***Fixed Point Theory***, 18(2017), No. 2, 625-640
DOI 10.24193/fpt-ro.2017.2.50
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

# SOME FIXED POINT RESULTS FOR A NEW THREE STEPS ITERATION PROCESS IN BANACH SPACES

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**Abstract.** In this paper, we introduce a three step iteration method and show that this method can be used to approximate fixed point of weak contraction mappings. Furthermore, we prove that this iteration method is equivalent to Mann iterative scheme and converges faster than Picard-S iterative scheme for the class of weak contraction mappings. We also present tables and three graphics to support this result. Finally, we prove a data dependence result for weak contraction mappings using this three step iterative scheme.

**Key Words and Phrases**: A new iterative scheme, strong convergence, weak contraction mapping.
**2010 Mathematics Subject Classification**: 47H10, 47H09.

## 1. Introduction

Let $X$ be a Banach space, and $C$ be a nonempty, closed, convex subset of $X$. Let $T$ be a mapping from a set $C$ to itself. An element $x$ in $C$ is said to be a fixed point of $T$ if $Tx = x$.

The iterative approximation of a fixed point for certain classes of operators is one of the main tools in the fixed point theory. Therefore, a lot of iterative methods have been defined and studied by numerous mathematicians (see [2], [5], [6], [7], [9]-[13], [15], [16]).

Recently, Gürsoy and Karakaya [8] introduced Picard-S iterative process as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)\, Tx_n + \alpha_n T z_n \\ z_n = (1-\beta_n)\, x_n + \beta_n T x_n \ (n \in \mathbb{N}), \end{cases} \tag{1.1}$$

where $(\alpha_n)_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$.

**Lemma 1.1.** [19] *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1-\mu_n)a_n + b_n, \tag{1.2}$$

*where $\mu_n \in (0,1)$ for all $n \geq n_0$, $\sum\limits_{n=1}^{\infty} \mu_n = \infty$ and $\frac{b_n}{\mu_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

**Lemma 1.2.** [18] *Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative real sequence and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ satisfying the following condition:*

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n \eta_n, \tag{1.3}$$

*where $\mu_n \in (0,1)$ such that $\sum\limits_{n=1}^{\infty} \mu_n = \infty$ and $\eta_n \geq 0$. Then the following inequality holds:*

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup \eta_n. \tag{1.4}$$

**Definition 1.3.** [4] *The self-map $T : C \to C$ is called weak-contraction if there exist $\delta \in (0,1)$ and $L \geq 0$ such that*

$$\|Tx - Ty\| \leq \delta \|x-y\| + L \|y - Tx\|.$$

**Theorem 1.4.** [4] *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a weak contraction for which there exist $\delta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$\|Tx - Ty\| \leq \delta \|x-y\| + L_1 \|x - Tx\|. \tag{1.5}$$

*Then, $T$ has a unique fixed point.*

**Definition 1.5.** [3] *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative real convergent sequences with limits a and b respectively. Then, $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$ if*

$$\lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right| = 0. \tag{1.6}$$

**Definition 1.6.** [18] *Let $T$, $S : C \to C$ be two operators. We say that $S$ is an approximate operator of $T$ for all $x \in C$ and a fixed $\varepsilon > 0$ if $\|Tx - Sx\| \leq \varepsilon$.*

In this paper, we introduce the following new iterative scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Ty_n \\ y_n = (1-\alpha_n) z_n + \alpha_n T z_n \\ z_n = T x_n \ (n \in \mathbb{N}) \end{cases} \tag{1.7}$$

where $(\alpha_n)_{n=1}^{\infty} \in [0,1]$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$.

## 2. Main results

**Theorem 2.1.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $T : C \to C$ be a weak-contraction map satisfying condition (1.5). Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1.7) with a real sequence $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$ satisfying $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point $p_*$ of $T$.*

*Proof.* We will follow the same scheme of proof as for a similar result on overlaps ([8], Theorem 1). It can easily be seen from (1.5) that $p_*$ is the unique fixed point of $T$. We have to show that $x_n \to p_*$ as $n \to \infty$. From (1.7) and (1.5), we have

$$\|z_n - p_*\| = \|Tx_n - p_*\| \leq \delta \|x_n - p_*\|,$$

and

$$\begin{aligned} \|y_n - p_*\| &= \|(1-\alpha_n) z_n + \alpha_n T z_n - p_*\| \\ &\leq (1-\alpha_n)\|z_n - p_*\| + \alpha_n \|Tz_n - Tp_*\| \\ &\leq \delta[1-\alpha_n(1-\delta)]\|x_n - p_*\|. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \|x_{n+1} - p_*\| &= \|Ty_n - p_*\| \leq \delta \|y_n - p_*\| \\ &\leq \delta^2[1-\alpha_n(1-\delta)]\|x_n - p_*\|. \end{aligned}$$

Repeating this process n-time, we obtain the following inequalities:

$$\begin{aligned} \|x_n - p_*\| &\leq \delta^2[1-\alpha_{n-1}(1-\delta)]\|x_{n-1} - p_*\| \\ \|x_{n-1} - p_*\| &\leq \delta^2[1-\alpha_{n-2}(1-\delta)]\|x_{n-2} - p_*\| \\ &\ \ ... \\ \|x_1 - p_*\| &\leq \delta^2[1-\alpha_1(1-\delta)]\|x_1 - p_*\|. \end{aligned} \tag{2.1}$$

From inequalities (2.1), we have

$$\|x_{n+1} - p_*\| \leq \|x_1 - p_*\| \delta^{2n} \prod_{i=1}^{n}[1-\alpha_i(1-\delta)]. \tag{2.2}$$

Since $\delta \in (0,1)$, we obtain $[1-\alpha_n(1-\delta)] < 1$.

From classical analysis, we know that $1-x \leq e^{-x}$ for all $x \in [0,1]$. By using this inequality with (2.2), we obtain

$$\begin{aligned} \|x_{n+1}-p_*\| &\leq \|x_1-p_*\| \, \delta^{2n} \prod_{i=1}^{n} e^{-(1-\delta)\alpha_i} \qquad (2.3) \\ &= \|x_1-p_*\| \, \delta^{2n} e^{-(1-\delta)\sum\limits_{i=1}^{n}\alpha_i}. \end{aligned}$$

Taking the limit in both sides of inequality (2.3), it can be seen that $x_n \to p_*$ as $n \to \infty$.

**Theorem 2.2.** *Let $X$ be a Banach space, $C$ be a nonempty, closed, convex subset of $X$ and $T: C \to C$ be a weak-contraction map satisfying condition (1.5) with a fixed point $p_*$. Let $\{u_n\}_{n=1}^{\infty}$ be the Mann iteration process defined in* [12] *with $u_1 \in C$ and $\{x_n\}_{n=1}^{\infty}$ defined by (1.7) with $x_1 \in C$ and a real sequence $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$ satisfying $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then the following assertions are equivalent:*

*i) The Mann (see* [12]*) iteration converges to $p_*$.*
*ii) The new iteration method (1.7) converges to $p_*$.*

*Proof.* We will show that $(i) \Rightarrow (ii)$, that is, if the Mann iteration method converges, then the iteration method (1.7) does too. Now, by using Mann iteration and (1.7) we have

$$\begin{aligned} \|u_{n+1}-x_{n+1}\| &= \|(1-\alpha_n)\,u_n+\alpha_n T u_n - T y_n\| \\ &\leq (1-\alpha_n)\,\|u_n - T y_n\| + \alpha_n \|T u_n - T y_n\| \qquad (2.4) \\ &\leq (1-\alpha_n)\,\{\|u_n - T u_n\| + \|T u_n - T y_n\|\} + \alpha_n \|T u_n - T y_n\| \\ &\leq [1-\alpha_n+L]\,\|u_n - T u_n\| + \delta\,\|u_n - y_n\|, \end{aligned}$$

and

$$\begin{aligned} \|u_n - y_n\| &\leq (1-\alpha_n)\,\|u_n - z_n\| + \alpha_n \|u_n - T z_n\| \\ &\leq (1-\alpha_n)\,\|u_n - z_n\| + \alpha_n\{\|u_n - T u_n\| \qquad (2.5) \\ &\quad + \|T u_n - T z_n\|\} \\ &\leq [1-\alpha_n(1-\delta)]\,\|u_n - z_n\| + \alpha_n(1+L)\,\|u_n - T u_n\|, \end{aligned}$$

and

$$\begin{aligned} \|u_n - z_n\| &\leq \|u_n - T u_n\| + \|T u_n - T x_n\| \qquad (2.6) \\ &\leq \delta\,\|u_n - x_n\| + (1+L)\,\|u_n - T u_n\|. \end{aligned}$$

Substituting (2.6) in (2.5) and (2.5) in (2.4) respectively, we obtain

$$\begin{aligned} \|u_{n+1}-x_{n+1}\| &\leq \{1-\alpha_n+L+\alpha_n\delta(1+L) \\ &\quad +\delta[1-\alpha_n(1-\delta)](1+L)\}\,\|u_n - T u_n\| \\ &\quad +\delta^2[1-\alpha_n(1-\delta)]\,\|u_n - x_n\| \\ &\leq \{1-\alpha_n+L+\delta\,(1+L)\,(1+\alpha_n\delta)\}\,\|u_n - T u_n\| \\ &\quad +[1-\alpha_n(1-\delta)]\,\|u_n - x_n\|. \end{aligned}$$

Let

$$\begin{aligned}\mu_n &= \alpha_n(1-\delta) \in (0,1)\\ a_n &= \|u_n - x_n\|\\ b_n &= \{1-\alpha_n + L + \delta(1+L)(1+\alpha_n\delta)\}\|u_n - Tu_n\|.\end{aligned}$$

Furthermore using $Tp_* = p_*$ and $\|u_n - p_*\| \to 0$ , we have

$$\begin{aligned}\|u_n - Tu_n\| &= \|u_n - p_* + Tp_* - Tu_n\|\\ &\leq \|u_n - p_*\| + \delta\|u_n - p_*\| + L\|p_* - Tp_*\|\\ &= (1+\delta)\|u_n - p_*\|.\end{aligned}$$

Then, $\|u_n - Tu_n\| \to 0$. Because of these results, we obtain $b_n \to 0$. By applying Lemma 1.1, we have $a_n = \|u_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\|u_{n+1} - x_{n+1}\| \to 0 \text{ as } n \to \infty.$$

Now, we show that $(ii) \to (i)$ :

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leq (1-\alpha_n)\|Ty_n - u_n\| + \alpha_n\|Ty_n - Tu_n\| \qquad (2.7)\\ &\leq (1-\alpha_n+\alpha_n L)\|y_n - Ty_n\| + [1-\alpha_n(1-\delta)]\|y_n - u_n\|,\end{aligned}$$

and

$$\begin{aligned}\|y_n - u_n\| &= (1-\alpha_n)\|z_n - u_n\| + \alpha_n\|Tz_n - u_n\|\\ &\leq (1-\alpha_n)\|z_n - u_n\| + \alpha_n\|Tz_n - z_n\| + \alpha_n\|z_n - u_n\| \qquad (2.8)\\ &= \|z_n - u_n\| + \alpha_n\|Tz_n - z_n\|,\end{aligned}$$

and

$$\begin{aligned}\|z_n - u_n\| &= \|Tx_n - u_n\| \qquad (2.9)\\ &\leq \|Tx_n - x_n\| + \|x_n - u_n\|.\end{aligned}$$

Substituting (2.9) in (2.8) and (2.8) in (2.7) respectively, we obtain

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| \leq\ & [1-\alpha_n(1-\delta)]\|x_n - u_n\|\\ &+[1-\alpha_n(1-\delta)]\|x_n - Tx_n\| \qquad (2.10)\\ &+(1-\alpha_n+\alpha_n L)\|y_n - Ty_n\|\\ &+[1-\alpha_n(1-\delta)]\alpha_n\|z_n - Tz_n\|.\end{aligned}$$

Using $Tp_* = p_*$ and $\|x_n - p_*\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - p_*\| + \|Tp_* - Tx_n\|\\ &\leq \|x_n - p_*\| + \delta\|x_n - p_*\| + L\|p_* - Tp_*\|\\ &= (1+\delta)\|x_n - p_*\|.\end{aligned}$$

Then, $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Similarly,

$$\begin{aligned} \|y_n - Ty_n\| &\leq \|y_n - p_*\| + \|Tp_* - Ty_n\| \\ &\leq \|y_n - p_*\| + \delta \|y_n - p_*\| + L \|p_* - Tp_*\| \\ &= (1+\delta) \|y_n - p_*\| \\ &= (1+\delta) \|(1-\alpha_n) z_n + \alpha_n T z_n - p_*\| \\ &\leq (1+\delta)(1-\alpha_n) \|z_n - p_*\| + (1+\delta)\alpha_n \|Tz_n - Tp_*\| \\ &\leq (1+\delta)[1 - \alpha_n(1-\delta)] \|z_n - p_*\|, \end{aligned}$$

and

$$\|z_n - p_*\| = \|Tx_n - p_*\| \leq \delta \|x_n - p_*\|,$$

then $\|z_n - p_*\| \to 0$ as $n \to \infty$. Thus, $\|y_n - Ty_n\| \to 0$ as $n \to \infty$. Moreover,

$$\begin{aligned} \|z_n - Tz_n\| &\leq \|z_n - p_*\| + \|Tp_* - Tz_n\| \\ &\leq \|z_n - p_*\| + \delta \|z_n - p_*\| + L \|p_* - Tp_*\| \\ &= (1+\delta) \|z_n - p_*\|, \end{aligned}$$

and hence $\|z_n - Tz_n\| \to 0$ as $n \to \infty$. Denote,

$$\begin{aligned} \mu_n &= \alpha_n(1-\delta) \in (0,1) \\ a_n &= \|x_n - u_n\| \\ b_n &= [1 - \alpha_n(1-\delta)] \|x_n - Tx_n\| \\ &\quad + (1 - \alpha_n + \alpha_n L) \|y_n - Ty_n\| \\ &\quad + [1 - \alpha_n(1-\delta)]\alpha_n \|z_n - Tz_n\|. \end{aligned}$$

Thus, from Lemma 1.1, $a_n = \|x_n - u_n\| \to 0$ as $n \to \infty$. From (2.10),

$$\|x_{n+1} - u_{n+1}\| \to 0 \text{ as } n \to \infty.$$

**Corollary 2.3.** *Let $X$ be a Banach space, $C$ be a nonempty, closed, convex subset of $X$ and $T : C \to C$ be a weak contraction mapping satisfying condition (1.5) with a fixed point $p_*$. If the initial point is the same for all iterations, then the following assertions are equivalent:*

(1) *the Picard-S iterative scheme (1.1) converges to $p_*$,*

(2) *the new iteration (1.7) converges to $p_*$,*

(3) *the CR iteration (see* [5]*) converges to $p_*$,*

(4) *the Ishikawa iteration (see* [9]*) converges to $p_*$,*

(5) *the $S^*$ iteration (see* [10]*) converges to $p_*$,*

(6) *the Mann iteration (see* [12]*) converges to $p_*$,*

(7) *the Noor iteration (see* [13]*) converges to $p_*$,*

(8) *the SP iteration (see* [15]*) converges to $p_*$,*

(9) *the Picard iteration (see* [16]*) converges to $p_*$.*

In the following theorem, we compare the rate of convergence of iterative scheme (1.7) and Picard-S iterative process (1.1). Also, in order to support the analytical proof of Theorem 2.4 and to demonstrate the efficiency of new iteration (1.7), we give some numerical examples.

**Theorem 2.4.** *Let $X$ be a Banach space, and $C$ be a closed, convex subset of $X$, and $T : C \to C$ be a weak contraction mapping satisfying condition (1.5) with a fixed point $p_*$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in* [0,1] *satisfying (*) $\alpha_1 \leq \alpha_n \leq 1$, $\beta_1 \leq \beta_n \leq 1$ for all $n \in \mathbb{N}$ and some $\alpha_1$, $\alpha_2 > 0$. For given $u_1 = x_1 \in C$, consider the iterative sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ defined by (1.7) and (1.1) respectively. Then, $\{x_n\}_{n=1}^{\infty}$ converges to $p_*$ faster than $\{u_n\}_{n=1}^{\infty}$ does.*

*Proof.* From (2.2) in Theorem 2.1, we have the following inequality

$$\|x_{n+1} - p_*\| \leq \|x_1 - p_*\| \, \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_i(1-\delta)]. \tag{2.11}$$

It is easy to see that,

$$\|u_{n+1} - p_*\| \leq \|u_1 - p_*\| \, \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_i\beta_i(1-\delta)]. \tag{2.12}$$

Applying assumption (*) to (2.11) and (2.12) respectively, we obtain

$$\begin{aligned} \|x_{n+1} - p_*\| &\leq \|x_1 - p_*\| \, \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_1(1-\delta)] \qquad (2.13) \\ &= \|x_1 - p_*\| \, \delta^{2n} [1 - \alpha_1(1-\delta)]^n, \end{aligned}$$

$$\begin{aligned} \|u_{n+1} - p_*\| &\leq \|u_1 - p_*\| \, \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_1\beta_1(1-\delta)] \qquad (2.14) \\ &= \|u_1 - p_*\| \, \delta^{2n} [1 - \alpha_1\beta_1(1-\delta)]^n. \end{aligned}$$

Define

$$a_n = \|x_1 - p_*\| \, \delta^{2n} [1 - \alpha_1(1-\delta)]^n,$$
$$b_n = \|u_1 - p_*\| \, \delta^{2n} [1 - \alpha_1\beta_1(1-\delta)]^n,$$

and

$$\begin{aligned} \psi_n &= \frac{a_n}{b_n} = \frac{\|x_1 - p_*\| \, \delta^{2n} [1 - \alpha_1(1-\delta)]^n}{\|u_1 - p_*\| \, \delta^{2n} [1 - \alpha_1\beta_1(1-\delta)]^n} \\ &= \left( \frac{1 - \alpha_1(1-\delta)}{1 - \alpha_1\beta_1(1-\delta)} \right)^n. \end{aligned}$$

Since $\delta$ and $\beta_1 \in (0,1)$, we have

$$\begin{aligned} \beta_1 &< 1 \\ &\Rightarrow \alpha_1\beta_1 < \alpha_1 \\ &\Rightarrow \alpha_1\beta_1(1-\delta) < \alpha_1(1-\delta) \\ &\Rightarrow \frac{[1 - \alpha_1(1-\delta)]}{[1 - \alpha_1\beta_1(1-\delta)]} < 1. \end{aligned}$$

Therefore, $\lim_{n\to\infty}\psi_n = 0$. From Definition 1.5, we obtain that $\{x_n\}_{n=1}^{\infty}$converges faster than $\{u_n\}_{n=1}^{\infty}$.

**Example 2.5.** Let $X = \mathbb{R}$ and $C = [1,\infty)$. Let $T : C \to C$ be a mapping defined by $T(x) = \frac{3}{4}(x + \frac{1}{x})$ for all $x \in C$. It is easy to show that $T$ is a weak contraction with fixed point $p_* = 1,73205080756888$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{4}$ with the initial value $x_1 = 1$. The following Tables 1-2-3 show that the new iteration method (1.7) converges faster than all S [2], CR [5], Ishikawa [9], S* [10], Mann [12], Noor [13], SP [15], Picard [16], Picard-Mann [17], Picard-S (1.1) iteration methods including the iteration method due to Abbas and Nazir [1].

Table 1. Comparison rate of convergence among some iteration methods

| $x_n$ | Mann | Ishikawa | Noor | SP |
|---|---|---|---|---|
| $x_1$ | 1 | 1 | 1 | 1 |
| $x_2$ | 1,12500000000000 | 1,12760416666667 | 1,12770753644630 | 1,29853765491738 |
| $x_3$ | 1,22135416666667 | 1,22835269106877 | 1,22874546233073 | 1,45852260608018 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{83}$ | 1,73204106834632 | 1,73204843765511 | 1,73204881698286 | 1,73205080756887 |
| $x_{84}$ | 1,73204228575256 | 1,73204877092441 | 1,73204910079701 | 1,73205080756888 |
| $x_{85}$ | 1,73204335098222 | 1,73204905732768 | 1,73204934414543 | 1,73205080756888 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{218}$ | 1,73205080756873 | 1,73205080756887 | 1,73205080756888 | 1,73205080756888 |
| $x_{219}$ | 1,73205080756875 | 1,73205080756887 | 1,73205080756888 | 1,73205080756888 |
| $x_{220}$ | 1,73205080756877 | 1,73205080756887 | 1,73205080756888 | 1,73205080756888 |
| $x_{221}$ | 1,73205080756877 | 1,73205080756887 | 1,73205080756888 | 1,73205080756888 |
| $x_{222}$ | 1,73205080756879 | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{249}$ | 1,73205080756887 | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 |
| $x_{250}$ | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 |

Table 1 shows that SP iteration reaches the fixed point at the $84^{th}$ step while Noor, Ishikawa and Mann iterations reach $218^{th}$step, $221^{th}$ step, and $250^{th}$ step, respectively.

Table 2. Comparison rate of convergence among some iteration methods

| $x_n$ | Picard | Picard-Mann | $S^*$ | S |
|---|---|---|---|---|
| $x_1$ | 1 | 1 | 1 | 1 |
| $x_2$ | 1,50000000000000 | 1,51041666666667 | 1,53152164346837 | 1,50260416666667 |
| $x_3$ | 1,62500000000000 | 1,64207959798239 | 1,65074040007186 | 1,62936277984372 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_{38}$ | 1,73205080756595 | 1,73205080756885 | 1,73205080756886 | 1,73205080756795 |
| $x_{39}$ | 1,73205080756742 | 1,73205080756887 | 1,73205080756887 | 1,73205080756843 |
| $x_{40}$ | 1,73205080756815 | 1,73205080756887 | 1,73205080756887 | 1,73205080756866 |
| $x_{41}$ | 1,73205080756851 | 1,73205080756888 | 1,73205080756888 | 1,73205080756877 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_{46}$ | 1,73205080756887 | 1,73205080756888 | 1,73205080756888 | 1,73205080756887 |
| $x_{47}$ | 1,73205080756887 | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 |
| $x_{48}$ | 1,73205080756887 | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 |
| $x_{49}$ | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 |

Table 2 shows that Picard iteration reaches the fixed point at the $49^{th}$ step while S iteration reaches at the $47^{th}$ step, Picard-Mann and $S^*$ iterations reach at the $41^{th}$step.

Table 3. Comparison rate of convergence among some iteration methods

| $x_n$ | Our iteration | Picard-S | Abbas and Nazır | CR |
|---|---|---|---|---|
| $x_1$ | 1 | 1 | 1 | 1 |
| $x_2$ | 1,63823341836735 | 1,62608657387348 | 1,59716909707178 | 1,53347476846837 |
| $x_3$ | 1,71238829126157 | 1,70759059791304 | 1,69489357952688 | 1,65363032158024 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_{22}$ | 1,73205080756887 | 1,73205080756883 | 1,73205080756650 | 1,73205080137048 |
| $x_{23}$ | 1,73205080756888 | 1,73205080756887 | 1,73205080756819 | 1,73205080494182 |
| $x_{24}$ | 1,73205080756888 | 1,73205080756887 | 1,73205080756868 | 1,73205080645546 |
| $x_{25}$ | 1,73205080756888 | 1,73205080756888 | 1,73205080756882 | 1,73205080709698 |
| $x_{26}$ | 1,73205080756888 | 1,73205080756888 | 1,73205080756886 | 1,73205080736887 |
| $x_{27}$ | 1,73205080756888 | 1,73205080756888 | 1,73205080756887 | 1,73205080748411 |
| $x_{28}$ | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 | 1,73205080753295 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_{39}$ | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 | 1,73205080756887 |
| $x_{40}$ | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 | 1,73205080756888 |

Table 3 shows that our iteration reaches fixed point at the $23^{th}$ step while Abbas and Nazir, Picard-S and CR iterations reach $28^{th}$step, $25^{th}$step, $40^{th}$step, respectively.

The following figures are graphical presentations of the above results:

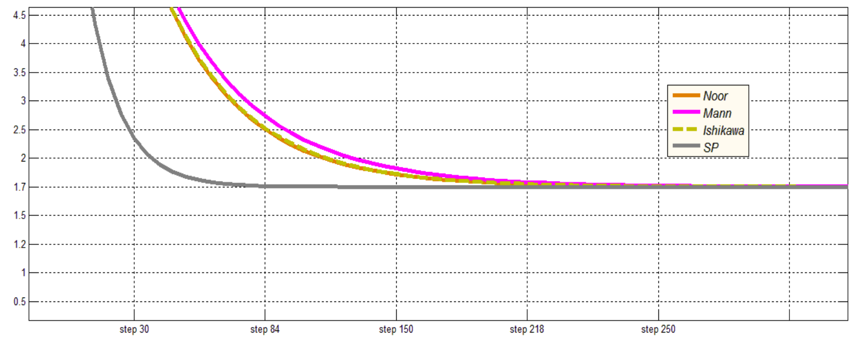


Figure 1. Comparison of rate of convergence among Noor, Mann, Ishikawa and SP

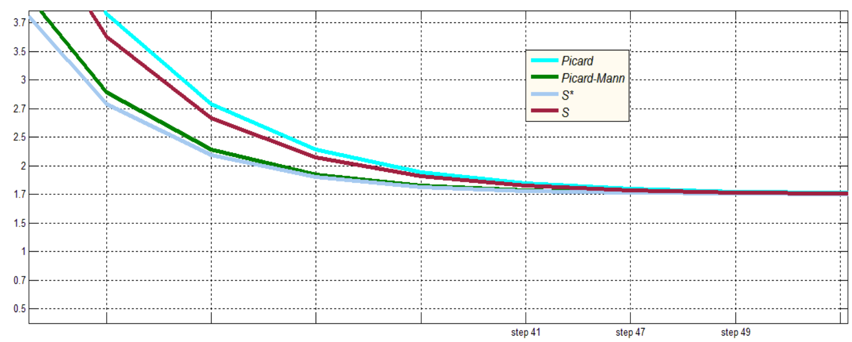


Figure 2. Comparison of rate of convergence among Picard, Picard-Mann,S* and S

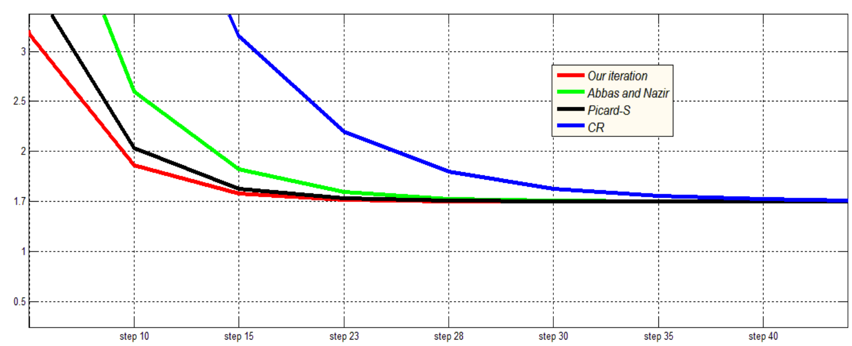


Figure 3. Comparison of rate of convergence among our iteration, Abbas-Nazir, Picard-S and CR

**Example 2.6.** Let $X = \mathbb{R}$ and $C = [0, \infty)$. Let $T : C \to C$ be a mapping defined by $T(x) = \sqrt{x^2 - 8x + 40}$ for all $x \in C$. It is easy to show that $T$ has a unique fixed point $p_* = 5$. Choose $\alpha_n = 0.25$, $\beta_n = 0.40$, $\gamma_n = 0.70$ with the initial value $x_1 = 200$. The following tables show that the new iteration method (1.7) converges faster than all iteration methods which are mentioned in the previous example.

Table 4. Comparison rate of convergence among some iteration methods

| $x_n$ | Mann | Ishikawa | Noor | SP |
|---|---|---|---|---|
| $x_1$ | 200 | 200 | 200 | 200 |
| $x_2$ | 199,0153037326100 | 198,6215491998820 | 198,3460095130540 | 194,6833352080640 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{63}$ | 139,13287449695200 | 114,94795603465800 | 98,11575260075080 | 5,00000000000001 |
| $x_{64}$ | 138,15506757872800 | 113,58617742269400 | 96,49055989897040 | 5,00000000000000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{264}$ | 5,00001351339154 | 5,00000000000246 | 5,00000000000000 | 5,00000000000000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{290}$ | 5,00000004084180 | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{362}$ | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 |

Table 4 shows that SP iteration reaches the fixed point at the $64^{th}$ step while Noor, Ishikawa and Mann iterations reach $264^{th}$step, $290^{th}$ step, and $362^{th}$ step, respectively.

Table 5. Comparison rate of convergence among some iteration methods

| $x_n$ | Picard | Picard-Mann | $S^*$ | S |
|---|---|---|---|---|
| $x_1$ | 200 | 200 | 200 | 200 |
| $x_2$ | 196,0612149304400 | 195,0768276600840 | 194,8012949067630 | 195,6674603977120 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{58}$ | 5,00102675607005 | 5,00000000000175 | 5,00000000000003 | 5,00000023248493 |
| $x_{59}$ | 5,00020545241574 | 5,00000000000028 | 5,00000000000000 | 5,00000004277723 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{62}$ | 5,00000164382032 | 5,00000000000000 | 5,00000000000000 | 5,00000000026648 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{69}$ | 5,00000000002104 | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{75}$ | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 |

Table 5 shows that Picard iteration reaches the fixed point at the $75^{th}$ step while S, Picard-Mann and S* iterations reach at the $69^{th}$ step, $62^{th}$ step and $59^{th}$step respectively.

Table 6. Comparison rate of convergence among some iteration methods

| $x_n$ | Our iteration | Picard-S | Abbas and Nazır | CR |
|---|---|---|---|---|
| $x_1$ | 200 | 200 | 200 | 200 |
| $x_2$ | 191,1396240553450 | 191,7300586118630 | 191,5923857531320 | 193,9744119792280 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{34}$ | 5,00000000000001 | 5,00000000000490 | 5,00000000000747 | 9,56932256205855 |
| $x_{35}$ | 5,00000000000000 | 5,00000000000018 | 5,00000000000034 | 6,68283376485750 |
| $x_{36}$ | 5,00000000000000 | 5,00000000000001 | 5,00000000000002 | 5,36905825343145 |
| $x_{37}$ | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 | 5,05357891261933 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{52}$ | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 | 5,00000000000000 |

Table 6 shows that our iteration reaches fixed point at the $35^{th}$ step, Picard-S and Abbas and Nazir iterations reach the fixed point at the $37^{th}$step while CR iteration reaches the fixed point at the $52^{th}$ step.

**Example 2.7.** Let $X = \mathbb{R}$ and $C = [0, \infty)$. Let $T : C \to C$ be a mapping defined by $T(x) = x - 1 + \frac{1}{e^x}$ for all $x \in C$. It is easy to show that $T$ has a unique fixed point $p_* = 0$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{9}$ with the initial value $x_1 = 3$. The following tables show that the new iteration method (1.7) converges faster than all iteration methods which are mentioned in the above examples.

Table 7. Comparison rate of convergence among some iteration methods

| $x_n$ | Mann | Ishikawa | Noor | SP |
|---|---|---|---|---|
| $x_1$ | 3 | 3 | 3 | 3 |
| $x_2$ | 2,89442078537421 | 2,88330576108344 | 2,88213947357638 | 2,68517474137639 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{104}$ | 0,00012335855652 | 0,00008877462777 | 0,00008681443723 | 0,000000000000000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{305}$ | 0,00000000000001 | 0,00000000000000 | 0,00000000000000 | 0,00000000000000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $x_{328}$ | 0,00000000000000 | 0,00000000000000 | 0,00000000000000 | 0,00000000000000 |

Table 7 shows that SP iteration reaches the fixed point at the $110^{th}$ step, Mann iteration reaches the fixed point at the $328^{th}$ and Noor and Ishikawa iterations reach the fixed point at the $324^{th}$ step, respectively.

Table 8. Comparison rate of convergence among some iteration methods

| $x_n$ | Picard | Picard-Mann | $S^*$ | S |
|---|---|---|---|---|
| $x_1$ | 3 | 3 | 3 | 3 |
| $x_2$ | 2,04978706836786 | 1,94975184975094 | 1,95190778766629 | 2,03867204407709 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_8$ | 0,00000000007486 | 0,00000000000001 | 0,00000000000008 | 0,00000000003124 |
| $x_9$ | 0,00000000000000 | 0,00000000000000 | 0,00000000000000 | 0,00000000000000 |

Table 8 shows that Picard, Picard-Mann, $S^*$ and S iteration reach the fixed point at the $9^{th}$ step.

Table 9. Comparison rate of convergence among some iteration methods

| $x_n$ | Our iteration | Picard-S | Abbas and Nazır | CR |
|---|---|---|---|---|
| $x_1$ | 3 | 3 | 3 | 3 |
| $x_2$ | 1,09483319871399 | 1,16887354205371 | 1,25563779846452 | 1,94202776607450 |
| $x_3$ | 0,06726292068775 | 0,09644256312723 | 0,16086015122482 | 1,00321225851571 |
| $x_4$ | 0,00000193275768 | 0,00000969017760 | 0,00116773275219 | 0,33108061715233 |
| $x_5$ | 0,00000000000000 | 0,00000000000000 | 0,00000005984445 | 0,04304896429304 |
| $x_6$ | 0,00000000000000 | 0,00000000000000 | 0,00000000000000 | 0,00079355690812 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_9$ | 0,00000000000000 | 0,00000000000000 | 0,00000000000000 | 0,00000000000000 |

Table 9 shows that our iteration and Picard-S iteration reach the fixed point at the $5^{th}$ step, but the $4^{th}$ step shows that our iteration process is faster than Picard-S iteration method. Abbas and Nazir iteration reaches the fixed point at the $6^{th}$step while CR iteration reaches the fixed point at the $9^{th}$ step.

**Theorem 2.7.** *Let $S$ be an approximate operator of $T$. Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1.7) for $T$ and define an iterative sequence $\{u_n\}_{n=1}^{\infty}$ as follows:*

$$\begin{cases} u_1 \in C, \\ u_{n+1} = Sv_n \\ v_n = (1-\alpha_n)\, w_n + \alpha_n S w_n \\ w_n = Su_n. \end{cases}$$

*where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying $\frac{1}{2} \leq \alpha_n$ for all $n \in \mathbb{N}$. If $Tp_* = p_*$ and $Sx_* = x_*$ such that $u_n \to x_*$ as $n \to \infty$, then we have*

$$\|p_* - x_*\| \leq \frac{5\varepsilon}{1-\delta},$$

*where $\varepsilon > 0$ is a fixed number.*

*Proof.* Let us consider the following iteration method defined by (1.7) according to $S$,

$$\begin{cases} u_0 \in C, \\ u_{n+1} = Sv_n \\ v_n = (1-\alpha_n)\, w_n + \alpha_n S w_n \\ w_n = S u_n \ (n \in \mathbb{N}). \end{cases} \tag{2.15}$$

From (1.7), (1.5) and (2.15), we have

$$\begin{aligned} \|z_n - w_n\| &= \|Tx_n - Su_n\| \leq \|Tx_n - Tu_n\| + \|Tu_n - Su_n\| \\ &\leq \delta \|x_n - u_n\| + L \|x_n - Tx_n\| + \varepsilon, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \|y_n - v_n\| &= \|(1-\alpha_n)\, z_n + \alpha_n T z_n - (1-\alpha_n)\, w_n - \alpha_n S w_n\| \\ &\leq (1-\alpha_n)\|z_n - w_n\| + \alpha_n \|Tz_n - Sw_n\| \\ &\leq (1-\alpha_n)\|z_n - w_n\| + \alpha_n \|Tz_n - Tw_n\| \\ &\quad +\alpha_n \|Tw_n - Sw_n\| \\ &\leq (1-\alpha_n)\|z_n - w_n\| + \alpha_n\delta \|z_n - w_n\| \\ &\quad +\alpha_n L \|z_n - Tz_n\| + \alpha_n\varepsilon \\ &= [1-\alpha_n(1-\delta)]\|z_n - w_n\| + \alpha_n L \|z_n - Tz_n\| \\ &\quad +\alpha_n\varepsilon. \end{aligned} \tag{2.17}$$

Substituting (2.16) in (2.17), we obtain

$$\begin{aligned} \|y_n - v_n\| &\leq [1-\alpha_n(1-\delta)]\{\delta \|x_n - u_n\| + L\|x_n - Tx_n\| + \varepsilon\} \\ &\quad +\alpha_n L \|z_n - Tz_n\| + \alpha_n\varepsilon, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|Ty_n - Sv_n\| \\ &\leq \|Ty_n - Tv_n\| + \|Tv_n - Sv_n\| \\ &\leq \delta \|y_n - v_n\| + L\|y_n - Ty_n\| + \varepsilon. \end{aligned} \tag{2.19}$$

Substituting (2.18) in (2.19), we obtain

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &\leq \delta[1-\alpha_n(1-\delta)]\{\delta\|x_n - u_n\| + L\|x_n - Tx_n\| + \varepsilon\} \\ &\quad +\alpha_n\delta L\|z_n - Tz_n\| + \alpha_n\delta\varepsilon + L\|y_n - Ty_n\| + \varepsilon \\ &= \delta^2[1-\alpha_n(1-\delta)]\|x_n - u_n\| \\ &\quad +\delta[1-\alpha_n(1-\delta)]L\|x_n - Tx_n\| \\ &\quad +\delta[1-\alpha_n(1-\delta)]\varepsilon + \alpha_n\delta L\|z_n - Tz_n\| \\ &\quad +\alpha_n\delta\varepsilon + L\|y_n - Ty_n\| + \varepsilon. \end{aligned} \tag{2.20}$$

Since $\delta \in (0,1)$ and $\alpha_n \in [0,1]$ for all $n \in \mathbb{N}$ we have

$$\begin{aligned} 1-\alpha_n(1-\delta) &< 1, \\ \delta[1-\alpha_n(1-\delta)] &< 1, \\ (1-\alpha_n) &< 1, \\ \alpha_n\delta &< 1, \end{aligned}$$

and using hypothesis, we obtain

$$1-\alpha_n \leq \alpha_n.$$

Hence, from (2.20) and the above inequalities, we have

$$\begin{aligned} \|x_{n+1}-u_{n+1}\| &\leq [1-\alpha_n(1-\delta)]\|x_n-u_n\| + \alpha_n\delta(1+\delta)L\|x_n-Tx_n\| \\ &\quad +2\alpha_n\varepsilon + \delta\alpha_n L\|z_n-Tz_n\| + \alpha_n\varepsilon \\ &\quad +2\alpha_n L\|y_n-Ty_n\| + 2\alpha_n\varepsilon \\ &= [1-\alpha_n(1-\delta)]\|x_n-u_n\| \\ &\quad +\alpha_n(1-\delta)\left\{\frac{\left\{\begin{array}{c}\delta(1+\delta)L\|x_n-Tx_n\|+2L\|y_n-Ty_n\| \\ +\delta L\|z_n-Tz_n\|+5\varepsilon\end{array}\right\}}{(1-\delta)}\right\} \end{aligned}$$

Denote that

$$\begin{aligned} a_n &= \|x_n-u_n\| \\ \mu_n &= \alpha_n(1-\delta) \in (0,1) \\ \eta_n &= \left\{\frac{\left\{\begin{array}{c}\delta(1+\delta)L\|x_n-Tx_n\|+2L\|y_n-Ty_n\| \\ +\delta L\|z_n-Tz_n\|+5\varepsilon\end{array}\right\}}{(1-\delta)}\right\}. \end{aligned}$$

It follows from Lemma 1.1 that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty}\sup\|x_n-u_n\| \\ &\leq \lim_{n\to\infty}\sup\left\{\frac{\left\{\begin{array}{c}\delta(1+\delta)L\|x_n-Tx_n\|+2L\|y_n-Ty_n\| \\ +\delta L\|z_n-Tz_n\|+5\varepsilon\end{array}\right\}}{(1-\delta)}\right\} \\ &= \frac{5\varepsilon}{(1-\delta)}. \end{aligned}$$

We know from Theorem 2.1 that $x_n \to p_*$ and using hypothesis, we obtain

$$\|p_*-x_*\| \leq \frac{5\varepsilon}{1-\delta}.$$

## 3. Conclusion

After comparing the new iteration defined in this paper with the aforementioned iterations, we conclude that (1.7) is the fastest one among three step iteration methods in current literature.

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*Received: March 6, 2015; Accepted: October 30, 2015.*