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GENERAL FIXED POINT RESULTS IN DISLOCATED METRIC SPACES

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Abstract. The purposes of this paper are twofold. The first is to indicate that fixed point theorems, for single maps, or pairs of maps, defined on metric spaces, remain true for dislocated metric (d-metric) spaces, and that fixed point theorems for weakly compatible maps on d-spaces, are actually special cases of a general theorem on dislocated quasi-metric spaces (dq spaces).

Key Words and Phrases: Dislocated metric space, dislocated quasi-metric space, fixed point theorem, occasionally weakly compatible.
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1. Main results

Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function. If d satisfies
d1: $d(x, x) = 0$,
d2: $d(x, y) = d(y, x) = 0 \quad$ implies that $\quad x = y$,
d3: $d(x, y) = d(y, x)$,
d4: $d(x, y) \leq d(x, z) + d(z, y)$.
for all $x, y, z \in X$, then d is a metric on X. If conditions $d2-d4$ are satisfied, then d is called a dislocated metric (or d-metric space). If d satisfies only conditions $d2$ and $d4$, then is it called a dislocated quasi-metric (or a dq metric space). Dislocated metrics appear in certain problems in topology, logic programming and electronic engineering.

We first observe that general fixed point theorems for a single maps do not require axiom $d1$.

Theorem 1.1. *Let f be a selfmap of a complete d-metric space (X, d). If there exists a constant $h, 0 \leq h < 1$ such that, for each $x, y \in X$*

$$d(fx, fy) \leq h \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \tag{1.1}$$

then f has a unique fixed point in X.

Proof. An examination of the proof of Theorem 1 of [5] (the metric-space version of Theorem 1.1) shows that, proving the existence of a fixed point uses only axioms d2 - d4.

Let p be a fixed point of f. Substituting into (1.1) gives

$$d(p,p) \le hd(p,p),$$

which implies that $d(p,p) = 0$.

We shall now prove uniqueness. Suppose that p, q are fixed points of f. From (1.1), and using d3,

$$d(p,q) = d(fp, fq) = h \max\{d(p,q), 0, 0, d(p,q), d(q,p)\} = hd(p,q),$$

which implies that d(p, q) = 0. From d3, d(q, p) = 0. Using d2, $p = q$, and the fixed point is unique. □

A function $\psi : [0,\infty) \to [0,\infty)$ is called an alternating distance function if ψ is monotone increasing, continuous, and $\psi(t) = 0$ if and only if $t = 0$. Then T is called a generalized weakly contractive mapping if, for all $x, y \in X$,

$$\psi(d(Tx,Ty)) \le \psi(m(x,y)) - \phi(\max\{d(x,y), d(y,Ty)\}), \tag{1.2}$$

where

$$m(x,y) \doteq \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\Big\},$$

where ψ is an alternating distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Theorem 1.2. *Let T be a generalized weakly contractive selfmap of a complete d-metric space (X,d). Then T has a unique fixed point.*

Proof. The metric space version of Theorem 1.2 is Theorem 3.1 of [4]. The proof of Theorem 3.1, that T has a fixed point, only uses axioms d2 - d4.

Let p be a fixed point of T, and assume that $d(p,p) \ne 0$. From (1.2),

$$\psi(d(p,p)) = \psi(d(Tp,Tp) \le \psi(d(p,p)) - \phi(d(p,p)) < \psi(d(p,p)),$$

which implies that $d(p,p) < d(p,p)$, a contradiction. Therefore $d(p,p) = 0$.

Let p, q be fixed points of T, and assume that $d(p,q) \ne 0$. Using (1.2),

$$\begin{aligned}\psi(d(p,q)) = \psi(d(Tp,Tq)) &\le \psi(\max\{d(p,q), 0, 0, [d(p,q) + d(q,p)]/2\}) \\ &\quad - \phi(\max\{d(p,q), 0\}) \\ &= \psi(d(p,q)) - \phi(d(p,q)) < \psi(d(p,q)),\end{aligned}$$

which implies that $d(p,q) < d(p,q)$, a contradiction. Therefore $d(p,q) = 0$, which implies that $p = q$, and the fixed point is unique. □

Some theorems only require $d2$ and $d4$.

Theorem 1.3. *Let T be a selfmap of a dq space (X,d) and let $0 \le \lambda < 1$. If there exists a point $u \in X$ such that the orbit $\overline{O(u)}$ is complete and*

$$d(Tx,Ty) \le \lambda d(x,y) \tag{1.3}$$

holds for any $x, y = Tx \in O(u)$, then $\{T^n u\}$ converges to some point $p \in X$, and

$$d(T^n u, p) \leq \frac{\lambda^n}{1-\lambda} d(u, Tu) \quad for \quad n \geq 1.$$

Further, if T is orbitally continuous at p, or if (1.3) holds for any $x, y \in \overline{O(u)}$, then p is a fixed point of T.

Proof. Define $x_n = T^n u$ for $n \geq 1$. Then, substituting into (1.3) yields

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \leq \cdots \leq \lambda^n d(u, Tu).$$

Thus

$$d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \leq d(u, Tu)\lambda^n \sum_{k=0}^{m-n-1} \lambda^k \leq \frac{1}{1-\lambda} d(u, Tu)\lambda^n,$$

and, from the definition of a Cauchy sequence, $\{x_n\}$ is Cauchy. Since $\overline{O(u)}$ is complete, $\{x_n\}$ converges to a point p. Using (1.3) it follows that

$$d(T^n u, p) \leq \frac{\lambda^n}{1-\lambda} d(u, Tu) \quad for \quad n \geq 1.$$

Thus $\lim T^n u = p$. If T is continuous at p, then p is a fixed point of T. On the other hand, if (1.3) holds for any $x, y \in \overline{O(u)}$, then one has

$$d(T^{n+1} u, Tp) \leq \lambda d(T^n u, p) \quad for\, any \quad n \geq 1,$$

which implies that $p = Tp$. □

The metric space version of Theorem 1.3 is Theorem 2 of [11].

Theorem 3.3 of [15] states the following.

Corollary 1.1. *Let T be a continuous self mapping defined on a complete dq metric space (X, d). Further, let T satisfy the contractive condition*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(yTy)}{d(x, y)} + \beta d(x, y) \tag{1.4}$$

for all $x, y \in X, x \neq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point.

Proof. Substituting $y = Tx$ in (1.4) gives

$$d(Tx, T^2 x) \leq (\alpha + \beta) d(x, Tx) = \lambda d(x, Tx),$$

where $\lambda = \alpha + \beta$. Since $\lambda < 1$, (1.3) is satisfied. Therefore, by Theorem 1.3, T has a fixed point. The uniqueness of p follows from (1.4), so Theorem 3 of [15] is a special case of Theorem 1.3. □

Corollary 1.2. ([15], Theorem 3.5) *Let $X, d)$ be a complete dq metric space (X, d). Let $T : X \to X$ be a continuous mapping satisfying the condition*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \tag{1.5}$$
$$\gamma[d(x, Tx) + d(y, Ty)] + \delta[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X, x \neq y$ and $\alpha, \beta, \gamma \geq 0$, with $\alpha + 2\beta + 2\gamma + 2\delta < 1$. Then T has a unique fixed point.

Proof. Substituting $y = Tx$ in (1.5) implies that

$$d(Tx, T^2x) \leq (\alpha + 2\beta + 2\gamma + 2\delta) \max\{d(x, Tx), d(Tx, T^2x). \\ [d(x, Tx) + d(Tx, T^2x)]/2, [d(x, T^2x) + 0]/2\},$$

which implies that

$$d(Tx, T^2x) \leq \lambda \max\{d(x, Tx), d(Tx, T^2x)\},$$

where $k = \alpha + 2\beta + 2\gamma + 2\delta$, and

$$\lambda = \max\left\{k, \frac{k}{2-k}\right\}.$$

If $d(Tx, T^2x) = 0$, then Tx is a fixed point of T. Otherwise, the above inequality yields (1.3) and T has a fixed point. Contractive condition (1.5) implies the uniqueness of the fixed point □

Corollary 1.3. ([1], Theorem 3.3) *Let (X, d) be a complete dq space, T a continuous selfmap of X satisfying*

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \tag{1.6}$$

for all $x, y \in X$ and $0 \leq \alpha < 1/2$. Then T has a unique fixed point.

Proof. From (1.6), for any $x \in X$,

$$d(Tx, T^2x) \leq \alpha[d(x, Tx) + d(Tx, T^2x)],$$

or

$$d(Tx, T^2x) \leq \lambda d(x, Tx),$$

where $\lambda := \alpha/(1-\alpha)$, which is a special case of (1.3). Therefore, by Theorem 1.2, T has a unique fixed point p. □

Corollary 1.4. *Let (X, d) be a complete dq-metric space, T a continuous selfmap of X satisfying*

$$d(Tx, Ty) \leq \frac{ad(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + bd(x, y) + cd(y, Ty) \tag{1.7}$$

for all $x, y \in X$, where $a, b, c > 0$ with $a + b + c < 1$. Then T has a unique fixed point.

Proof. From (1.7),

$$d(Tx, T^2x) \leq ad(Tx, T^2x) + bd(x, Tx) + cd(Tx, T^2x) \leq \frac{b}{1-1-c} d(x, Tx),$$

which implies (1.3). The result then follows from Theorem 1.3. □

We now prove two theorems for a pair of maps.

Theorem 1.4. *Let f, g be selfmaps of a complete d-metric space (X, d) satisfying, for some $h, 0 \leq h < 1$, and for all $x, y \in X$,*

$$d(fx, gy) \leq h \max\{d(x, y), d(x, fx), d(y, gy), [d(x, gy) + d(y, fx)]/2\}. \tag{1.8}$$

Then f and g have a unique common fixed point p.

Proof. The metric space version of Theorem 1.3 (Theorem 14 of [14]) only uses d2 - d4 in proving that f and g have a common fixed point.

Let p be a common fixed point of f and g. Substituting into (1.8) we have

$$d(p, p) \leq hd(p, p)$$

which implies that $d(p, p) = 0$.

To prove uniqueness, suppose that p and q are common fixed points of f and g. From (1.3),

$$d(p, q) = d(fp, gq) \leq h \max\{d(p, q), 0, 0, [d(p, q) + d(q, p)]/2\} = hd(p, q),$$

which implies that $d(p, q) = 0$. From $d3$, $d(q, p) = 0$. Using $d2$, $p = q$ and the fixed point is unique. □

The following result is Theorem 3.6 of [15].

Corollary 1.5. *Let (X, d) be a complete dislocated metric space. Let $S, T : X \to X$ be continuous mappings satisfying*

$$d(Sx, Ty) \leq \alpha \max\{d(x, Sx) + d(y, Ty), d(y, Ty) + d(x, y), d(x, Sx) + d(x, Ty)\} \tag{1.9}$$

for all $x, y \in X$ and $\alpha \in [0, 1/2)$. Then S and T have a common fixed point.

Proof. Note that $d(x, Sx) + d(y, Ty) \leq 2 \max\{d(x, Sx), d(y, Ty)\}$. Similarly, $d(y, Ty) + d(x, y) \leq 2 \max\{d(y, Ty), d(x, y)\}$ and $d(x, Sx) + d(x, y) \leq 2 \max\{d(x, Sx), d(x, y)\}$. Thus (1.9) implies that

$$d(Sx, Ty) \leq 2\alpha \max\{d(x, y), d(x, Sx), d(y, Ty)\},$$

which is a special case of (1.8). The result now follows from Theorem 1.4. Note that the assumption of continuity is not needed. □

The following is Theorem 3.3 of [8].

Corollary 1.6. *Let (X, d) be a complete dislocated metric space. Let f, g be continuous selfmaps of X satisfying (1.9) for all $x, y \in X$, where $0 < h < 1$. Then f and g have a common fixed point.*

Proof. The result follows from Theorem 1.4. Note that continuity of the maps is not needed. □

The following is Theorem 3.6 of [1].

Corollary 1.7. *Let (X, d) be a compete dislocated metric space, f, g continuous selfmaps of X satisfying*

$$d(fx, gy) \leq h \max\{d(x, y), d(x, fx), d(y, gy)\} \tag{1.10}$$

for all $x, \in X$, where $0 < h < 1$. Then f and g have a common fixed point.

Proof. Inequality (1.10) is a special case of (1.8). Again, the assumption of continuity is not needed. □

Theorem 1.5. *Let (X, d) be a complete d-metric space, and S, T two selfmaps of X such that, for all $x, y \in X$,*

$$\psi(d(Sx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{1.11}$$

where

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,

(b) $\varphi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$,

(c) $M(x, y) := \max\{d(x, y), d(x, Sx), d(y, Ty), [d(x, Ty) + d(y, Sx)]/2\}$.

Then S and T have a unique common fixed point.

Proof. The metric space version of Theorem 1.5 is Theorem 2.1 of [7], and only d2 - d4 are used to establish the existence of a common fixed point.

Let p be a common fixed point of S and T, and assume that $d(p, p) \neq 0$. Substituting in (1.6) gives

$$\psi(d(p, p)) \leq \psi(d(p, p)) - \varphi(d(p, p)) < \psi(d(p, p)),$$

a contradiction.

Suppose that p and q are common fixed points of S and T with $d(p, q) \neq 0$.

$$M(p, q) = \max\{d(p, q), 0, 0, [d(p, q) + d)q, p)]/2\} = d(p, q).$$

Substituting into (1.11),

$$\psi(d(p, q)) = \psi(d(Sx, Ty)) \leq \psi(d(p, q)) - \varphi(d(p, q)) < \psi(d(p, q)),$$

which implies that $d(p, q) < d(p, q)$, a contradiction. It then follows,using $d3$ and $d2$, that the common fixed point is unique. □

Park [12] proved a metric space version of Theorem 1.2 for two maps. Phrasing this theorem in a dq space, it reads as follows:

Theorem 1.6. *Let $S$and T be selfmaps of a dq space X. If there exists a sequence $\{u_n\}$ in X such that*

$$u_{2n+1} = Su_{2n}, \quad u_{23n+2} = Tu_{2n+1} \quad for \quad n \geq 0$$

and $\overline{\{u_n\}}$ is complete, and if there exists a $\lambda \in [0, 1)$ such that

$$d(Sx, Ty) \leq \lambda d(x, y) \tag{1.12}$$

for each distinct $x, y \in \overline{\{u_n\}}$ satisfying either $x = Ty$ or $y = Sx$, then either (1) S or T has a fixed point in $\{u_n\}$, or (2) $\{u_n\}$ converges to some $p \in X$, and

$$d(u_n, p)\frac{\lambda^n}{1 - \lambda}d(u_0, u_n), \quad for \quad n > 0.$$

Further, if one of S or T is continuous at p and (1.11) holds for any distinct $x, y \in \overline{\{u_n\}}$, then p is a common fixed point of S and T.

Proof. Set $c_n = d(u_n, u_{n+1})$. If $c_n = 0$ for any n, then (1.1) holds. Suppose that $c_n > 0$ for all n. Then we have $c_{n+1} \leq \lambda c_n$ for all $n \geq 0$. Therefore $\{u_n\}$ is Cauchy, and hence (1.11) holds. Suppose that T is continuous at p and (1.11) holds for any distinct $x, y \in \overline{\{u_n\}}$. Since p is a limit point of $\overline{\{u_n\}}$, we have $Sp = Tp$ and, since $u_{2n} \to p$ and $u_{2n+1} = Tu_{2n} \to Tp$, it follows that $p = Sp = Tp$. □

A pair of maps $\{f, g\}$ is called occasionally weakly compatible (owc) if there exists a common coincidence point at which the maps commute. A dislocated symmetric on a space X is a mapping $r : X \times X \to [0, \infty)$ such that

$$x = y \quad \text{if} \quad r(x, y) = 0, \quad \text{and} \quad r(x, y) = r(y, x) \quad \text{for} \quad x, y \in X.$$

Theorem 1.7. *Let X be a set with a dislocated symmetric r Suppose that f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each owc. If*

$$r(fx, gy) < M(x, y) \tag{1.13}$$

for each $x, y \in X$ for which $fx \neq gy$, and where

$$M(x, y) := \max\{r(x, y), r(Sx, fx), d(Ty, gy), r(Sx, gy), r(Ty, fx)\}.$$

then there is a unique point $w \in X$ such that $fw = gw = w$ and a unique point $z \in X$ such that $gz = Tz = z$. Moreover, $z = w$, so that $f, g, S,$ and T have a unique common fixed point.

The symmetric space version of Theorem 1.5 is Theorem 1 of [9]. The proof of a fixed point in Theorem 1 of [9] uses only the definition of a dislocated symmetric. Uniqueness follows as in the previous theorems of this paper.

The following is Theorem 2.6 of [3].

Corollary 1.8. *Let (X, d) be a complete d-metric space. Let $A, B, S, T : X \to X$ be continuous mappings satisfying*

1. *$T(X) \subset A(X), S(X) \subset B(X)$,*
2. *The pairs (S, A) and (T, B) are weakly compatible and*
3. *For all $, y \in X$ and $\alpha, \beta, \gamma > 0$ satisfying $\alpha + \beta + \gamma \leq 1/4$, we have*

$$d(Sx, Ty) \leq \alpha[d(Ax, Ty) + d(By, Sx)] + \beta[d(Ax, Sx) + d(By, Ty)] + \gamma d(Ax, By)$$

4. *The range of one of the mappings A, B, S or T is a complete subspace of X.*

Then A, B, S and T have a unique common fixed point.

Proof. Condition 3 is a special case of (1.13). In the course of the proof of Theorem 2.6 it is shown that (S, A) and (B, T) have common coincidence points. By 2 they are owc. The result follows from Theorem 1.7. □

The following is Theorem 3.1 of [13].

Corollary 1.9. ([13], Theorem 3.1) *Let (X, d) be a complete d-metric space, A, B, S, T, L, M selfmaps of X satisfying*

1. *$M(X) \subset AB(X)$ and $L(X) \subset ST(X)$*
2. *The pairs (L, AB) and (M, ST) are occasionally weakly compatible and*

3. *For all* $x, y \in X, \alpha, \beta \geq 0$ *and* $\gamma > 0$ *satisfying* $\alpha + \beta + \gamma \leq 1/4$,

$$d(Lx, My) \leq \alpha d(ABx, My) + \beta d(STy, Lx)] + \gamma d(ABx, STy)$$

for all $x, y \in X$, *where* $\alpha, \beta, \gamma \geq 0, 0 \leq \alpha + \beta + \gamma < 1/2$.

4. *The range of one of the mappings* A, B, S *or* T *is a complete subspace of* X. *Then* $AB, ST.L$ *and* M *have a unique common fixed point.*

Proof. Although (1.3) involves six maps, the fact that ST and AB are always together makes the theorem a result about four maps, which satisfy an inequality that is a special case of (1.13). Since (L, AB) and (M, ST) are assumed to be occasionally weakly compatible, the result follows from Theorem 1.7. □

Let $\Phi = \{\varphi | \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semicontinuous, $\varphi(t) > 0$ for each $t > 0$, and $\varphi(0) = 0\}$. Define $\Psi = \{\psi | \psi : [0, \infty) \to]0, \infty)$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0.\}$

Corollary 1.10. ([2], Theorem 2.1) *Suppose that* f, g, S *and* T *are selfmaps of a complete metric space* (X, d) *satisfying* $f(X) \subset T(X)$ *and* $g(X) \subset S(X)$ *and such that the pair* $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible. If*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{1.14}$$

for each $x, y \in X$, *where* $\varphi \in \Phi, \psi \in \Psi$, *and where*

$$M(x, y) := \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), [d(Sx, gy) + d(fx, Ty)]/2\}.$$

Then f, g, S *and* T *have a unique common fixed point in* X *provided one of the ranges* $f(X), g(X), S(X)$ *and* $T(X)$ *is closed.*

Proof. If there exist x, y such that $M(x, y) = 0$, then $Sx = Ty, fx = Sx, gy = Ty, Sx = gy$, and $fx = Ty$. Therefore $fx = Sx$ and $gy = Ty$, which implies that $\{f, S\}$ and $\{g, T\}$ have coincidence points. Since they are assumed to be weakly compatible, they are owc. Also, $M(x, y) = 0$ reduces (1.14) to the triviality $0 \leq 0$. Therefore (1.14) is only meaningful if x and y are such that $M(x, y) \neq 0$. Then (1.14) implies

$$\psi(d(fx, gy)) < \psi(M)x, y)),$$

and, since ψ is nondecreasing, it follows that

$$d(fx, gy) < M(x, y),$$

and (1.14) is a special case of (1.13).

In the course of the proof of Theorem 2.1 it is shown that $\{f, S\}$ and $\{g, T\}$ each have a coincidence point. Since (f, S) and (g, T) are assumed to be weakly compatible, they are owc. The result then follows from Theorem 1.7. □

Corollary 1.11. ([6], Theorem 2.3) *Let* A, B, S, *and* T *be self mappings of a metric space* (X, d) *such that* $AX \subset TX$ *and* $BX \subset SX$. *Assume that there exists a* $\psi : [0, \infty) \to [0, \infty)$ *such that*

(i) F *is nondecreasing, continuous, and* $F(0) = 0 < F(t)$ *for every* $t > 0$;
(ii) ψ *is nondecreasing, right continuous, and* $\psi(t) < t$ *for every* $t > 0$;
(iii) $F(d(Ax, By)) \leq \psi(F(M(x, y)))$ *for all* $x, y \in X$.

If one of AX, TX, BX, and SX is a complete subspace of X, then the following hold:

(iv) *A and S have a coincidence point;*

(v) *T and B have a coincidence point.*

Further, if A and S as well as B and T are weakly compatible, then A, B, S, and T have a unique common fixed point.

Proof. Since $\psi(t) < t$ for each $t > 0$, (iii) implies that

$$F(d(Ax, By)) < F(M(x, y)).$$

Since F is nondecreasing, the above inequality implies that

$$d(Ax, By) < M(x, y),$$

which is a special case of inequality (1.13). Conditions (iv) and (v) and the fact that the maps are pairwise weakly compatible imply that they are pairwise owc. The result then follows from Theorem 1.7. □

Corollary 1.12. [6], Theorem 2.4) *Let A, B, S, and T be self mappings of a metric space (X, d) such that $AX \subset TX, BX \subset SX$. Assume that there exists a nondecreasing right continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(t) < t$ for all $t > 0$, such that*

$$\int_0^{d(Ax,By)} \varphi(t)dt \le \psi\Big(\int_0^{M(x,y)} \varphi(t)dt\Big), \tag{1.15}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function which is nonnegative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0, \quad \text{for every} \quad \epsilon > 0.$$

If one of AX, TX, BX, and SX is a complete subspace of X, then the following hold:

(i) *A and S have a coincidence point;*

(ii) *T and B have a coincidence ;point.*

Further, if A and S as well as B and T are weakly compatible, then A, B, S, and T have a unique common fixed point.

Proof. Since $\psi(t) < t$, inequality (1.15) implies that

$$\int_0^{d(Ax,By)} \varphi(t)dt < \int_)^{M(x,y)} \varphi(t)dt,$$

which in turn implies that

$$d(Ax, By) < M(x, y),$$

which is a special case of (1.13). Conditions (i) and (ii) and the fact that the maps are pairwise weakly compatible implies that they are owc. The conclusion follows from Theorem 1.7. □

Remark. Although only two fixed point theorems for a single map have been extended from a metric space to a dislocated space, it is a reasonable conjecture that every fixed point theorem for a single map defined on a metric space is extendable to the corresponding fixed point theorem on a d-metric space. The same remarks apply to a pair of maps.

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