(2\Lambda\mathcal{D} - 2f\big) + \frac{1}{2}\widetilde{\rho}\big(2\Lambda\mathcal{G} - 2f\big)\\ &\leq \frac{M}{2}\widetilde{\rho}\big(\Lambda\mathcal{D} - f\big) + \frac{M}{2}\widetilde{\rho}\big(\Lambda\mathcal{G} - f\big)\\ &\leq \frac{ML}{1-L} < \infty.\end{aligned}$$

Since $\Lambda$ is $\widetilde{\rho}$–strict contraction, we get

$$\widetilde{\rho}(\mathcal{D} - \mathcal{G}) = \widetilde{\rho}\big(\Lambda\mathcal{D} - \Lambda\mathcal{G}\big) \leq L\widetilde{\rho}(\mathcal{D} - \mathcal{G}),$$

which implies that $\widetilde{\rho}(\mathcal{D} - \mathcal{G}) = 0$ or $\mathcal{D} = \mathcal{G}$ since $\widetilde{\rho}(\mathcal{D} - \mathcal{G}) < \infty$, which proves the uniqueness of $\mathcal{D}$. This completes the proof.

**Corollary 2.2.** *Let $\mathcal{A}$ be a normed algebra, let $\mathcal{B}$ be a Banach algebra and let $0 < r < 1$ and $\varepsilon$ be nonnegative real numbers. If $f : \mathcal{A} \to \mathcal{B}$ is a mapping such that*

$$\|\Delta_\mu f\,(x, y)\,\| \leq \varepsilon(\|x\|^r + \|y\|^r), \qquad \|\Delta_{m,n}f\,(x, y)\,\| \leq \varepsilon\|x\|^r \cdot \|y\|^r$$

*for all $x, y \in \mathcal{A}$ and all $\mu \in \mathbb{T}_{1/n_0}$, then there exists a unique linear module left $(m, n)$-derivation $\mathcal{D} : \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x) - \mathcal{D}(x)\| \leq \frac{\varepsilon}{2^{1-r} - 1}\|x\|^r$$

*for all $x \in \mathcal{A}$.*

*Proof.* It is known that every normed space is modular space with the modular $\rho(x) = \|x\|$ and $M = 2$. Now, the proof follows from Theorem 2.1 by taking

$$\phi(x, y) := \varepsilon(\|x\|^r + \|y\|^r)$$

and

$$\psi(x, y) := \varepsilon\|x\|^r \cdot \|y\|^r$$

for all $x, y \in \mathcal{A}$ and putting $L = 2^{r-1}$.

Now, we formulate and prove a theorem in hyperstability of linear module left $(m, n)$-derivations.

**Theorem 2.3.** *Let $\mathcal{A}$ be a unital algebra and let $\mathcal{X}$ be a unital $\mathcal{A}$-module for which $\mathcal{X}_\rho$ is a $\rho$–complete modular space. Suppose $f : \mathcal{A} \to \mathcal{X}_\rho$ is a mapping for which there is a function $\phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying* (2.1) *and*

$$\rho\left(\Delta_{m,n} f\,(x, y)\right) \leq \phi(x, y), \tag{2.9}$$

$$\lim_{k\to\infty} \frac{1}{2^k}\phi(2^k x, 2^k y) = 0 \tag{2.10}$$

*for all $x, y \in \mathcal{A}$ and all $\mu \in \mathbb{T}_{1/n_0}$. If there exists $0 < L < 1$ such that $\phi(2x, 0) \leq 2L\phi(x, 0)$ for all $x \in \mathcal{A}$, then $f$ is a linear module left $(m, n)$-derivation.*

*Proof.* Notice that $m$ and $n$ are nonnegative integers with $m + n \neq 0$, without loss of generality, let us assume $m \neq 0$. Based on the proof of Theorem 2.1, we can find the linear module left $(m, n)$-derivation $\mathcal{D}$ given by

$$\mathcal{D}(x) = \rho - \lim_{k\to\infty} \Lambda^k f(x) = \rho - \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$$

for all $x \in \mathcal{A}$. It follows from (2.9) and (2.10) that

$$\rho - \lim_{k\to\infty} \left( \frac{m+n}{2^k} f(2^k xy) - \frac{2m2^k x}{2^k} \cdot f(y) - \frac{2ny}{2^k} \cdot f(2^k x) \right) = 0$$

for all $x, y \in \mathcal{A}$. Since $\mathcal{D}$ is a linear module left $(m, n)$-derivation, we have

$$2mx \cdot \mathcal{D}(y) + 2ny \cdot \mathcal{D}(x) = (m+n)\mathcal{D}(xy) = 2mx \cdot f(y) + 2ny \cdot \mathcal{D}(x)$$

for all $x, y \in \mathcal{A}$. Therefore, $mx \cdot \mathcal{D}(y) = mx \cdot f(y)$ for all $x, y \in \mathcal{A}$. If $x = e$, we have $f = \mathcal{D}$, hence $f$ is a linear module left $(m, n)$-derivation.

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